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H^p, $p \in \mathbb{N}^*$, H^∞-DENSITIES IN NUMBER THEORY. PART II: H^p, $p \in \mathbb{N}^*$, H^∞-DENSITIES OF ARITHMETIC FUNCTIONS

(In memory of Professor Aimé Fuchs (1925-2006)
my teacher in probabilistic number theory)

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Abstract

In this paper, we generalize a process of iteration of a positive arithmetic function. We introduce densities: H^p, $p \in \mathbb{N}^*$, H^∞ and study these densities for positive arithmetic functions. Some applications are obtained.

1. Introduction

To resolve the problem of the nonexistence of asymptotic density in the first-digit problem, we have used in [6] the following method:

Let E be the subset of $\mathbb{N}^*$ of the first-digit problem. Consider, for all

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integer $n \geq 1$, the expression defined by

$$(Hf)(n) := \nu_n^{(1)}(I_E) := \frac{1}{n}\sum_{k=1}^{n} I_E(k),$$

where $I_E(k)$ is the indicator function of the set E.

If the limit, $\lim\limits_n \nu_n^{(1)}(I_E)$ exists, as n tends to infinity, then E admits an asymptotic density, else, we pass to the next iterate. Namely, we form

$$\nu_n^{(2)}(I_E) := \frac{1}{n}\sum_{s=1}^{n} \nu_s^{(1)}(I_E).$$

To remove the abnormal behavior of this mean, Flehinger [7] suggests that if we proceed to one infinity of iterates, we come to compute a density of the set E of the first-digit problem.

The above leads us to consider the process of iterates for positive arithmetic functions.

In this paper, introducing the notions of H^p, $p \in \mathbb{N}^*$, H^∞-densities, we study these for a class of positive bounded arithmetic functions. Further, we generalize some results obtained in [6]. We further obtain that

$$H^1\text{-density} \Leftrightarrow H^p(p \in \mathbb{N}^*)\text{-density} \Rightarrow H^\infty\text{-density}$$

for positive bounded arithmetic functions. Finally, some applications are demonstrated.

Definition 1.1. Let f be a positive arithmetic function and let $p \in \mathbb{N}^*$. We say that f has the number $\ell \geq 0$ as an H^p-*density*, if, $\ell = \lim\limits_n (H^p f)(n)$, when n tends to infinity. If this is the case, we shall denote this density by $d_p(f)$.

Definition 1.2. Let f be a positive arithmetic function, with the same notations as above. We say that f has the number $\ell \geq 0$ as an H^∞-*density*, if the limits

$$\lim_{(p\to+\infty)} (\liminf_{(n\to+\infty)}) \nu_n^{(p)}(f)$$

and

$$\lim_{(p\to+\infty)} (\limsup_{(n\to+\infty)}) \nu_n^{(p)}(f)$$

exist and are equal to ℓ. If this is the case, we shall denote this density by $d_\infty(f)$.

2. Main Results

Next, the concept of H^∞-density is defined with the Hölder's transformation H. These averages are known as *Césaro means* [1, p. 103].

We prove that asymptotic density and H^p-density, for $p \in \mathbb{N}^*$, are equivalent for the class of positive arithmetic functions.

2.1. Generalities on the transformation H

Denote the set of real sequences by $\mathfrak{F}$, and the set of limited real sequences by $\mathfrak{F}^\circ$. Denote the elements of $\mathfrak{F}$ by f, g, h, ...

For $f \in \mathfrak{F}$, we define

$$\bar{f} := \limsup_{(n\to+\infty)} f(n); \quad \underline{f} := \liminf_{(n\to+\infty)} f(n).$$

Proposition 2.1. *For* $f \in \mathfrak{F}^\circ$,

$$\| f \| = \bar{f} - \underline{f}$$

defines a semi-norm on $\mathfrak{F}^\circ$.

Definition 2.1. We call *Hölder's transformation* all maps

$$\begin{cases} H : \mathfrak{F} \to \mathfrak{F} \\ \ f \mapsto Hf, \end{cases}$$

where

$$(Hf)(n) := \frac{1}{n}\sum_{k=1}^{n} f(k), \quad n = 1, 2, \ldots .$$

In other words, $(Hf)(n)$ is the expectation of f relatively to the probability measure ν_n introduced in [3]. Namely,

$$E_{\nu_n}(f) = (Hf)(n).$$

Like that, convergence of the sequence (Hf) is equivalent to the existence of an asymptotic density of f.

