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GENERALIZED SPECTRAL RELATIVE PERTURBATION BOUNDS FOR POSITIVE DEFINITE MATRICES

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Abstract

In the field of image identification and vibrancy theory, we often consider the equations for the generalized eigenvalues of matrix. We discuss perturbation questions on generalized eigenvalues of the complex positive definite matrix (possibly non-Hermitian). Some relative perturbation bounds for generalized eigenvalues of the complex positive definite matrix are established.

1. Introduction

We denote the set of $n \times n$ complex matrices by M_n. Let $A = (a_{ij}) \in M_n$, A^* be the conjugate transpose of matrix A and A^{-*} be the conjugate transpose of matrix A^{-1} if A is invertible. $\| A \|_2 = (\lambda_{\max}(A^* A))^{1/2}$ is the spectral norm of matrix A and $\kappa(A) = \| A \|_2 \| A^{-1} \|_2$ is the spectral norm condition number of matrix A with respect to inversion.

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In the field of image identification and vibrancy theory, we often consider the equation

$$Ax = \lambda Bx, \quad 0 \neq x \in C^n,$$

where λ is a scalar, and both A and B are $n \times n$ matrices. If a scalar λ and a non-zero vector $x \in C^n$ happen to satisfy this equation, then λ is called an *eigenvalue* of A with respect to B (it is called *generalized eigenvalue* of A for short). If B is nonsingular, then λ is an eigenvalue of A with respect to B if and only if λ is the eigenvalue of $B^{-1}A$. A matrix $A \in M_n$ is called a *normal matrix* with respect to H (it is called *generalized normal matrix* for short) if there exists a positive definite Hermitian matrix H such that $A^*HA = AHA^*$ (see [10, 11, 12]). A matrix $A \in M_n$ is called a *complex positive definite matrix* if $\mathrm{Re}(x^*Ax) > 0$, for any non-zero $x \in C^n$ (see [6]). We denote the Toeplitz decomposition $A = H(A) + K(A)$, where $H(A) = \frac{1}{2}(A + A^*)$, $K(A) = \frac{1}{2}(A - A^*)$. Obviously, A is a complex positive definite matrix if and only if $H(A)$ is positive definite.

Now, there are many famous results on the perturbation problems for eigenvalues of the positive definite Hermitian matrix (see [1], [3], [7-9]). In this paper, we discuss perturbation questions on the generalized eigenvalues of complex positive definite matrix (possibly non-Hermitian). Some relative perturbation bounds for generalized eigenvalues of the complex positive definite matrix are established.

2. Main Result

Lemma 1 (see [10, Theorem 5])**.** *Suppose $A \in M_n$ and H is a positive definite Hermitian matrix. Then A is a normal matrix with respect to H if and only if there exists a nonsingular matrix P such that*

$$PAP^* = \Lambda, \tag{1}$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Furthermore, $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are n eigenvalues of* HA *with* $H = P^*P$ (*or n eigenvalues of A with respect to* H^{-1}).

Remark 1. By the proof of Lemma 1, we see that $P = H^{-1/2}U^*$ with U is unitary in (1) (see [10], the proof of Theorem 5).

Lemma 2. *If* $A \in M_n$ *is a positive definite matrix, then*

$$A^*H(A)^{-1}A = AH(A)^{-1}A^*, \tag{2}$$

and A is a normal matrix with respect to $H(A)^{-1}$, *where* $H(A) = \frac{1}{2}(A + A^*)$.

Proof. By a simple calculation, we have

$$\begin{aligned} A^*H(A)^{-1}A &= AH(A)^{-1}A^* \\ &= H(A) - K(A)H(A)^{-1}K(A). \end{aligned}$$

Theorem 1. $A \in M_n$ *is a complex positive definite matrix if and only if there exists a nonsingular matrix P, such that*

$$PAP^* = \Lambda, \tag{3}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $H(A)^{-1} = P^*P$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of A with respect to* $H(A)$, *where* $H(A) = \frac{1}{2}(A + A^*)$.

Proof. If A is a complex positive definite matrix, then by Lemma 1 and Lemma 2, it is easy to get the necessity. Conversely, since P is nonsingular and $H(A)^{-1} = P^*P$, we know that $H(A)$ is positive definite, so A is a positive definite matrix.

