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ON A CLASS OF PARTIAL FUNCTORS BETWEEN CATEGORIES WHICH PROVIDE A CLOSED AND ORDERED PARTIALLY ADDITIVE CATEGORY OF CATEGORIES

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Abstract

We single out a class of partial functors, the partial supersaturated functors, together with the associated partial natural transformations between them, which allows us to get a symmetric monoidal closed, non-cartesian, and ordered partially additive category with small categories as objects and morphisms the partial supersaturated functors. Moreover, we extend some notions and constructions from semigroup theory, e.g., Rees congruences, Rees homomorphisms and ideal extensions, to categories and partial supersaturated functors. Also, we state a Yoneda-Grothendieck Lemma for the above class of partial functors and generalize the concept of adjunction to that of partial adjunction, providing for this last concept one fundamental example from the field of algebraic logic.

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1. Introduction

In mathematics, leaving out some fields, partiality has a bad reputation, probably as a consequence of the fact that, e.g., one and the same concept, when seen from the standpoint of partiality, splits up into a non well-ordered multiplicity of different concepts. Besides, and this is perhaps worse, partiality is regarded as needless, this last being wrongly based on the existence of a reduction process from partiality to totality. However, as witnessed, e.g., by recursion theory and universal algebra, partiality not only provides subtle conceptual distinctions that otherwise, almost surely, would have remained hidden for ever, but also shows itself as unavoidable when approaching some problems, specially those that by nature have a computational character. To this we add that during the last few decades there have been some interesting, although isolated, contributions to the investigation of partiality, most remarkably those by Poythress in [10], by Burmeister in [2], and by Robinson-Rosolini in [11], among others. The mentioned works by Poythress and by Burmeister are written from the universal algebra standpoint, and we will compare, in the first section of this paper, the definition of p-morphism of Poythress, as applied to categories, with our notion of partial supersaturated functor. As to the above mentioned work by Robinson-Rosolini, which is written from the category-theoretical standpoint, we should say that it contains an attempt to reconcile various abstract notions of category of partial maps which appear in the literature, through the concept of p-category; nevertheless, it is also imperative to point out that, strangely, in such a work there is not any example of category of partial functors.

One of the goals of this paper is to specify a class of partial functors which allows us to get structured categories of the same type as those for sets and partial mappings. As it happens, there is a wide variety of classes of partial functors between categories which we could consider, e.g., the p-functors, as particular examples of the concept of partial morphism on partial algebras defined by Poythress in [10], as well as those we could get, more generally, starting from pairs of partial mappings from a category to another, when their domains of definition determine some type of subcategory of the domain category, and also satisfy some definite conditions related to the preservation or reflection of some properties. However, almost none of the above classes of partiality leads to reach the type of structured categories we are interested in.

In the second section of this paper we define the partial supersaturated functors, and for this type of partiality we show that a structured category can be obtained, which is symmetric, monoidal closed (abbreviated to closed), and is such that its underlying category is not cartesian closed, exactly as it happens for sets and partial mappings. Also, we show in this section how to extend to categories and partial supersaturated functors some notions and constructions from semigroup theory, particularly those of Rees congruence, Rees homomorphism, and ideal extension of a semigroup by another. Moreover, we prove a Yoneda-Grothendieck Lemma for partial supersaturated functors, and also generalize the concept of adjoint situation to the case of partial supersaturated functors.

In the third section of this paper we show that the set of all partial supersaturated functors between two categories, when partially ordered by extension, becomes a coherent algebraic complete partial order, and also that for partial supersaturated functors a structured category can be obtained, which is ordered partially additive in the sense of [8], once again as for sets and partial mappings. However, as we shall see, the notions of finiteness and compactness are not equivalent for partial supersaturated functors, while, as it is well known (see, e.g., [1]), they are so for partial mappings.

As to examples of partial supersaturated functors we remark that many of the functors in [9], could be considered in a natural way as partial supersaturated functors. This is due to the fact that, as we show in the second section, to obtain partial supersaturated functors from a category $\mathbf{C}$ to another $\mathbf{D}$ it is enough to give a supersaturated subcategory of $\mathbf{C}$ and an ordinary functor from such a supersaturated subcategory to $\mathbf{D}$. Actually, the procedure used several times by Petrich in [9], starts by considering a groupoid $\mathbf{C}$ together with a property Φ of the objects of $\mathbf{C}$ that is invariant under isomorphisms, then it goes on by forming the set of all objects of $\mathbf{C}$ that fall under Φ, together with all the isomorphisms between them, arriving at a supersaturated subcategory of $\mathbf{C}$, and finishes by defining a partial functor from $\mathbf{C}$, having as domain of definition the supersaturated subcategory of $\mathbf{C}$ obtained from the abstract property Φ. Additional examples of partial supersaturated functors from the fields of recursion theory and of algebraic logic will be provided in the second section, after having characterized partial supersaturated functors in terms of the concept of supersaturated subcategory of a category and of the usual concept of functor.

Finally, let us say that even though partial supersaturated functors are, in fact, a generalization of functors, giving rise to a closed and ordered partially additive category, we do not hold in any way that such a notion is the best generalization of the concept of functor. However, we believe that the notion of partiality given by partial supersaturated functors is the weakest one giving rise to a closed and ordered partially additive category. We should also note, paraphrasing and in complete agreement with what Ljapin says (in [7, p. 32]) about the reduction of semigroups of partial transformations to semigroups of transformations, that it is not always expedient to reduce partial functors to ordinary functors, because, in the transition, some properties can be lost.

In this paper, $\mathcal{U}$ will be a Grothendieck universe, fixed once and for all. On the other hand, an expression such as $F : X \rightharpoonup Y$ means that F is a partial mapping from X to Y, and in this case $\mathrm{Dom}(F)$ is the domain of definition of F; finally, if $\mathbf{C}$ is a category, d_0, d_1, id and $\circ$ denote the structural operations of $\mathbf{C}$. It is to be noted that the composition circle is omitted when composing morphisms.

2. Partial Supersaturated Functors

To begin with, we state the concept of partial supersaturated functor between two categories.

Definition 2.1. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a partial *supersaturated* functor from $\mathbf{C}$ to $\mathbf{D}$ is a triple $F = (\mathbf{C}, (F_0, F_1), \mathbf{D})$, denoted by $F : \mathbf{C} \rightharpoonup \mathbf{D}$, such that F_0 is a partial mapping from $\mathrm{Ob}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{D})$, F_1 is a partial mapping from $\mathrm{Mor}(\mathbf{C})$ to $\mathrm{Mor}(\mathbf{D})$, and subject to satisfy the following axioms:

(1) The domains of definition of the partial mappings $F_0 \circ \mathrm{d}_0$ and $\mathrm{d}_0 \circ F_1$ are identical, and for every morphism f in this common domain we have that

$$\mathrm{d}_0(F_1(f)) = F_0(\mathrm{d}_0(f)).$$

(2) The domains of definition of the partial mappings $F_0 \circ \mathrm{d}_1$ and $\mathrm{d}_1 \circ F_1$ are identical, and for every morphism f in this common domain we have that

$$\mathrm{d}_1(F_1(f)) = F_0(\mathrm{d}_1(f)).$$

(3) The domains of definition of the partial mappings $F_1 \circ \mathrm{id}$ and $\mathrm{id} \circ F_0$ are identical, and for every object x in this common domain we have that

$$\mathrm{id}_{F_0(x)} = F_1(\mathrm{id}_x).$$

(4) For every pair $(f, g) \in \mathrm{Mor}(\mathbf{C})^2$, if f and g are composable and fg is in $\mathrm{Dom}(F_1)$, then (f, g) is in $\mathrm{Dom}(F_1^2)$, $F_1(f)$ and $F_1(g)$ are composable and

$$F_1(fg) = F_1(f)F_1(g).$$

We agree to denote by $\mathrm{P}(\mathbf{C}, \mathbf{D})$ the set of all partial supersaturated functors from $\mathbf{C}$ to $\mathbf{D}$, and if $F : \mathbf{C} \rightharpoonup \mathbf{D}$, then $\mathrm{Dom}(F)$, the *domain of definition* of F, is the pair $(\mathrm{Dom}(F_0), \mathrm{Dom}(F_1))$.

From axiom (4) in the above definition we get the following property: For every pair $(f, g) \in \mathrm{Mor}(\mathbf{C})^2$, if f and g are composable, $(f, g) \in \mathrm{Dom}(F_1^2)$, and $fg \in \mathrm{Dom}(F_1)$, then $F_1(fg) = F_1(f)F_1(g)$ and $F_1(f)$ and $F_1(g)$ are composable. Reciprocally, from this last property together with the first axiom from Definition 2.1, we get axiom (4). Therefore, if, in the above definition, we left invariant the first two axioms, then the last axiom could be replaced equivalently by the above property.