2.2. Properties of the transformation H

The Hölder's transformation H has some characteristic properties given by the following propositions:

Proposition 2.2. *For all* $f \in \mathfrak{F}$, *we have*

$$-\infty \leq \underline{f} \leq \underline{H}f \leq \overline{H}f \leq \bar{f} \leq +\infty.$$

Proof. (a) Obviously, we have $\underline{H}f \leq \overline{H}f$.

(b) Remains to prove that $\underline{f} \leq \underline{H}f$ and $\overline{H}f \leq \bar{f}$.

$(b)_1$ We prove that $\overline{H}f \leq \bar{f}$. For this purpose; for all $m, n \in \mathbb{N}^*$ such that $n > m$, we have

$$(Hf)(n) = \frac{1}{n}\sum_{k=1}^{n} f(k)$$

$$= \frac{1}{n}\sum_{k=1}^{m} f(k) + \frac{1}{n}\sum_{k=m+1}^{n} f(k)$$

$$\leq \frac{1}{n}\sum_{k=1}^{m} f(k) + \frac{n-m}{n}(\sup_{k>m} f(k))$$

the integer m is fixed, for all $(\varepsilon > 0)$, there exists an integer $N(\varepsilon) > m$ such that

$$\forall (n \geq N(\varepsilon)), (Hf)(n) \leq \varepsilon + \sup_{k>m} f(k),$$

which implies that for all $m \in \mathbb{N}^*$, for all $(\varepsilon > 0)$

$$\limsup_{(n\to+\infty)} (Hf)(n) \le \varepsilon + \sup_{k>m} f(k).$$

Namely,

$$\limsup_{(n\to+\infty)} (Hf)(n) \le \limsup_{(n\to+\infty)} f(n). \text{ Thus } \overline{H}f \le \bar{f}.$$

$(b)_2$ We prove that $\underline{f} \le \underline{H}f$.

Holds by the same way.

Remark 2.1. (a) The Hölder's transformation H transforms limited sequence to a limited sequence.

(b) The Hölder's transformation H transforms a convergent sequence to a convergent sequence.

Proposition 2.3. *If $f \in \mathfrak{F}^\circ$, then the set of limit points of the sequence (Hf) is the interval $[\underline{H}f, \overline{H}f]$.*

Proof. Let $f \in \mathfrak{F}^\circ$, and let $M \ge 0$ be such that for all $n \in \mathbb{N}^*$, $|f(n)| \le M$. We have

$$\begin{aligned}(Hf)(n+1) - (Hf)(n) &= \frac{1}{n+1}\sum_{k=1}^{n+1} f(k) - (Hf)(n)\\ &= \frac{1}{n+1} f(n+1) + \frac{1}{n+1}\sum_{k=1}^{n} f(k) - (Hf)(n)\\ &= \frac{1}{n+1}(f(n+1) - (Hf)(n)).\end{aligned}$$

So

$$|(Hf)(n+1) - (Hf)(n)| \le \frac{1}{n+1}(|f(n+1)| - |(Hf)(n)|) \le \frac{2M}{n+1}.$$

It holds that

$$\lim_{(n\to+\infty)} |(Hf)(n+1) - (Hf)(n)| = 0.$$

Proposition 2.4 [8]. *For all couples* (α, β) *of real numbers such that* $0 \leq \alpha \leq \beta \leq 1$, *there exists a sequence f with valued in* $\{0; 1\}$ *such that* $\underline{H}f = \alpha$ *and* $\overline{H}f = \beta$.

Proof. For all $n \in \mathbb{N}^*$ and for all f, we have

$$(Hf)(n+1) = \frac{n}{n+1}(Hf)(n) + \frac{f(n+1)}{n+1}.$$

Put

$$e(n) := (Hf)(n+1) - (Hf)(n) = \frac{1}{n+1}(f(n+1) - (Hf)(n)).$$

So, if f is with valued in $\{0; 1\}$, the difference $e(n)$ proves that

$$|e(n)| \leq \frac{1}{n+1}.$$

Further,

$$e(n) \geq 0 \quad \text{if } f(n+1) = 1.$$

$$e(n) \leq 0 \quad \text{if } f(n+1) = 0.$$

These relations allow us to construct a sequence f which answers to Proposition 2.4. Put

$$\varepsilon_n = \frac{1}{n}, \quad n \in \mathbb{N}^*$$

and

$$f(k) = \begin{cases} 0 & \text{for } k = 1, \\ 1 & \text{for } k = 2, \ldots, n_1, \end{cases}$$

where n_1 is the least integer ≥ 2, such that $(Hf)(n_1) \in \left]\beta - \varepsilon_1, \beta + \varepsilon_1\right[$ (here, as $\varepsilon_1 = 1$, $n_1 = 2$).