Remark 2. Since (3) is equivalent to $H(A)^{-1}A = P^*\Lambda P^{-*}$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A with respect to $H(A)$, so we say that (3) is a *generalized eigenvalue decomposition* of matrix A and any positive definite matrix A has generalized eigenvalue decomposition (3).

If H is a positive definite Hermitian matrix and $A \in M_n$ is a normal matrix with respect to H, we say that A is an *H-normal matrix*.

Theorem 2. *Let* $A \in M_n$ *be an H-normal matrix with* (1). *If* $\tilde{\lambda}$ *is an eigenvalue of* $\tilde{A} = A + E$ *with respect to* H^{-1}, *then*

$$\min_{1 \le i \le n} | \tilde{\lambda} - \lambda_i | \le \| PEP^* \|_2 \le \kappa(P) \| HE \|_2, \tag{4}$$

where $H = P^* P$.

Proof. By equality (1), we have $P\tilde{A}P^* = \Lambda + PEP^*$, so

$$H(A + E) = P^*(\Lambda + PEP^*)P^{-*}.$$

Then $\tilde{\lambda}$ is an eigenvalue of $\Lambda + PEP^*$ from the assumption. Hence $\tilde{\lambda}I - \Lambda - P^{-1}EP^{-*}$ is a singular matrix, where I is the identity matrix. If $\tilde{\lambda}I - \Lambda$ is singular, then there is some i, such that $\tilde{\lambda} = \lambda_i$ and the bound (4) holds. If $\tilde{\lambda}I - \Lambda$ is nonsingular, then

$$(\tilde{\lambda}I - \Lambda)^{-1}(\tilde{\lambda}I - \Lambda - PEP^*) = I - (\tilde{\lambda}I - \Lambda)^{-1} PEP^*$$

is singular, and hence it must be that $\| (\tilde{\lambda}I - \Lambda)^{-1} PEP^* \|_2 \ge 1$ (see [5, Corollary 5.6.16]). Thus

$$\begin{aligned} 1 &\le \| (\tilde{\lambda}I - \Lambda)^{-1} PEP^* \|_2 \le \| PEP^* \|_2 \| (\tilde{\lambda}I - \Lambda)^{-1} \|_2 \\ &= \| PEP^* \|_2 \max_{1 \le i \le n} | \tilde{\lambda} - \lambda_i |^{-1} = \| PEP^* \|_2 (\min_{1 \le i \le n} | \tilde{\lambda} - \lambda_i |)^{-1}. \end{aligned}$$

Since $H = P^* P$, apply the property of spectral norm to get

$$\min_{1 \le i \le n} | \tilde{\lambda} - \lambda_i | \le \| PEP^* \|_2 \le \| P^{-*} \|_2 \| HE \|_2 \| P^* \|_2.$$

(4) holds by $k(P) = k(P^*)$.

Theorem 3. *Let* $A \in M_n$ *be a positive definite matrix with generalized eigenvalue decomposition* (3). *If* $\tilde{\lambda}$ *is an eigenvalue of* $\tilde{A} = A + E$ *with respect to* $H(A)$, *then*

$$\min_{1 \le i \le n} | \tilde{\lambda} - \lambda_i | \le \| PEP^* \|_2 \le \kappa(P) \| H(A)^{-1} E \|_2, \tag{5}$$

where $H(A)^{-1} = P^* P$.

Proof. Applying Lemma 2 and Theorem 2, Theorem 3 can easily be proved.

Theorem 4. *Let $A \in M_n$ be a positive definite matrix with generalized eigenvalue decomposition* (3). *If $\widetilde{\lambda}$ is an eigenvalue of $\widetilde{A} = A + E$ with respect to $H(A)$, then*

$$\min_{1 \leq i \leq n} \frac{|\lambda_i - \widetilde{\lambda}|}{|\lambda_i|} \leq k(P) \| A^{-1} E \|_2. \tag{6}$$