In order to compare the above concept of partial supersaturated functor with that of p-functor, which falls under the concept of

p-morphism between partial algebras defined by Poythress in [10], we recall it in the following

Definition 2.2. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *p-functor* from $\mathbf{C}$ to $\mathbf{D}$ is a triple $F = (\mathbf{C}, (F_0, F_1), \mathbf{D})$, such that F_0 is a partial mapping from $\mathrm{Ob}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{D})$, F_1 is a partial mapping from $\mathrm{Mor}(\mathbf{C})$ to $\mathrm{Mor}(\mathbf{D})$, and subject to satisfy the following axioms:

(1) For every morphism f, if $f \in \mathrm{Dom}(F_1)$, then, on the one hand, $\mathrm{d}_0(f) \in \mathrm{Dom}(F_0)$ and $\mathrm{d}_0(F_1(f)) = F_0(\mathrm{d}_0(f))$, and, on the other hand, $\mathrm{d}_1(f) \in \mathrm{Dom}(F_0)$ and $\mathrm{d}_1(F_1(f)) = F_0(\mathrm{d}_1(f))$.

(2) For every object x in $\mathrm{Dom}(F_0)$, $\mathrm{id}_x \in \mathrm{Dom}(F_1)$ and $\mathrm{id}_{F_0(x)} = F_1(\mathrm{id}_x)$.

(3) For every pair $(f, g) \in \mathrm{Mor}(\mathbf{C})^2$, if $(f, g) \in \mathrm{Dom}(F_1^2)$ and $F_1(f)$, $F_1(g)$, are composable, then f, g are composable, $fg \in \mathrm{Dom}(F_1)$, and $F_1(fg) = F_1(f)F_1(g)$.

From this we see that axioms (1) and (2) in Definition 2.1 are a strengthening of the corresponding axioms in Definition 2.2, whereas axiom (4) in Definition 2.1 and axiom (3) in Definition 2.2 are dual.

Following this we introduce the notion of supersaturated subcategory of a category which will allow us, among other things, to characterize the partial supersaturated functors through it and that of functor.

Definition 2.3. Given a category $\mathbf{C}$, a subset O of $\mathrm{Ob}(\mathbf{C})$ and a subset M of $\mathrm{Mor}(\mathbf{C})$, we say that (O, M) is a *supersaturated subcategory* of $\mathbf{C}$ if it satisfies the following axioms:

(1) For every morphism f in $\mathbf{C}$, $f \in M$ if, and only if, $\mathrm{d}_0(f) \in O$.

(2) For every morphism f in $\mathbf{C}$, $f \in M$ if, and only if, $\mathrm{d}_1(f) \in O$.

We denote by $\mathrm{Ssat}(\mathbf{C})$ the set of all supersaturated subcategories of $\mathbf{C}$.

Let us observe that (O, M) is a supersaturated subcategory of $\mathbf{C}$ if, and only if, for every morphism f in $\mathbf{C}$, if $f \in M$, then $\mathrm{d}_0(f) \in O$ and $\mathrm{d}_1(f) \in O$, and, if $\mathrm{d}_0(f) \in O$ or $\mathrm{d}_1(f) \in O$, then $f \in M$.

From the above definition we immediately get the following

Corollary 2.4. *If* (O, M) *is a supersaturated subcategory of* $\mathbf{C}$, *then*

(1) *For every* $x \in \mathrm{Ob}(\mathbf{C})$, $x \in O$ *if, and only if,* $\mathrm{id}_x \in M$.

(2) *For every pair* $(f, g) \in \mathrm{Mor}(\mathbf{C})^2$, *if* f *and* g *are composable, then* $fg \in M$ *if, and only if,* $(f, g) \in M^2$.

After this we state in the following proposition some of the relations between the concept of supersaturated subcategory and those of connected component, full, and abstract (for this concept see [4, p. 104]) subcategory of a given category.

Proposition 2.5. *Let* $\mathbf{C}$ *be a category. Then*

(1) *Every connected component of* $\mathbf{C}$ *is a supersaturated subcategory of* $\mathbf{C}$.

(2) *Every supersaturated subcategory of* $\mathbf{C}$ *is an abstract and full subcategory of* $\mathbf{C}$.

Moreover, the converses of 1 *and* 2 *are not valid in general.*

We gather together in the following proposition the main characterizations of the concept of supersaturated subcategory. But before that, for a category $\mathbf{C}$ and an object x of $\mathbf{C}$, we agree that $\downarrow x$ and $\uparrow x$ are the subsets of $\mathrm{Ob}(\mathbf{C})$ defined, respectively, as follows:

$$\downarrow x = \{y \in \mathrm{Ob}(\mathbf{C}) \mid \mathrm{Hom}(y, x) \neq \varnothing\}$$

and

$$\uparrow x = \{y \in \mathrm{Ob}(\mathbf{C}) \mid \mathrm{Hom}(x, y) \neq \varnothing\}.$$

Proposition 2.6. *Given a category* $\mathbf{C}$, *a subset* O *of* $\mathrm{Ob}(\mathbf{C})$, *and a subset* M *of* $\mathrm{Mor}(\mathbf{C})$, *the following conditions are equivalent*:

(1) *The pair* (O, M) *is a supersaturated subcategory of* $\mathbf{C}$.

(2) *For every* $x \in O$, $\downarrow x$, $\uparrow x \subseteq O$, *and* $M = \bigcup_{a, b \in O} \mathrm{Hom}(a, b)$.

(3) *The pair* (O, M) *is closed under the structural operations* d_0, d_1, *and* id *of* $\mathbf{C}$, *and*, *for every* $(f, g) \in \mathrm{Mor}(\mathbf{C})^2$, *if f and g are composable and f or g is in M*, *then fg is in M*.

(4) *The pair* (O, M) *can be represented as the union of a set of connected components of the category* $\mathbf{C}$.

From this we obtain immediately the following

Proposition 2.7. *If* $F : \mathbf{C} \rightharpoonup \mathbf{D}$, *then* $\mathrm{Dom}(F)$ *is a supersaturated subcategory of* $\mathbf{C}$.

As it is well known, the set of all subcategories of a category $\mathbf{C}$ is an algebraic closure system on the two-sorted set $(\mathrm{Ob}(\mathbf{C}), \mathrm{Mor}(\mathbf{C}))$, but, in general, such a set is neither closed under taking finite unions, nor under taking complements, and identical considerations could also be applied to the sets of all full or abstract subcategories of $\mathbf{C}$. But for the system of all supersaturated subcategories of a category we have something more, as stated in the following

Proposition 2.8. *The set of all supersaturated subcategories of a category* $\mathbf{C}$ *is a complete Boolean subalgebra of the Boolean set algebra of subsets of the two-sorted set* $(\mathrm{Ob}(\mathbf{C}), \mathrm{Mor}(\mathbf{C}))$.

Next we characterize the partial supersaturated functors through the concepts of supersaturated subcategory and functor.

Proposition 2.9. *Given two categories* $\mathbf{C}$ *and* $\mathbf{D}$, *there is a bijection from the set* $\mathrm{P}(\mathbf{C}, \mathbf{D})$ *to the set of all those pairs* $((O, M), F)$ *such that* (O, M) *is a supersaturated subcategory of* $\mathbf{C}$, *and F is a functor from the category canonically associated to* (O, M) *to* $\mathbf{D}$. *Therefore*, *for the final category* $\mathbf{1}$, $\mathrm{P}(\mathbf{C}, \mathbf{1})$ *and* $\mathrm{Ssat}(\mathbf{C})$ *are isomorphic sets*.

Proof. It is readily seen that the mapping that to a partial supersaturated functor $F : \mathbf{C} \rightharpoonup \mathbf{D}$ assigns $(\mathrm{Dom}(F), F \upharpoonright \mathrm{Dom}(F))$, is the desired bijection.

Example 2.10. Let *f* be a partial mapping from a set *X* to a set *Y*.

Then, denoting by $\mathbf{Dis}(X)$ and $\mathbf{Dis}(Y)$ the discrete categories associated to X and Y, respectively, we have that $\mathrm{Dom}(f)$ can be identified to a supersaturated subcategory of $\mathbf{Dis}(X)$ and, therefore, that f can be identified to a supersaturated functor from $\mathbf{Dis}(X)$ to $\mathbf{Dis}(Y)$. By restricting our attention to computable mappings, and since there is a partial recursive mapping f such that f cannot be extended to a recursive mapping, we see that, within recursion theory, there is a supersaturated (recursive) functor which cannot be extended to a (recursive) functor. Thus, confirming what Ljapin says (in [7, p. 32]), it is not always expedient to reduce partial functors to ordinary functors, because, in the transition, some fundamental properties can be lost, e.g., in this case the recursiveness.

Our next goal is to provide another example of partial supersaturated functor from the field of algebraic logic. But to do it, and in order to make the paper as self-contained as possible, we should begin by recalling a series of basic notions, constructions, and results from such a field.

Let $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ be an arbitrary, but fixed, single-sorted signature, and V be a fixed but unspecified set of (sentential) variables, which we assume to be countably infinite. Then we denote by $\mathbf{T}_\Sigma(V)$ the free Σ-algebra on V and call its elements, in this context, sentential formulas.