And, put $f(k) = 0$ for $k = n_1 + 1, \ldots, n_2$, where n_2 is the least integer $> n_1$, such that $(Hf)(n_2) \in \left]\alpha - \varepsilon_1, \alpha + \varepsilon_1\right[$.

Thus, we construct by recurrence a strictly increasing sequence of integers $(n_k)_{k \geq 1}$, and we define successively the sequence f on the sets $\{n_k + 1, \ldots, n_{k+1}\}$ such that

(a) we have

$$f(k) = \begin{cases} 0 & \text{if } k = 1 \text{ or } k \in \{n_{2p+1} + 1, \ldots, n_{2p+2}\}, \\ 1 & \text{if } k \in \{n_{2p} + 1, \ldots, n_{2p+1}\}. \end{cases}$$

(b) . n_{2p+1} is the least integer $> n_{2p}$, such that

$$(Hf)(n_{2p+1}) \in \left]\beta - \varepsilon_p, \beta + \varepsilon_p\right[.$$

. n_{2p+2} is the least integer $> n_{2p+1}$, such that

$$(Hf)(n_{2p+2}) \in \left]\alpha - \varepsilon_p, \alpha + \varepsilon_p\right[.$$

Then, we have

$$\lim_{(p \to +\infty)} (Hf)(n_{2p}) = \alpha, \qquad \lim_{(p \to +\infty)} (Hf)(n_{2p+1}) = \beta.$$

$$\forall k \in \{n_{2p} + 1, \ldots, n_{2p+1}\}, \quad (Hf)(n_{2p}) \leq (Hf)(k) \leq (Hf)(n_{2p-1}).$$

$$\forall k \in \{n_{2p-1}, \ldots, n_{2p}\}, \quad (Hf)(n_{2p}) \leq (Hf)(k) \leq (Hf)(n_{2p-1}).$$

It holds that $\overline{H}f = \beta$, $\underline{H}f = \alpha$;

Proposition 2.2 shows that for all $f \in \mathfrak{F}^\circ$, $\| Hf \| \leq \| f \|$.

Consequently, if we denote H^p the pth iteration of Hölder's transformation H, namely,

$$H^0 = I,\ H^1 = H,\ H^{p+1} = H \circ H^p \quad \text{for } p = 1, 2, \ldots$$

then, for all $f \in \mathfrak{F}^\circ$, the sequence $(\|H^p f\|)_{p \geq 1}$ is decreasing, limited below by 0, and consequently it converges.

Definition 2.2. Let $f : \mathbb{N}^* \to \mathbb{R}^+$ be a positive arithmetic function and $p \in \mathbb{N}^*$. We call that f admits ℓ $(\ell \geq 0)$ as H^p*-density* if the limit, $\lim_n (H^p f)(n)$ exists and equal to ℓ, as $n \to +\infty$. If this is the case, we shall denote this density $d_p(f)$.

Definition 2.3. Let $f : \mathbb{N}^* \to \mathbb{R}^+$ be a positive arithmetic function. We say that *f* admits ℓ $(\ell \geq 0)$, as H^∞-density if limits,

$$\lim_{(p \to +\infty)} (\underline{H^p} f)(n)$$

and

$$\lim_{(p \to +\infty)} (\overline{H^p} f)(n)$$

exist and are equal to ℓ. If this is the case, we shall denote this density $d_\infty(f)$.

Remark 2.2. (a) The existence of one H^p-density, for $f \in \mathfrak{F}^\circ$, is equivalent to $\|H^p f\| = 0$.

(b) The existence of one H^∞-density, for $f \in \mathfrak{F}^\circ$, is equivalent to

$$\lim_{(p \to +\infty)} \|H^p f\| = 0.$$

(c) The case $p = 1$ agrees to the asymptotic density introduced in [3], denoted $d(f)$.

We have the following theorems which characterize these densities.

Theorem 2.5 (Comparison Theorem between: H^1 and H^p, $p \in \mathbb{N}^*$-densities)**.** *Let* $f : \mathbb{N}^* \to \mathbb{R}^+$ *be a limited and positive arithmetic function. Then, for all real number* $\ell \geq 0$, *the following two statements are equivalent*:

$(p_1) : f$ *admits* ℓ *as* H^1*-density and* $(d_1(f) = \ell)$;

$(p_2) : f$ *admits* ℓ *as* H^p*-density, for all* $p \in \mathbb{N}^*$ *and* $\left(d_p(f) = \ell\right)$.

Nevertheless this is not true for $p = +\infty$, as shows by the Theorem 2.7 below.

Proof. $(p_1) \Rightarrow (p_2)$ holds from Proposition 2.2.