Proof. Let $\hat{A} = \widetilde{\lambda} H(A) A^{-1} H(A)$ and $\hat{E} = -H(A) A^{-1} E$. If

$$H(A)^{-1}(\hat{A} + \hat{E})\widetilde{x} = \widetilde{x},$$

then

$$[\widetilde{\lambda} A^{-1} H(A) - A^{-1} E]\widetilde{x} = \widetilde{x},$$

so $(A + E)\widetilde{x} = \widetilde{\lambda} H(A)\widetilde{x}$, and $H(A)^{-1}(A + E)\widetilde{x} = \widetilde{\lambda}\widetilde{x}$. Hence, we can write $H(A)^{-1}(A + E)\widetilde{x} = \widetilde{\lambda}\widetilde{x}$ as

$$H(A)^{-1}(\hat{A} + \hat{E})\widetilde{x} = \widetilde{x}, \tag{7}$$

where

$$\hat{A} = \widetilde{\lambda} H(A) A^{-1} H(A) \quad \text{and} \quad \hat{E} = -H(A) A^{-1} E. \tag{8}$$

Since A is normal with respect to $H(A)^{-1}$, $AH(A)^{-1} A^* = A^* H(A)^{-1} A$. By (8), we have

$$\hat{A} H(A)^{-1} \hat{A}^* = \widetilde{\lambda}\widetilde{\lambda}^* H(A) A^{-1} H(A) A^{-*} H(A)$$

and

$$\hat{A}^* H(A)^{-1} \hat{A} = \widetilde{\lambda}^* \widetilde{\lambda} H(A) A^{-*} H(A) A^{-1} H(A).$$

Then $\hat{A} H(A)^{-1} \hat{A}^* = \hat{A}^* H(A)^{-1} \hat{A}$, so $\hat{A}$ is normal with respect to $H(A)^{-1}$. Since

$$\hat{A} = \widetilde{\lambda} H(A) A^{-1} H(A) \quad \text{and} \quad H(A) = (P^* P)^{-1},$$

by (3), we get

$$P\hat{A}P^* = \tilde{\lambda}PH(A)A^{-1}H(A)P^*$$

$$= \tilde{\lambda}P^{-*}A^{-1}P^{-1} = \tilde{\lambda}\Lambda^{-1}. \tag{9}$$

So $\hat{A}$ has the generalized eigenvalue decomposition (9) with n eigenvalues $\tilde{\lambda}/\lambda_i$ $(i = 1, 2, \ldots, n)$ with respect to $H(A)$. By (7), we know that 1 is the eigenvalue of $\hat{A} + \hat{E}$ with respect to $H(A)$. Hence, applying Theorem 2 to the eigenvalues $\tilde{\lambda}/\lambda_i$ $(i = 1, 2, \ldots, n)$ of $\hat{A}$ with respect to $H(A)$ and the eigenvalue 1 of $\hat{A} + \hat{E}$ with respect to $H(A)$, and by (8), we have

$$\min_{1 \le i \le n} | 1 - \tilde{\lambda}/\lambda_i | \le k(p) \| H(A)^{-1}\hat{E} \|_2$$

$$= k(P) \| A^{-1}E \|_2.$$

So inequality (6) holds.

Let $B, C \in M_n$. Then $[B, C] = BC - CB$ is called the *commutator*, B and C commute if and only if $[B, C] = 0$.

Theorem 5. *Let* $A \in M_n$ *be positive definite with generalized eigenvalue decomposition* (3) *and let*

$$A = A_1A_2 \quad and \quad [H(A)^{-1/2}A_1, \; A_2H(A)^{-1/2}] = 0. \tag{10}$$

If $\tilde{\lambda}$ *is an eigenvalue of* $\tilde{A} = A + E$ *with respect to* $H(A)$, *then*

$$\min_{1 \le i \le n} \frac{| \lambda_i - \tilde{\lambda} |}{| \lambda_i |} \le \| A_1^{-1}EA_2^{-1} \|_2. \tag{11}$$