Definition 2.11 (Cf. [5, p. 23]). A *sentential logic* is a pair $\mathcal{S} = (\mathbf{T}_\Sigma(V), \vdash_{\mathcal{S}})$, where $\mathbf{T}_\Sigma(V)$ is the free Σ-algebra on V and $\vdash_{\mathcal{S}} \subseteq \mathrm{Sub}(\mathrm{T}_\Sigma(V)) \times \mathrm{T}_\Sigma(V)$ a relation satisfying, for every $\varphi \in \mathrm{T}_\Sigma(V)$ and every $\Gamma,\ \Delta \subseteq \mathrm{T}_\Sigma(X)$, the following conditions:

(1) If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$.

(2) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathcal{S}} \varphi$.

(3) If $\Gamma \vdash_{\mathcal{S}} \varphi$ and, for every $\gamma \in \Gamma$, $\Delta \vdash_{\mathcal{S}} \gamma$, then $\Delta \vdash_{\mathcal{S}} \varphi$.

(4) If $\Gamma \vdash_{\mathcal{S}} \varphi$, then $f[\Gamma] \vdash_{\mathcal{S}} f(\varphi)$, for every endomorphism f of $\mathbf{T}_\Sigma(V)$.

(5) If $\Gamma \vdash_{\mathcal{S}} \varphi$, then there exists a finite subset Θ of Γ such that $\Theta \vdash_{\mathcal{S}} \varphi$.

The main distinctive feature of the approach to the algebraization of sentential logic on the part of Font-Jansana in [5] lies in the mathematical objects used as models of a sentential logic. Concretely, they propose to use abstract logics, instead of logical matrices, as models of the sentential logic under consideration, essentially, because being abstract logics structurally more richer than logical matrices, in them the metalogical properties of the sentential logic will be reflected more faithfully than in logical matrices. Moreover, they believe that abstract logics provide an explanation of the connection, both to the logical and metalogical level, between a sentential logic and the particular class of models associated with it.

Definition 2.12 (Cf. [5, p. 15])**.** An *abstract logic* is a pair $(\mathbf{A}, J)$, where $\mathbf{A}$ is a Σ-algebra and J is a closure operator on A.

Let us observe that since there is an anti-isomorphism between the ordered set of all closure operators on a set A and the ordered set of all closure systems on A, an abstract logic can be defined alternative, but equivalently, as a pair $(\mathbf{A}, \mathcal{C})$, where $\mathbf{A}$ is a Σ-algebra and $\mathcal{C}$ is a closure system on A.

After having fixed the objects, some of which, selected by means of a suitable property, will be the models of a given sentential logic, we proceed next to define, the admissible morphisms between abstract logics which we will consider in what follows.

Definition 2.13 (Cf. [5, p. 18])**.** Given two abstract logics $(\mathbf{A}, J)$ and $(\mathbf{A}', J')$, a *bilogical morphism* from $(\mathbf{A}, J)$ to $(\mathbf{A}', J')$ is a surjective homomorphism f from $\mathbf{A}$ onto $\mathbf{A}'$ such that, for every $X \subseteq A$, $J(X) = f^{-1}[J'(f[X])]$.

These bilogical morphisms are particularly relevant, among other reasons, because they are to logical congruences on an abstract logic, defined immediately below, as the surjective homomorphisms between universal algebras are to congruences on a universal algebra.

Definition 2.14 (Cf. [5, p. 16])**.** If $(\mathbf{A}, J)$ is an abstract logic, then a congruence Φ on $\mathbf{A}$ is a *logical congruence* on $(\mathbf{A}, J)$ when, for every

$x \in A$, if $(x, y) \in \Phi$, then $J(\{x\}) = J(\{y\})$. The ordered set $\mathbf{Cgr}(\mathbf{A}, J) = (\mathrm{Cgr}(\mathbf{A}, J), \subseteq)$ is a complete lattice and a principal ideal of the algebraic lattice $\mathbf{Cgr}(\mathbf{A})$. Actually, the generator of the principal ideal is the so-called *Tarski congruence* on $(\mathbf{A}, J)$ and it is denoted by $\widetilde{\Omega}(\mathbf{A}, J)$.

Observe that a congruence Φ on $\mathbf{A}$ is a logical congruence on $(\mathbf{A}, J)$ precisely if, for every fixed point $F = J(F)$ of the closure operator J, Φ saturates F.

The process of reduction of an abstract logic consists in factoring an abstract logic by its Tarski congruence. The result of the action of this process on an abstract logic is a new abstract logic of the type specified in the following

Definition 2.15 (Cf. [5, p. 21])**.** An abstract logic $(\mathbf{A}, J)$ is *reduced* when it has only one logical congruence, i.e., when $\widetilde{\Omega}(\mathbf{A}, J)$, the greatest logical congruence on $(\mathbf{A}, J)$, is precisely Δ_A. We write $(\mathbf{A}^*, J^*)$ for the quotient of $(\mathbf{A}, J)$ by $\widetilde{\Omega}(\mathbf{A}, J)$ and we call it the *reduction* of $(\mathbf{A}, J)$.

If an abstract logic $(\mathbf{B}, K)$ is already reduced, then it is trivially isomorphic to its reduction $(\mathbf{B}^*, K^*)$.

Proposition 2.16 (Cf. [5, p. 21])**.** *If there is a bilogical morphism between two abstract logics* $(\mathbf{A}, J)$ *and* $(\mathbf{A}', J')$, *then their reductions are isomorphic*.

Therefore, the only possible bilogical morphisms between two reduced abstract logics are logical isomorphisms.

From a sentential logic $\mathcal{S}$ and an abstract logic $(\mathbf{A}, J)$ we obtain the binary relation $\vDash_{(\mathbf{A}, J)} \subseteq \mathrm{Sub}(\mathrm{T}_\Sigma(V)) \times \mathrm{T}_\Sigma(V)$ defined, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{T}_\Sigma(V)$, as $\Gamma \vDash_{(\mathbf{A}, J)} \varphi$ if, and only if, for every homomorphism f from $\mathbf{T}_\Sigma(V)$ to $\mathbf{A}$, we have that $f(\varphi) \in J(f[\Gamma])$. The binary relations of the type $\vDash_{(\mathbf{A}, J)}$ are at the basis of the concept of model of a sentential logic as stated in the following

Definition 2.17 (Cf. [5, p. 30]). An abstract logic $(\mathbf{A}, \mathcal{J})$ is a *model* of a sentential logic $\mathcal{S}$ when, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{T}_{\Sigma}(V)$, from $\Gamma \vdash_{\mathcal{S}} \varphi$ it follows that $\Gamma \vDash_{(\mathbf{A}, \mathcal{J})} \varphi$, or, what is equivalent, when $\vdash_{\mathcal{S}} \subseteq \vDash_{(\mathbf{A}, \mathcal{J})}$.

Proposition 2.18 (Cf. [5, p. 30]). *If there is a bilogical morphism between two abstract logics* $(\mathbf{A}, \mathcal{J})$ *and* $(\mathbf{A}', \mathcal{J}')$, *then* $(\mathbf{A}, \mathcal{J})$ *is a model of* $\mathcal{S}$ *if, and only if,* $(\mathbf{A}', \mathcal{J}')$ *is a model of* $\mathcal{S}$; *in particular,* $(\mathbf{A}, \mathcal{J})$ *is a model of* $\mathcal{S}$ *if, and only if,* $(\mathbf{A}^{*}, \mathcal{J}^{*})$ *is a model of* $\mathcal{S}$.

We denote by $\mathbf{M}(\mathcal{S})$ the category which has as objects precisely those abstract logics $(\mathbf{A}, \mathcal{J})$ that are models of $\mathcal{S}$ and as morphisms all bilogical morphisms between abstract logics.

To define the full models of a sentential logic we state next the concept of deductive filter of a sentential logic on an abstract logic.

Definition 2.19 (Cf. [5, p. 24]). Given a sentential logic $\mathcal{S}$ and a Σ-algebra $\mathbf{A}$, subset F of A is an $\mathcal{S}$*-deductive filter on* $\mathbf{A}$ if, and only if, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{T}_{\Sigma}(V)$ and every homomorphism f from $\mathbf{T}_{\Sigma}(V)$ to $\mathbf{A}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $f[\Gamma] \subseteq F$, then $f(\varphi) \in F$. We denote by $\mathrm{DF}_{\mathcal{S}}(\mathbf{A})$ the set of all $\mathcal{S}$-deductive filters on $\mathbf{A}$.

Proposition 2.20 (Cf. [5, p. 25]). *If f is a bilogical morphism from* $(\mathbf{A}, \mathrm{DF}_{\mathcal{S}}(\mathbf{A}))$ *onto* $(\mathbf{B}, \mathcal{C})$, *then* $\mathcal{C} = \mathrm{DF}_{\mathcal{S}}(\mathbf{B})$. *In particular,* $\mathrm{DF}_{\mathcal{S}}(\mathbf{A})^{*} = \mathrm{DF}_{\mathcal{S}}(\mathbf{A}^{*})$. *Moreover, if two abstract logics* $(\mathbf{A}, \mathcal{C})$ *and* $(\mathbf{A}', \mathcal{C}')$ *are isomorphic, then* $\mathcal{C} = \mathrm{DF}_{\mathcal{S}}(\mathbf{A})$ *if, and only if,* $\mathcal{C}' = \mathrm{DF}_{\mathcal{S}}(\mathbf{A}')$.