$(p_2) \Rightarrow (p_1)$: It is enough to prove that if $\lim\limits_{(n\to+\infty)} H^2 f(n) = \ell$, then $\lim\limits_{(n\to+\infty)} H^1 f(n) = \ell$.

For this purpose, we use Theorem 2.2 [3, Theorem 2.2, p. 340]; the general term of the sequence $H^1 f$ is

$$g(k) = \frac{1}{k}\sum_{s=1}^{k} f(s).$$

It proves

$$\begin{aligned} n(g(n) - g(n+1)) &= n\left(\frac{1}{n}\sum_{k=1}^{n} f(k) - \frac{1}{n+1}\sum_{k=1}^{n+1} f(k)\right) \\ &= n\left(\left(\frac{1}{n} - \frac{1}{n+1}\right)\sum_{k=1}^{n} f(k) - \frac{f(n+1)}{n+1}\right) \\ &= \frac{1}{n+1}\sum_{k=1}^{n} f(k) - \frac{n}{n+1} f(n+1). \end{aligned}$$

Since $f(k)$ is limited by assumption, the above quantity is limited, namely,

$$n(g(n) - g(n+1)) \leq M$$

and the proof of theorem holds.

We have the following corollary:

Corollary 2.6. *Let E be a subset of $\mathbb{N}^*$. Then, for $\ell \in [0, 1]$, the following two statements are equivalent*:

(p_1) : *E admits ℓ as H^1-density and $(d_1(E) = \ell)$;*

(p_2) : *E admits ℓ as H^p-density, for all $p \in \mathbb{N}^*$ and $(d_p(E) = \ell)$.*

Theorem 2.7 (Comparison Theorem between: $H^p,\ p \in \mathbb{N}^*$ and H^∞- densities)**.** *Let $f : \mathbb{N}^* \to \mathbb{R}^+$ be a limited and positive arithmetic function. Consider, for all real number $\ell \geq 0$, the following two statements*:

(p_1) : *f admits* ℓ *as* H^p*-density, for all* $p \in \mathbb{N}^*$ *and* $(d_p(f) = \ell)$;

(p_2) : *f admits* ℓ *as* H^∞*-density and* $(d_\infty(f) = \ell)$.

Then $(p_1) \Rightarrow (p_2)$. *The converse is not true.*

Moreover, the H^∞*-density is an extension of the* H^p*-density, for all* $p \in \mathbb{N}^*$, *for the class of limited and positive arithmetic functions.*

Proof. $(p_1) \Rightarrow (p_2)$ holds from Proposition 2.2.

$(p_2) \Rightarrow (p_1)$: For the converse, we take $f(k) = I_E(k)$, where E is the subset of $\mathbb{N}^*$ introduced by Proposition 3.2 [6]. Then f admits an H^∞-density and $d_\infty(f) = \frac{d^* - d}{a}$, but it does not admit an H^p-density [2, Theorem 2.6, pp. 516-517].

Corollary 2.8. *Let E be a subset of* $\mathbb{N}^*$ *and let* $\ell \in [0, 1]$. *If E admits* ℓ *as an* H^p*-density, for one* $p \in \mathbb{N}^*$, *then E admits* ℓ *as an* H^∞*-density. The converse does not true.*

2.3. Generalizations

Put

$$\mathfrak{F}^1 := \{f \in \mathfrak{F} : E_s(f) \text{ exists for all } s > 1\}.$$

$\mathfrak{F}^1$ denotes the class of Dirichlet's series of the form

$$\sum_{n \geq 1} \frac{f(n)}{n^s},$$

with the absolute convergence abscissa is ≤ 1.

Lemma 2.9 [8, p. 51]**.** *If* $f \in \mathfrak{F}^1$, *then for all* $s > 1$ *and all* $p \in \mathbb{N}$, $E_s(H^p f)$ *exists and we have*

$$E_s(H^p f) - sE_s(H^{p+1} f) = o(1), \quad as \ (s \to 1^+).$$

Proof. From the relation $|H(f)| \leq H(|f|)$, it holds that if $f \in \mathfrak{F}^1$, then, for all $p \in \mathbb{N}$ and all $s > 1$, the series

$$\sum_{n \geq 1} \frac{(H^p f)(n)}{n^s}$$

is absolutely convergent. Let $p \in \mathbb{N}$, put

$$g = H^p f, \quad g^+ = \sup(g, 0), \quad g^- = \sup(-g, 0).$$

From inequalities

$$s \sum_{n \geq k+1} \frac{1}{n^{s+1}} < \frac{1}{k^s} < s \sum_{n \geq k} \frac{1}{n^{s+1}},$$

it holds that

$$E_s(g) \leq \frac{s}{\zeta(s)} \left(\sum_{k \leq 1} g^+(k) \sum_{n \geq k} \frac{1}{n^{s+1}} - \sum_{k \geq 1} g^-(k) \sum_{n \geq k+1} \frac{1}{n^{s+1}} \right), \tag{2.1}$$

$$E_s(g) \geq \frac{s}{\zeta(s)} \left(\sum_{k \geq 1} g^+(k) \sum_{n \geq k+1} \frac{1}{n^{s+1}} - \sum_{k \geq 1} g^-(k) \sum_{n \geq k} \frac{1}{n^{s+1}} \right). \tag{2.2}$$