Proof. Define

$$\tilde{A}_1 = H(A)^{-1/2}A_1, \quad \tilde{A}_2 = A_2H(A)^{-1/2}.$$

Then $H(A)^{-1/2}AH(A)^{-1/2} = \tilde{A}_1\tilde{A}_2$, and we get $\tilde{A}_1\tilde{A}_2 = \tilde{A}_2\tilde{A}_1$ from (10). From the hypothesis, A is a normal matrix with respect to $H(A)^{-1}$, we have $AH(A)^{-1}A^* = A^*H(A)^{-1}A$, then

$$[H(A)^{-1/2}AH(A)^{-1/2}][H(A)^{-1/2}AH(A)^{-1/2}]^*$$

$$= [H(A)^{-1/2}AH(A)^{-1/2}]^*[H(A)^{-1/2}AH(A)^{-1/2}]$$

and hence $H(A)^{-1/2}AH(A)^{-1/2}$ is normal. Obviously, λ_i is the eigenvalue of $H(A)^{-1/2}AH(A)^{-1/2}$ and $\widetilde{\lambda}$ is the eigenvalue of $H(A)^{-1/2}\widetilde{A}H(A)^{-1/2}$. Notice that

$$H(A)^{-1/2}\widetilde{A}H(A)^{-1/2} = H(A)^{-1/2}AH(A)^{-1/2} + H(A)^{-1/2}EH(A)^{-1/2},$$

applying [3, Corollary 3.1], we get

$$\min_{1\le i\le n}\frac{|\lambda_i - \widetilde{\lambda}|}{|\lambda_i|} \le \| \widetilde{A}_1^{-1}H(A)^{-1/2}EH(A)^{-1/2}\widetilde{A}_2^{-1}\|_2 = \| A_1^{-1}EA_2^{-1}\|_2.$$

Theorem 6. *Let $A \in M_n$ be positive definite and $H(A)^{-1/2}AH(A)^{-1/2} = PU$ be a polar factorization with P positive definite Hermitian and U be unitary. Then*

(1) *matrix A has a QU factorization $A = QUQ^*$, where $Q = H(A)^{1/2}P^{1/2}$ is a positive definite symmetrizable matrix, i.e., there exists a nonsingular matrix D, such that $D^{-1}QD$ is a positive definite diagonal matrix.*

(2) *if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigenvalues of A with respect to $H(A)$, and $\widetilde{\lambda}$ is an eigenvalue of $\widetilde{A} = A + E$ with respect to $H(A)$, then*

$$\min_{1\le i\le n}\frac{|\lambda_i - \widetilde{\lambda}|}{|\lambda_i|} \le \| P^{-1/2}H(A)^{-1/2}EH(A)^{-1/2}P^{-1/2}\|_2. \tag{12}$$

Proof. (1) Because $H(A)^{-1/2}AH(A)^{-1/2}$ is normal (see the proof of Theorem 5) with a polar factorization $H(A)^{-1/2}AH(A)^{-1/2} = PU$ and $PU = P^{1/2}UP^{1/2}$ (see [3, Lemma 3.2]), so $A = QUQ^*$, where $Q = H(A)^{1/2}P^{1/2}$. Since both P and $H(A)$ are positive definite Hermitian matrices, $P^{1/2}$ and $H(A)^{1/2}$ are positive definite Hermitian matrices, too. Hence $Q = H(A)^{1/2}P^{1/2}$ is similar to a positive definite diagonal matrix.

(2) Matrix A has a QU factorization $A = QUQ^*$, where $Q = H(A)^{1/2}P^{1/2}$. Set

$$A_1 = QU,\ A_2 = Q^*,$$

then $A = A_1A_2$. By $Q^*H(A)^{-1}Q = P$ and $H(A)^{-1/2}AH(A)^{-1/2} = P^{1/2}UP^{1/2}$, we have

$$\begin{aligned}[H(A)^{-1/2}A_1,\ A_2H(A)^{-1/2}] &= H(A)^{-1/2}AH(A)^{-1/2} - A_2H(A)^{-1}A_1\\ &= P^{1/2}UP^{1/2} - Q^*H(A)^{-1}QU\\ &= P^{1/2}UP^{1/2} - PU = 0.\end{aligned}$$

Notice that U is unitary, apply Theorem 5 to get that

$$\begin{aligned}\min_{1\le i\le n}\frac{|\lambda_i - \tilde{\lambda}|}{|\lambda_i|} &\le \| A_1^{-1}EA_2^{-1}\|_2 = \| U^{-1}Q^{-1}EQ^{-*}\|_2\\ &= \| P^{-1/2}H(A)^{-1/2}EH(A)^{-1/2}P^{-1/2}\|_2.\end{aligned}$$