Font and Jansana in [5] associate with each sentential logic $\mathcal{S}$ a class of abstract logics called the *full models* of $\mathcal{S}$ with the conviction that (some of) the interesting metalogical properties of the sentential logic are precisely those shared by its full models. Moreover, they also claim that the concept of full model is a “right” notion of model of a sentential logic. Since these statements are actually verified by the results contained in [5], we should consequently allow a fundamental and privileged place to full models in algebraic logic.

Definition 2.21 (Cf. [5, p. 31]). An abstract logic $(\mathbf{A}, \mathcal{C})$ is a *full model* of a sentential logic $\mathcal{S}$ if, and only if, $(\mathbf{A}^*, \mathcal{C}^*) = (\mathbf{A}^*, \mathrm{DF}_{\mathcal{S}}(\mathbf{A}^*))$. We denote by $\mathrm{M_f}(\mathcal{S})$ the set of all full models of $\mathcal{S}$.

From Proposition 2.20 it follows immediately the following

Corollary 2.22 (Cf. [5, p. 32]). *The set* $\mathrm{M_f}(\mathcal{S})$ *is closed under bilogical morphisms*, *i.e.*, *if there is a bilogical morphism between two abstract logics* $(\mathbf{A}, \mathcal{C})$ *and* $(\mathbf{B}, \mathcal{K})$, *then* $(\mathbf{A}, \mathcal{C})$ *is a full model of* $\mathcal{S}$ *if*, *and only if*, $(\mathbf{B}, \mathcal{K})$ *is a full model of* $\mathcal{S}$. *In particular*, *an abstract logic* $(\mathbf{A}, \mathcal{J})$ *is a full model of* $\mathcal{S}$ *if*, *and only if*, *its reduction* $(\mathbf{A}^*, \mathcal{J}^*)$ *is.*

Therefore, $\mathrm{M_f}(\mathcal{S})$ together with all bilogical morphisms between full models of $\mathcal{S}$ is a supersaturated subcategory of $\mathbf{M}(\mathcal{S})$, hence a union of connected components in $\mathbf{M}(\mathcal{S})$. We denote by $\mathbf{M}_{\mathrm{f}}(\mathcal{S})$ the category which has as objects precisely those abstract logics $(\mathbf{A}, \mathcal{J})$ that are full models of $\mathcal{S}$ and as morphisms all bilogical morphisms between abstract logics.

Full models are indeed models of the sentential logic under consideration, however, by [5, pp. 99-100], it is not, generally, true that every model is a full model.

We state next the concept of reduced full model of a sentential logic.

Definition 2.23 (Cf. [5, p. 31]). An abstract logic $(\mathbf{A}, \mathcal{C})$ is a *reduced full model* of a sentential logic $\mathcal{S}$ if, and only if, $(\mathbf{A}, \mathcal{C})$ is reduced and full, i.e., if, and only if, $(\mathbf{A}, \mathcal{C})$ is reduced and $\mathcal{C} = \mathrm{DF}_{\mathcal{S}}(\mathbf{A})$.

We denote by $\mathbf{M}_{\mathrm{rf}}(\mathcal{S})$ the category which has as objects the abstract logics $(\mathbf{A}, \mathcal{J})$ that are reduced full models of $\mathcal{S}$ and as morphisms all bilogical morphisms between abstract logics. Observe that by allowing as morphisms precisely the bilogical morphisms the category $\mathbf{M}_{\mathrm{rf}}(\mathcal{S})$ is a groupoid, i.e., every morphism in it is an isomorphism.

After these lengthy logical preliminaries, we can finally provide the following example of partial supersaturated functor which is fundamental for the field of algebraic logic.

Example 2.24. For a sentential logic $\mathcal{S}$, the partial functor of reduction $\mathrm{Rd}_{\mathcal{S}}$ from $\mathbf{M}(\mathcal{S})$, the category of models of $\mathcal{S}$ with all the bilogical morphisms, to $\mathbf{M}_{\mathrm{rf}}(\mathcal{S})$, the category of all reduced full models of $\mathcal{S}$ with all the bilogical morphisms, is supersaturated and has as domain of definition precisely $\mathbf{M}_{\mathrm{f}}(\mathcal{S})$, the category of all full models of $\mathcal{S}$ with all bilogical morphisms.

Remark. The concept of partial supersaturated functor can be seen as a special case of at least two, generally, non-equivalent concepts of partial homomorphism between partial many-sorted algebras of the same type of similarity.

Let S be a set of sorts and $\Sigma = (\Sigma_{(s_j)_{j\in n}, s})_{((s_j)_{j\in n}, s)\in S^{\star}\times S}$ be an S-sorted signature, where $S^{\star}$ is the underlying set of the free monoid on S. Then, given two partial many-sorted (S, Σ)-algebras $\mathbf{A} = (A, (F_{\sigma})_{\sigma\in\Sigma})$ and $\mathbf{B} = (B, (G_{\sigma})_{\sigma\in\Sigma})$, we could consider, at least, two classes of partial homomorphism from $\mathbf{A}$ to $\mathbf{B}$. On the one hand, the pairs (X, f) which satisfy the following (somewhat redundant) conditions:

(1) X is a subalgebra of $\mathbf{A}$ such that, for each σ in $\Sigma_{(s_j)_{j\in n}, s}$, and each a in $\prod_{j\in n} A_{s_j}$, if $a \in \mathrm{Dom}(F_{\sigma})$, then a in $\prod_{j\in n} X_{s_j}$ if, and only if, $F_{\sigma}(a) \in X_s$.

(2) f is a homomorphism from $\mathbf{X}$ to $\mathbf{B}$,

and, on the other hand, the pairs (X, f) which satisfy the following (somewhat redundant) conditions:

(1) X is a subalgebra of $\mathbf{A}$ such that, for each σ in $\Sigma_{(s_j)_{j\in n}, s}$, and each a in $\prod_{j\in n} A_{s_j}$, if $a \in \mathrm{Dom}(F_{\sigma})$ and for some $j \in n$, $a_j \in X_{s_j}$, then $F_{\sigma}(a) \in X_s$.

(2) f is a homomorphism from $\mathbf{X}$ to $\mathbf{B}$.

It is easy to see that for categories, which fall under the concept of partial many-sorted algebra, both notions of partial homomorphism are equivalent.

Proposition 2.25. *If* $F : \mathbf{C} \rightharpoonup \mathbf{D}$ *and* $G : \mathbf{D} \rightharpoonup \mathbf{E}$, *then* $G \circ F$ *is a partial supersaturated functor from* $\mathbf{C}$ *to* $\mathbf{E}$. *Moreover, the composition of partial supersaturated functors is associative, for every category* $\mathbf{C}$, $\mathrm{Id}_{\mathbf{C}}$, *the identity functor of* $\mathbf{C}$, *is neutral with respect to composition. Therefore, taking as objects the* $\mathcal{U}$*-small categories and as morphisms the partial supersaturated functors between* $\mathcal{U}$*-small categories, we obtain a category which we agree to denote by* $\mathbf{Cat}_{\mathrm{p}}$. *Moreover, if* $\mathbf{Y}$ *is a supersaturated subcategory of* $\mathbf{D}$, *then* $F^{-1}[\mathbf{Y}]$ *is a supersaturated subcategory of* $\mathbf{C}$.

Proposition 2.26. *The category* $\mathbf{Cat}_{\mathrm{p}}$ *has a unique zero object, the empty category, and is not cartesian closed.*

Proof. The first assertion is obvious. The second follows easily from the fact that $\mathbf{Cat}_{\mathrm{p}}$ is a pointed category (because of the existence of the zero object), and from the standard Yoneda-Grothendieck Lemma.

The category $\mathbf{Set}_{\mathrm{p}}$, of $\mathcal{U}$-small sets and partial mappings between $\mathcal{U}$-small sets, which is naturally embedded into the category $\mathbf{Cat}_{\mathrm{p}}$, by Example 2.10, also has a unique zero object, the empty set, and is not cartesian closed. Later on we will point out some other properties shared by $\mathbf{Set}_{\mathrm{p}}$ and $\mathbf{Cat}_{\mathrm{p}}$, as well as differences between them. These differences derive, essentially, from the fact that sets are homogeneous, while categories are heterogeneous entities.

Next, before investigating the structural properties of the category $\mathbf{Cat}_{\mathrm{p}}$, we show that just as in semigroup theory (see e.g., [3], [6], or [7]), in category theory there is also one type of functor, a Rees functor, which does correspond very closely to a supersaturated subcategory of the source category of the functor. But in order to verify it we should consider, instead of the ordinary congruences on a category which only classify the morphisms of the category under consideration, congruences which simultaneously and coherently classify both the objects and the morphisms of the given category.

Definition 2.27. Let $\mathbf{C}$ be a category. Then we say that a pair $\Phi = (\Phi_0, \Phi_1)$, where Φ_0 is a binary relation on the set of objects of $\mathbf{C}$

and Φ_1 is a binary relation on the set of morphisms of $\mathbf{C}$, is a congruence on $\mathbf{C}$ if it satisfies the following axioms:

(1) Φ_0 is an equivalence relation on $\mathrm{Ob}(\mathbf{C})$.