By writing

$$\sum_{n \geq k+1} \frac{1}{n^{s+1}} = \sum_{n \geq k} \frac{1}{n^{s+1}} - \frac{1}{k^{s+1}},$$

inequalities (2.1) and (2.2) give

$$E_s(g) \leq \frac{s}{\zeta(s)} \left(\sum_{k \geq 1} g(k) \sum_{n \geq k} \frac{1}{n^{s+1}} + \sum_{k \geq 1} \frac{g^-(k)}{k^{s+1}} \right)$$

and

$$E_s(g) \geq \frac{s}{\zeta(s)} \left(\sum_{k \geq 1} g(k) \sum_{n \geq k} \frac{1}{n^{s+1}} - \sum_{k \geq 1} \frac{g^+(k)}{k^{s+1}} \right).$$

Then

$$E_s(g) \leq s E_s(Hg) + \frac{s}{\zeta(s)} \sum_{k \geq 1} \frac{g^-(k)}{k^{s+1}}, \tag{2.3}$$

$$E_s(g) \geq s E_s(Hg) - \frac{s}{\zeta(s)} \sum_{k \geq 1} \frac{g^+(k)}{k^{s+1}}. \tag{2.4}$$

Or, since $g \in \mathfrak{F}^1$ we deduce that g^+ and $g^- \in \mathfrak{F}^1$ and consequently

$$\lim_{(s \to 1^+)} \frac{1}{\zeta(s)} \sum_{k \geq 1} \frac{g^+(k)}{k^{s+1}} = \lim_{(s \to 1^+)} \frac{1}{\zeta(s)} \sum_{k \geq 1} \frac{g^-(k)}{k^{s+1}} = 0.$$

And from relations (2.3) and (2.4) it holds that $E_s(g) - sE_s(Hg) = o(1)$.

Immediately, from Lemma 2.9 follows the following theorem:

Theorem 2.10. *If $f \in \mathfrak{F}^1$ with f positive and there exists $p_0 \in \mathbb{N}$ such that $H^{p_0} f$ admits an analytic density $\delta(H^{p_0} f) = \ell$, then for all $p \in \mathbb{N}$, $H^p f$ admits an analytic density $\delta(H^p f)$ which is equal to ℓ.*

By combining Theorem 3.12 [4, Theorem 3.12, p. 220], with Theorem 2.10 above, which gives first generalization of Theorem 3.7 [4, Theorem 3.7, p. 214].

Theorem 2.11. *If $f \in \mathfrak{F}^1$ and if there exists $p \in \mathbb{N}$ such that $H^p f$ converges, then f admits an analytic density $\delta(f)$ and*

$$\delta(f) = \lim_{(n \to +\infty)} (H^p f)(n).$$

We can prove that if $f \in \mathfrak{F}$ admits an H^∞-density, then there exists $p \in \mathbb{N}$ such that $H^p f \in \mathfrak{F}^0$.

The following theorem is another generalization of Theorem 3.7 [4, p. 214]. It proves that we can replace the hypothesis of mean convergence of rank 1, in Theorem 2.5 above by the hypothesis of mean convergence of rank ∞.

Theorem 2.12. *If an arithmetic function f admits an H^∞-density, $d_\infty(f)$, then it admits an analytic density $\delta(f)$, and $d_\infty(f) = \delta(f)$.*

Remark 2.3. The converse of this theorem is not true.

Proof. Let

$$E = \bigcup_{k \geq 1} [p_k, q_k[,$$

where

$$\begin{cases} p_k = b^{P(k)}, \\ q_k = b^{Q(k)}, \end{cases} \quad \text{with} \quad \begin{cases} P(k) = k^2, \\ Q(k) = \left(k + \dfrac{1}{2}\right)^2 \end{cases}$$

and $b \in \{2, 3, ..., 9\}$. By taking $f(k) = I_E(k)$, f does not admit an H^∞-density [6, Proposition 3.12], but it admits an analytic density $\delta(f) = \frac{1}{2}$ [6, Remark 3.2].

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