Theorem 7. *Let $A \in M_n$ be positive definite. Suppose*

$$H(A)^{-1/2}AH(A)^{-1/2} = PU$$

is a polar factorization with P positive definite Hermitian, and U is unitary. If $\tilde{\lambda}$ is an eigenvalue of $\tilde{A} = A + E$ with respect to $H(A)$, and D is nonsingular such that

$$E = DE_1D^*,\quad P = DM_1D^*,$$

then

$$\min_{1\le i\le n}\frac{|\lambda_i - \tilde{\lambda}|}{|\lambda_i|} \le \| E_1\|_2\| G\|_2^2, \tag{13}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n eigenvalues of A with respect to $H(A)$ and $G = M_1^{-1/2}D^{-1}H(A)^{-1/2}D$.

Proof. Since P is positive definite Hermitian, $M_1 = D^{-1}PD^{-*}$ is also positive definite Hermitian. Since

$$P = P^{1/2}P^{1/2} = DM_1^{1/2}(DM_1^{1/2})^*,$$

$P^{1/2}$ and $DM_1^{1/2}$ are both "Cholesky factorizations" of P, they are related by a unitary matrix Q, i.e., $P^{1/2} = (DM_1^{1/2})Q$.

Since $\| . \|_2$ is unitarily invariant norm and Q is unitary, by

$$P^{-1/2} = Q^* M_1^{-1/2} D^{-1} = D^{-*} M_1^{-1/2} Q = (P^{-1/2})^*$$

and applying Theorem 6, we have

$$\min_{1 \le i \le n} \frac{|\lambda_i - \widetilde{\lambda}|}{|\lambda_i|} \le \| P^{-1/2} H(A)^{-1/2} E H(A)^{-1/2} P^{-1/2} \|_2$$

$$= \| M_1^{-1/2} D^{-1} H(A)^{-1/2} D E_1 D^* H(A)^{-1/2} D^{-*} M_1^{-1/2} \|_2$$

$$= \| G E_1 G^* \|_2 \le \| E_1 \|_2 \| G \|_2^2,$$

where $G = M_1^{-1/2} D^{-1} H(A)^{-1/2} D.$

Corollary 7.1. *Let* $A \in M_n$ *be positive definite. Let*

$$H(A)^{-1/2} A H(A)^{-1/2} = PU$$

be a polar factorization with P Hermitian positive definite and U be unitary. Let D be nonsingular and

$$A = DMD^*, \quad E = DE_1D^*, \quad P = DM_1D^*.$$

If $\widetilde{\lambda}$ *is an eigenvalue of* $\widetilde{A} = A + E$ *with respect to* $H(A)$, *and* $\| E_1 \|_2 \le \varepsilon \| M \|_2$, *where* ε *is a small positive number, then*

$$\min_{1 \le i \le n} \frac{|\lambda_i - \widetilde{\lambda}|}{|\lambda_i|} \le \varepsilon \| M \|_2 \| G \|_2^2, \tag{14}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are n eigenvalues of A with respect to* $H(A)$ *and*

$$G = M_1^{-1/2} D^{-1} H(A)^{-1/2} D.$$

Proof. Theorem 7 implies

$$\min_{1 \le i \le n} \frac{|\lambda_i - \widetilde{\lambda}|}{|\lambda_i|} \le \| E_1 \|_2 \| G \|_2^2 \le \varepsilon \| M \|_2 \| G \|_2^2,$$

where $G = M_1^{-1/2} D^{-1} H(A)^{-1/2} D.$

Corollary 7.2. *If, in addition to the assumptions of Corollary* 7.1, D *also commutes with* $H(A)^{1/2}$, *and* M *is unitary, then*

$$\min_{1 \le i \le n} \frac{|\lambda_i - \tilde{\lambda}|}{|\lambda_i|} \le \varepsilon \| M_1^{-1/2} H(A)^{-1/2} \|_2^2. \tag{15}$$

Proof. $\| M \|_2 = 1$ because M is unitary. $G = M_1^{-1/2} H(A)^{-1/2}$ because D commutes with $H(A)^{1/2}$. So Corollary 7.1 implies inequality (15).

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