(2) Φ_1 is an equivalence relation on $\mathrm{Mor}(\mathbf{C})$.

(3) For every $f,\ g \in \mathrm{Mor}(\mathbf{C})$,

$$\frac{f \equiv g \,(\mathrm{mod}\ \Phi_1)}{\mathrm{d}_0(f) \equiv \mathrm{d}_0(g)\,(\mathrm{mod}\ \Phi_0)} \text{ and } \frac{f \equiv g \,(\mathrm{mod}\ \Phi_1)}{\mathrm{d}_1(f) \equiv \mathrm{d}_1(g)\,(\mathrm{mod}\ \Phi_0)}.$$

(4) For every $x,\ y \in \mathrm{Ob}(\mathbf{C})$,

$$\frac{x \equiv y \,(\mathrm{mod}\ \Phi_0)}{\mathrm{id}_x \equiv \mathrm{id}_y \,(\mathrm{mod}\ \Phi_1)}.$$

(5) For every $(f', f),\ (g', g) \in \mathrm{Mor}(\mathbf{C})^2$, if f', f as well as g', g are composable, then

$$\frac{f \equiv g \,(\mathrm{mod}\ \Phi_1) \text{ and } f' \equiv g' \,(\mathrm{mod}\ \Phi_1)}{f'f \equiv g'g \,(\mathrm{mod}\ \Phi_1)}.$$

If $\Phi = (\Phi_0, \Phi_1)$ is a congruence on $\mathbf{C}$, then we denote by $\mathbf{C}/\Phi$ the corresponding quotient category, which has as set of objects $\mathrm{Ob}(\mathbf{C})/\Phi_0$ and as set of morphisms $\mathrm{Mor}(\mathbf{C})/\Phi_1$.

Remark. Every ordinary congruence Φ on a category $\mathbf{C}$ is a congruence on $\mathbf{C}$, since Φ can, obviously, be identified to the congruence $(\Delta_{\mathrm{Ob}(\mathbf{C})}, \Phi)$ on $\mathbf{C}$, where $\Delta_{\mathrm{Ob}(\mathbf{C})}$ is the diagonal of $\mathrm{Ob}(\mathbf{C})$. Moreover, the set of all congruences on $\mathbf{C}$ is an algebraic lattice.

Proposition 2.28. *If* $\mathbf{X} = (O, M)$ *is a nonempty supersaturated subcategory of a category* $\mathbf{C}$, *then the pair of equivalence relations associated to the pair of partitions*

$$\{O\} \cup \{\{x\} \mid x \in \mathrm{Ob}(\mathbf{C}) - O\} \text{ and } \{M\} \cup \{\{f\} \mid f \in \mathrm{Mor}(\mathbf{C}) - M\},$$

of $\mathrm{Ob}(\mathbf{C})$ *and* $\mathrm{Mor}(\mathbf{C})$, *respectively, denoted by* $\mathrm{R}(\mathbf{X})$, *is a congruence on* $\mathbf{C}$, *the Rees congruence on* $\mathbf{C}$ *determined by* $\mathbf{X}$; *and the canonical* *projection* $\mathrm{Pr}_{\mathrm{R}(\mathbf{X})}$ *from* $\mathbf{C}$ *to the Rees quotient* $\mathbf{C}/\mathrm{R}(\mathbf{X})$ *is the Rees functor determined by* $\mathbf{X}$.

Definition 2.29. A congruence Φ on a category $\mathbf{C}$ is a *Rees congruence* on $\mathbf{C}$ if there exists a nonempty supersaturated subcategory $\mathbf{X}$ of $\mathbf{C}$ such that $\Phi = \mathrm{R}(\mathbf{X})$. Moreover, a *Rees functor* from a category $\mathbf{C}$ to another $\mathbf{D}$ is a functor $F : \mathbf{C} \to \mathbf{D}$ such that $\mathrm{Ker}(F) = (\mathrm{Ker}(F_0), \mathrm{Ker}(F_1))$ is a Rees congruence.

As for semigroups we also have the following

Proposition 2.30. *Let* $\mathbf{X}$ *be a nonempty supersaturated subcategory of a category* $\mathbf{C}$. *Then*

(1) *There is an inclusion-preserving bijection, determined by* $\mathrm{Pr}_{\mathrm{R}(\mathbf{X})}$, *from the set of all supersaturated subcategories of* $\mathbf{C}$ *which contain* $\mathbf{X}$ *onto the set of all nonempty supersaturated subcategories of* $\mathbf{C}/\mathrm{R}(\mathbf{X})$.

(2) *If* $\mathbf{X}'$ *is another supersaturated subcategory of* $\mathbf{C}$ *such that* $\mathbf{X} \subseteq \mathbf{X}'$, *then*

$$\frac{\mathbf{C}/\mathrm{R}(\mathbf{X})}{\mathbf{X}'/\mathrm{R}(\mathbf{X})} \cong \frac{\mathbf{C}}{\mathrm{R}(\mathbf{X}')}.$$

(3) *If, however,* $\mathbf{X}'$ *is simply a subcategory of* $\mathbf{C}$, *then* $\mathbf{X} \cup \mathbf{X}'$ *is a subcategory of* $\mathbf{C}$, $\mathbf{X}$ *is a supersaturated subcategory of* $\mathbf{X} \cup \mathbf{X}'$, $\mathbf{X} \cap \mathbf{X}'$ *is a supersaturated subcategory of* $\mathbf{X}'$ *and*

$$\frac{\mathbf{X} \cup \mathbf{X}'}{\mathrm{R}(\mathbf{X})} \cong \frac{\mathbf{X}'}{\mathrm{R}(\mathbf{X} \cap \mathbf{X}')}.$$

Corollary 2.31. *Let* $\mathbf{X} = (O, M)$ *and* $\mathbf{X}' = (O', M')$ *be two supersaturated subcategories of a category* $\mathbf{C}$. *If* $\mathbf{X}'$ *is maximal in* $\mathbf{X}$ *and* $x \in O - O'$, *then, denoting by* $[x]_{\mathbf{C}}$ *the connected component of* x *in* $\mathbf{C}$, $[x]_{\mathbf{C}} \cup \mathbf{X}' = \mathbf{X}$, $[x]_{\mathbf{C}} \cap \mathbf{X}' = (\varnothing, \varnothing)$, *and there is a partial supersaturated functor from* $\mathbf{X}/\mathrm{R}(\mathbf{X}')$ *to* $[x]_{\mathbf{C}}$ *whose underlying partial mappings are bijections, therefore we also obtain a Rees isomorphism between* $\mathbf{X}/\mathrm{R}(\mathbf{X}')$ *and* $[x]_{\mathbf{C}} \amalg \mathbf{1}$.

Moreover, also as for ideal extensions in semigroup theory, we show that a category $\mathbf{C}$ may be reconstructed from a supersaturated subcategory $\mathbf{X}$ and its Rees quotient $\mathbf{C}/\mathrm{R}(\mathbf{X})$. But before that we need to state the following

Definition 2.32. A category $\mathbf{C}$ is an *extension* of a category $\mathbf{X}$ by a category $\mathbf{Q}$ if $\mathbf{X}$ is a supersaturated subcategory of $\mathbf{C}$ and $\mathbf{Q}$ is isomorphic to the Rees quotient of $\mathbf{C}$ by $\mathbf{X}$.

Proposition 2.33. *Let* $\mathbf{C}$ *and* $\mathbf{Q}$ *be categories. If* $\mathbf{C}$ *has a supersaturated subcategory* $\mathbf{X} = (O, M)$, $\mathbf{Q}$ *has a final supersaturated subcategory* $\mathbf{Y} = (\{o\}, \{m\})\,(\cong \mathbf{1})$, *there is a partial supersaturated functor* $F : \mathbf{Q} \rightharpoonup \mathbf{C}$ *such that* $\mathrm{Dom}(F) = \mathbf{Q} - \mathbf{Y}$, *and the unique* $\widetilde{F} : \mathbf{Q} \to \mathbf{C}/\mathrm{R}(\mathbf{X})$ *such that the following diagram commutes*

$$\begin{tikzcd}
\mathbf{Q} - \mathbf{Y} \arrow[r, "\mathrm{In}_{\mathbf{Q}-\mathbf{Y}}"] \arrow[d, "F"'] & \mathbf{Q} \arrow[d, "\widetilde{F}"] & \mathbf{Y} \arrow[l, "\mathrm{In}_{\mathbf{Y}}"'] \arrow[d] \\
\mathbf{C} \arrow[r, "\mathrm{Pr}_{\mathrm{R}(\mathbf{X})}"'] & \mathbf{C}/\mathrm{R}(\mathbf{X}) & (\{O\}, \{M\}) \arrow[l, "\mathrm{In}_{(\{O\},\{M\})}"]
\end{tikzcd}$$

is a functor, then $(\mathbf{Q} - \mathbf{Y}) \coprod \mathbf{X}$, *denoted by* $\mathbf{X} \rtimes_F \mathbf{Q}$, *is an extension of* $\mathbf{X}$ *by* $\mathbf{Q}$ *through* F.

Proposition 2.34. *Let* $\mathbf{C}$ *be an extension of a category* $\mathbf{X}$ *by a category* $\mathbf{Q}$, *where, for simplicity, we identify* $\mathbf{Q}$ *with the Rees quotient* $\mathbf{C}/\mathrm{R}(\mathbf{X})$. *Then*

(1) *There is a congruence* Φ *on* $\mathbf{C}$ *such that*

(a) *The restriction of* Pr_{Φ} *to* $\mathbf{X}$ *is injective, hence* $\mathbf{X} \cong \mathbf{X}/\Phi$.

(b) $\mathbf{X}/\Phi$ *is a supersaturated subcategory of* $\mathbf{C}/\Phi$.

(c) $\Delta_{\mathbf{C}/\Phi}$ *is the only congruence on* $\mathbf{C}/\Phi$ *such that the restriction of the canonical projection to* $\mathbf{X}$ *is injective.*

(2) *There is an* $F : \mathbf{C}/\mathrm{R}(\mathbf{X}) \rightharpoonup \mathbf{C}/\Phi$ *such that*

(a) $\mathrm{Dom}(F) = \mathbf{C}/\mathrm{R}(\mathbf{X}) - (\{O\}, \{M\})$.

(b) $\mathrm{Im}(F) = \mathbf{C}/\Phi - (O/\Phi_0,\ M/\Phi_1)$.

(c) *F has a unique extension* $\widetilde{F} : \mathbf{C}/\mathrm{R}(\mathbf{X}) \to (\mathbf{C}/\Phi)/\mathrm{R}(\mathbf{X}/\Phi)$ *such that the following diagram commutes*

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\mathrm{Pr}_{\Phi}"] \arrow[d, "\mathrm{Pr}_{\mathrm{R}(\mathbf{X})}"'] & \mathbf{C}/\Phi \arrow[d, "\mathrm{Pr}_{\mathrm{R}(\mathbf{X}/\Phi)}"] \\
\mathbf{C}/\mathrm{R}(\mathbf{X}) \arrow[r, "\widetilde{F}"'] & (\mathbf{C}/\Phi)/\mathrm{R}(\mathbf{X}/\Phi)
\end{tikzcd}$$

Moreover, $\mathbf{C} \cong \mathbf{X}/\Phi \rtimes_F \mathbf{C}/\mathrm{R}(\mathbf{X})$.

We consider now the concept of partial natural transformation between partial supersaturated functors, which will allow us to form, from two categories $\mathbf{C}$ and $\mathbf{D}$, the corresponding category $\mathbf{P}(\mathbf{C}, \mathbf{D})$ of partial supersaturated functors and partial natural transformations. From such a functorial category we will obtain another category $\mathbf{P}_0(\mathbf{C}, \mathbf{D})$ that we will use to prove that $\mathbf{Cat}_{\mathrm{p}}$ is a closed category.

Definition 2.35. Given two partial supersaturated functors F and G from a category $\mathbf{C}$ to another $\mathbf{D}$ such that $\mathrm{Dom}(F) = \mathrm{Dom}(G)$, a *partial* natural transformation from F to G is a triple (F, η, G), denoted by $\eta : F \rightharpoonup G$, such that $\eta : \mathrm{Ob}(\mathbf{C}) \rightharpoonup \mathrm{Mor}(\mathbf{D})$, and subject to satisfy the following axioms:

(1) $\mathrm{Dom}(\eta) = \mathrm{Dom}(F_0)\,(= \mathrm{Dom}(G_0))$.

(2) For every $\mathbf{C}$-object x in $\mathrm{Dom}(\eta)$, $\eta_x : F_0(x) \to G_0(x)$.

(3) For every $\mathbf{C}$-morphism $f : x \to y$, if $f \in \mathrm{Dom}(F_1)\,(= \mathrm{Dom}(G_1))$, then

$$G_1(f) \circ \eta_x = \eta_y \circ F_1(f).$$

We agree that a diagram such as the following:

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{D}, \end{tikzcd}$$

also indicates that $\eta : F \rightharpoonup G$. Moreover, we denote by $\mathrm{N}(F, G)$ the set of all partial natural transformations from F to G.

As for ordinary natural transformations we also have the following

Proposition 2.36. *If* $F, G : \mathbf{C} \rightharpoonup \mathbf{D}$ *are such that* $\mathrm{Dom}(F) = \mathrm{Dom}(G)$, *then there is an isomorphism from* $\mathrm{N}(F, G)$ *to the set of all partial supersaturated functors* $H : \mathbf{C} \to \mathbf{D}^{\to}$, *where* $\mathbf{D}^{\to}$, *also written as* $\mathbf{D}^{\mathbf{2}}$, *is the category of arrows of* $\mathbf{D}$, *such that the following diagram commutes*

$$\begin{tikzcd} & \mathbf{C} \arrow[dl, "G"'] \arrow[d, "H"] \arrow[dr, "F"] & \\ \mathbf{D} & \mathbf{D}^{\to} \arrow[l, "\mathrm{Tg}_{\mathbf{D}}"] \arrow[r, "\mathrm{Sc}_{\mathbf{D}}"'] & \mathbf{D} \end{tikzcd}$$

where $\mathrm{Sc}_{\mathbf{D}}$ *and* $\mathrm{Tg}_{\mathbf{D}}$ *are the functors source and target, respectively, for* $\mathbf{D}^{\to}$.

Proposition 2.37. *If* $F, G, H : \mathbf{C} \rightharpoonup \mathbf{D}$, $\alpha : F \rightharpoonup G$, *and* $\beta : G \rightharpoonup H$, *then the vertical composition of* α *and* β, *denoted by* $\beta \circ \alpha$, *defined, for each* $x \in \mathrm{Dom}(F_0)$, *as* $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$, *is a partial natural transformation from* F *to* H. *Moreover, the vertical composition of partial natural transformations is associative and has identities.*

Definition 2.38. Given two categories $\mathbf{C}$ and $\mathbf{D}$, $\mathbf{P}(\mathbf{C}, \mathbf{D})$ is the category whose objects are the partial supersaturated functors from $\mathbf{C}$ to $\mathbf{D}$, and for two supersaturated functors $F, G : \mathbf{C} \rightharpoonup \mathbf{D}$, $\mathrm{Hom}_{\mathbf{P}(\mathbf{C},\mathbf{D})}(F, G)$, the set of morphisms from F to G is the set defined as:

$$\mathrm{Hom}_{\mathbf{P}(\mathbf{C},\mathbf{D})}(F, G) = \begin{cases} \mathrm{N}(F, G), & \text{if } \mathrm{Dom}(F) = \mathrm{Dom}(G); \\ \varnothing, & \text{otherwise.} \end{cases}$$

Once defined the concept of vertical composition of partial natural transformations, we could take into account the concept of *horizontal* composition of partial natural transformations, in order to, eventually, obtain a 2-category, as for $\mathcal{U}$-small categories, functors and natural transformations. With regard to this we have the following

Proposition 2.39. *Given partial supersaturated functors and partial natural transformations as in the following diagram*

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{D} \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{E}, \end{tikzcd}$$

if $\mathrm{Im}(F_0) \cup \mathrm{Im}(G_0) \subseteq \mathrm{Dom}(R_0)$, *then*

(1) $\mathrm{Dom}(R \circ F) = \mathrm{Dom}(S \circ G)$.

(2) *For every* $x \in \mathrm{Dom}(\alpha)$, $\alpha_x \in \mathrm{Dom}(R_1)$.

(3) *The following diagram commutes*

$$\begin{tikzcd} R_0(F_0(x)) \arrow[r, "R_1(\alpha_x)"] \arrow[d, "\beta_{F_0(x)}"'] & R_0(G_0(x)) \arrow[d, "\beta_{G_0(x)}"] \\ S_0(F_0(x)) \arrow[r, "S_0(\alpha_x)"'] & S_0(G_0(x)) \end{tikzcd}$$

Moreover, the horizontal composition of α *and* β

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "R \circ F"] \arrow[r, bend right, "S \circ G"'] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & \mathbf{E}, \end{tikzcd}$$

defined, for each $x \in \mathrm{Dom}((R \circ F)_0)$, *as* $(\beta * \alpha)_x = \beta_{G_0(x)} \circ R_1(\alpha_x)$, *is a partial natural transformation from* $R \circ F$ *to* $S \circ G$.

But the situation described by the diagram:

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "\mathrm{Id}_{\mathbf{C}}"] \arrow[r, bend right, "\mathrm{Id}_{\mathbf{C}}"'] \arrow[r, phantom, "\Downarrow \mathrm{id}_{\mathrm{Id}_{\mathbf{C}}}"] & \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{D} \end{tikzcd}$$

in case $\mathrm{Dom}(F_0) \neq \mathrm{Ob}(\mathbf{C})$, shows that we can not obtain a 2-category.

Notwithstanding, given

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{D}, \end{tikzcd}$$

for $\mathrm{In}_{\mathrm{Dom}(F)}$, the partial supersaturated endofunctor of $\mathbf{C}$ whose domain of definition is $\mathrm{Dom}(F)$, we have that $\alpha * \mathrm{id}_{\mathrm{In}_{\mathrm{Dom}(F)}} = \alpha$. Moreover, the partial supersaturated endofunctor $\mathrm{In}_{\mathrm{Dom}(F)}$ (a type of subidentity) is maximal relative to such property. The existence and properties of these partial functors could be taken as a starting point in order to build a convenient generalization of the notion of 2-category, in which, among others, categories, partial supersaturated functors and partial natural transformations could live.

We return now to our principal task in this section, i.e., to show that $\mathbf{Cat}_{\mathrm{p}}$ is a closed category. But before that we agree on the following notation and terminology relative to a given partial supersaturated functor $T : \mathbf{A} \times \mathbf{C} \rightharpoonup \mathbf{D}$:

(1) The set of all *first* coordinates of $\mathrm{Dom}(T_0)$ is denoted by $\mathrm{Fst}(\mathrm{Dom}(T_0))$, and that of all *first* coordinates of $\mathrm{Dom}(T_1)$ by $\mathrm{Fst}(\mathrm{Dom}(T_1))$.

(2) The set of all *second* coordinates of $\mathrm{Dom}(T_0)$ is denoted by $\mathrm{Snd}(\mathrm{Dom}(T_0))$, and that of all *second* coordinates of $\mathrm{Dom}(T_1)$ by $\mathrm{Snd}(\mathrm{Dom}(T_1))$.

(3) $\mathrm{Fst}(\mathrm{Dom}(T)) = (\mathrm{Fst}(\mathrm{Dom}(T_0)), \mathrm{Fst}(\mathrm{Dom}(T_1)))$.

(4) $\mathrm{Snd}(\mathrm{Dom}(T)) = (\mathrm{Snd}(\mathrm{Dom}(T_0)), \mathrm{Snd}(\mathrm{Dom}(T_1)))$.

Lemma 2.40. *Let $T : \mathbf{A} \times \mathbf{C} \rightharpoonup \mathbf{D}$ be a partial supersaturated functor. Then*

(1) *For every $a, b \in \mathrm{Fst}(\mathrm{Dom}(T_0))$, if $\mathrm{Hom}_{\mathbf{A}}(a, b) \cup \mathrm{Hom}_{\mathbf{A}}(b, a) \neq \varnothing$, then, for every* $\mathbf{C}$*-object x, $(a, x) \in \mathrm{Dom}(T_0)$ if, and only if, $(b, x) \in \mathrm{Dom}(T_0)$.*

(2) *For every $x, y \in \mathrm{Snd}(\mathrm{Dom}(T_0))$, if $\mathrm{Hom}_{\mathbf{C}}(x, y) \cup \mathrm{Hom}_{\mathbf{C}}(y, x) \neq \varnothing$, then, for every* $\mathbf{A}$*-object a, $(a, x) \in \mathrm{Dom}(T_0)$ if, and only if, $(a, y) \in \mathrm{Dom}(T_0)$.*

Proof. It is enough to take into account that $\mathrm{Dom}(T)$ is a supersaturated subcategory of $\mathbf{A} \times \mathbf{C}$.

Remark. The first assertion from Lemma 2.40 is equivalent to: For every a, b in $\mathrm{Fst}(\mathrm{Dom}(T_0))$, if $\mathrm{Hom}_{\mathbf{A}}(a, b) \cup \mathrm{Hom}_{\mathbf{A}}(b, a) \neq \varnothing$, then, for every $\mathbf{C}$-morphism f, $(\mathrm{id}_a, f) \in \mathrm{Dom}(T_1)$ if, and only if, $(\mathrm{id}_b, f) \in \mathrm{Dom}(T_1)$. In the same way, the second assertion from Lemma 2.40 is equivalent to: For every x, y in $\mathrm{Snd}(\mathrm{Dom}(T_0))$, if $\mathrm{Hom}_{\mathbf{C}}(x, y) \cup \mathrm{Hom}_{\mathbf{C}}(y, x) \neq \varnothing$, then, for every $\mathbf{A}$-morphism t, $(t, \mathrm{id}_x) \in \mathrm{Dom}(T_1)$ if, and only if, $(t, \mathrm{id}_y) \in \mathrm{Dom}(T_1)$.

Lemma 2.41. (1) *If $F : \mathbf{A} \rightharpoonup \mathbf{B}$ and $G : \mathbf{C} \rightharpoonup \mathbf{D}$, then $F \times G : \mathbf{A} \times \mathbf{C} \rightharpoonup \mathbf{B} \times \mathbf{D}$.*

(2) *If $T : \mathbf{A} \times \mathbf{C} \rightharpoonup \mathbf{D}$, then $\mathrm{Fst}(\mathrm{Dom}(T))$ is a supersaturated subcategory of $\mathbf{A}$ and $\mathrm{Snd}(\mathrm{Dom}(T))$ is a supersaturated subcategory of $\mathbf{C}$.*

Proposition 2.42. *Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Then there is a category $\mathbf{P}_0(\mathbf{C}, \mathbf{D})$ and a partial supersaturated functor $\mathrm{Ev}_{\mathbf{C}, \mathbf{D}}$ from $\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C}$ to $\mathbf{D}$, also denoted by Ev, such that, for every category $\mathbf{A}$ and every partial supersaturated functor T from $\mathbf{A} \times \mathbf{C}$ to $\mathbf{D}$, there is a unique partial supersaturated functor $T^{\#}$ from $\mathbf{A}$ to $\mathbf{P}_0(\mathbf{C}, \mathbf{D})$ such that the following diagram commutes*

$$\begin{tikzcd}
\mathbf{A} \times \mathbf{C} \arrow[d, "T^{\#} \times \mathrm{Id}_{\mathbf{C}}"'] \arrow[dr, "T"] & \\
\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C} \arrow[r, "\mathrm{Ev}_{\mathbf{C},\mathbf{D}}"'] & \mathbf{D}
\end{tikzcd}$$

Proof. Let $\mathbf{P}_0(\mathbf{C}, \mathbf{D})$ be the category which has as objects the non-zero partial supersaturated functors from $\mathbf{C}$ to $\mathbf{D}$, and as morphisms from F to G the partial natural transformations from F to G.

Next, in order to define Ev, let, on the one hand, Ev_0 be the partial mapping from $\mathrm{Ob}(\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C})$ to $\mathrm{Ob}(\mathbf{D})$ whose domain of definition is

$$\mathrm{Dom}(\mathrm{Ev}_0) = \{(F, x) \in \mathrm{Ob}(\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C}) \mid x \in \mathrm{Dom}(F_0)\},$$

and is such that, for $(F, x) \in \mathrm{Dom}(\mathrm{Ev}_0)$, $\mathrm{Ev}_0(F, x) = F_0(x)$, and, on the other, let Ev_1 be the partial mapping from $\mathrm{Mor}(\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C})$ to $\mathrm{Mor}(\mathbf{D})$ whose domain of definition is

$$\mathrm{Dom}(\mathrm{Ev}_1) = \bigcup_{(F, x), (G, y) \in \mathrm{Dom}(\mathrm{Ev}_0)} \mathrm{Hom}_{\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C}}((F, x), (G, y)),$$

and is such that, for $(\eta, f) : (F, x) \to (G, y)$ in $\mathrm{Dom}(\mathrm{Ev}_1)$, $\mathrm{Ev}_1(\eta, f)$ is the diagonal of the commutative diagram

$$\begin{tikzcd}
F_0(x) \arrow[r, "\eta_x"] \arrow[d, "F_1(f)"'] & G_0(x) \arrow[d, "G_1(f)"] \\
F_0(y) \arrow[r, "\eta_y"'] & G_0(y)
\end{tikzcd}$$

Thus defined it is readily seen that Ev is a partial supersaturated functor from $\mathbf{P}_0(\mathbf{C}, \mathbf{D}) \times \mathbf{C}$ to $\mathbf{D}$.

Now, let $\mathbf{A}$ be a category and let us suppose that T is a non-zero partial supersaturated functor from $\mathbf{A} \times \mathbf{C}$ to $\mathbf{D}$ (because the case of the nowhere defined partial supersaturated functor is obvious). From this we want to show that there is precisely one $T^{\#} : \mathbf{A} \rightharpoonup \mathbf{P}_0(\mathbf{C}, \mathbf{D})$ such that $\mathrm{Ev} \circ (T^{\#} \times \mathrm{Id}_{\mathbf{C}}) = T$.

Next to define the two components $T_0^{\#}$ and $T_1^{\#}$ of $T^{\#}$ we proceed as follows. Let $T_0^{\#}$ be the partial mapping from $\mathrm{Ob}(\mathbf{A})$ to $\mathrm{Ob}(\mathbf{P}_0(\mathbf{C}, \mathbf{D}))$ whose domain of definition is

$$\mathrm{Dom}(T_0^{\#}) = \mathrm{Fst}(\mathrm{Dom}(T_0)),$$

and is such that, for each $a \in \mathrm{Dom}(T_0^{\#})$, $T_0^{\#}(a)$ is the partial supersaturated functor from $\mathbf{C}$ to $\mathbf{D}$ that has as partial object mapping the partial mapping $T_0^{\#}(a)_0$ from $\mathrm{Ob}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{D})$ whose domain of definition is

$$\mathrm{Dom}(T_0^{\#}(a)_0) = \{x \in \mathrm{Ob}(\mathbf{C}) \mid (a, x) \in \mathrm{Dom}(T_0)\},$$

and is such that, for each $x \in \mathrm{Dom}(T_0^{\#}(a)_0)$, $T_0^{\#}(a)_0(x) = T_0(a, x)$, and as partial morphism mapping the partial mapping $T_0^{\#}(a)_1$ from $\mathrm{Mor}(\mathbf{C})$ to $\mathrm{Mor}(\mathbf{D})$ whose domain of definition is:

$$\mathrm{Dom}(T_0^{\#}(a)_1) = \{f \in \mathrm{Mor}(\mathbf{C}) \mid (\mathrm{id}_a, f) \in \mathrm{Dom}(T_1)\},$$

and is such that, for each f in $\mathrm{Dom}(T_0^{\#}(a)_1)$, $T_0^{\#}(a)_1(f) = T_1(\mathrm{id}_a, f)$.

Before we define $T_1^{\#}$ we remark that, for each $t : a \to b \in \mathrm{Fst}(\mathrm{Dom}(T_1))$, we have that $\mathrm{Dom}(T_0^{\#}(a)) = \mathrm{Dom}(T_0^{\#}(b))$, by Lemma 2.40.

Let $T_1^{\#}$ be the partial mapping from $\mathrm{Mor}(\mathbf{A})$ to $\mathrm{Mor}(\mathbf{P}_0(\mathbf{C}, \mathbf{D}))$ whose domain of definition is

$$\mathrm{Dom}(T_1^{\#}) = \mathrm{Fst}(\mathrm{Dom}(T_1)),$$

and is such that, for each $t : a \to b \in \mathrm{Dom}(T_1^{\#})$, $T_1^{\#}(t)$ is the partial natural transformation from $T_0^{\#}(a)$ to $T_0^{\#}(b)$, whose domain of definition is $\mathrm{Dom}(T_0^{\#}(a)_0)$ and is such that, for each x in $\mathrm{Dom}(T_1^{\#}(t))$, $T_1^{\#}(t)_x = T_1(t, \mathrm{id}_x)$.

Thus defined $T^{\#}$ preserves identities and compositions, moreover, by Lemma 2.41, we have that $\mathrm{Dom}(T^{\#})$ is a partial supersaturated subcategory of $\mathbf{A}$, hence $T^{\#}$ is a partial supersaturated functor. On the other hand, also by Lemma 2.41, we have that $T^{\#} \times \mathrm{Id}_{\mathbf{C}}$ is a partial supersaturated functor from $\mathbf{A} \times \mathbf{C}$ to $\mathbf{P}_0(\mathbf{C}, \mathbf{D})$.

Finally, it is obvious that $T^{\#} : \mathbf{A} \rightharpoonup \mathbf{P}_0(\mathbf{C}, \mathbf{D})$ is the unique partial supersaturated functor such that $\mathrm{Ev} \circ (T^{\#} \times \mathrm{Id}_{\mathbf{C}}) = T$.

Corollary 2.43. *The category* $\mathbf{Cat}_{\mathrm{p}}$ *is closed.*

Proof. It is enough to consider $\times : \mathbf{Cat}_{\mathrm{p}} \times \mathbf{Cat}_{\mathrm{p}} \to \mathbf{Cat}_{\mathrm{p}}$, the final category $\mathbf{1}$, and to take into account that, for every $\mathcal{U}$-small category $\mathbf{C}$ the functor

$$(\cdot) \times \mathbf{C} : \mathbf{Cat}_{\mathrm{p}} \to \mathbf{Cat}_{\mathrm{p}}$$

has, by Proposition 2.42, a right adjoint.

Before stating the following proposition we agree that, for two partial supersaturated functors $F, G : \mathbf{C} \rightharpoonup \mathbf{D}$, $F \leq G$ means that $\mathrm{Dom}(F) \subseteq \mathrm{Dom}(G)$ and that, for every $x \in \mathrm{Dom}(F_0)$, $F_0(x) = G_0(x)$, and, for every $f \in \mathrm{Dom}(F_1)$, $F_1(f) = G_1(f)$. The binary relation $\leq$ on the set $\mathbf{P}(\mathbf{C}, \mathbf{D})$ will be accurately investigated in the following section where it will called the *extension order* on $\mathbf{P}(\mathbf{C}, \mathbf{D})$.

Proposition 2.44. *If* $T : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \rightharpoonup \mathbf{Set}$ *is such that* $T \leq \mathrm{Hom}_{\mathbf{C}}$, *then the partial supersaturated functor* $T^{\#} : \mathbf{C}^{\mathrm{op}} \rightharpoonup \mathbf{P}_0(\mathbf{C}, \mathbf{Set})$ *is a full embedding.*

We conclude this section by stating the Yoneda-Grothendieck Lemma and defining the concept of partial adjunction, both for partial supersaturated functors, and by providing an example of partial adjunction from the field of algebraic logic.

Definition 2.45. Given $F : \mathbf{C} \rightharpoonup \mathbf{Set}$ and $g : x \to y \in \mathrm{Dom}(F_1)$, we denote by $\mathrm{H}^F(x, \cdot)$ the partial supersaturated functor from $\mathbf{C}$ to $\mathbf{Set}$ whose domain of definition is that of F and is defined in the same way as is the ordinary covariant hom-functor at x; and by $\mathrm{H}^F(g, \cdot)$ the partial natural transformation from $\mathrm{H}^F(y, \cdot)$ to $\mathrm{H}^F(x, \cdot)$ also defined as classically.

To state the Yoneda-Grothendieck Lemma we need to consider, in addition to the partial supersaturated functor $\mathrm{Ev} : \mathbf{P}_0(\mathbf{C}, \mathbf{Set}) \times \mathbf{C} \rightharpoonup \mathbf{Set}$, another partial supersaturated functor N between the same categories, that we make explicit in the following

Definition 2.46. Let $\mathbf{C}$ be a category. Then we denote by N the partial supersaturated functor from $\mathbf{P}_0(\mathbf{C}, \mathbf{Set}) \times \mathbf{C}$ to $\mathbf{Set}$ which has as partial object mapping the partial mapping N_0 from $\mathrm{Ob}(\mathbf{P}_0(\mathbf{C}, \mathbf{Set}) \times \mathbf{C})$ to $\mathrm{Ob}(\mathbf{Set})$ whose domain of definition is identical to that of Ev_0, i.e., to the set

$$\mathrm{Dom}(\mathrm{N}_0) = \{(F, x) \in \mathrm{Ob}(\mathbf{P}_0(\mathbf{C}, \mathbf{Set}) \times \mathbf{C}) \mid x \in \mathrm{Dom}(F_0)\},$$

and is such that, for each $(F, x) \in \mathrm{Dom}(\mathrm{N}_0)$, $\mathrm{N}_0(F, x) = \mathrm{N}(\mathrm{H}^F(x, \cdot), F)$, and as partial morphism mapping the partial mapping N_1 from $\mathrm{Mor}(\mathbf{P}_0(\mathbf{C}, \mathbf{Set}) \times \mathbf{C})$ to $\mathrm{Mor}(\mathbf{Set})$ whose domain of definition is identical to that of Ev_1, i.e., to the set

$$\mathrm{Dom}(\mathrm{N}_1) = \textstyle\bigcup_{(F,x),(G,y) \in \mathrm{Dom}(\mathrm{N}_0)} \mathrm{Hom}_{\mathbf{P}_0(\mathbf{C},\mathbf{Set}) \times \mathbf{C}}((F, x), (G, y)),$$

and is such that, for each $(\eta, f) : (F, x) \to (G, y)$ in $\mathrm{Dom}(\mathrm{N}_1)$, $\mathrm{N}_1(\eta, f)$ is the mapping from $\mathrm{N}(\mathrm{H}^F(x, \cdot), F)$ into $\mathrm{N}(\mathrm{H}^G(y, \cdot), F)$ that to $\alpha : \mathrm{H}^F(x, \cdot) \rightharpoonup F$ assigns $\eta \circ \alpha \circ \mathrm{H}^F(g, \cdot) : \mathrm{H}^G(y, \cdot) \rightharpoonup G$.

Lemma 2.47 (Yoneda-Grothendieck)**.** *Let* $\mathbf{C}$ *be a category. Then there is a partial natural isomorphism*

$$\begin{tikzcd} \mathbf{P}_0(\mathbf{C}, \mathbf{Set}) \times \mathbf{C} \arrow[r, bend left, "\mathrm{Ev}"] \arrow[r, bend right, "\mathrm{N}"'] \arrow[r, phantom, "\Downarrow \mathrm{Y}"] & \mathbf{Set} \end{tikzcd}$$

Definition 2.48. Let $G : \mathbf{A} \rightharpoonup \mathbf{X}$ and $F : \mathbf{X} \rightharpoonup \mathbf{A}$ be partial supersaturated functors. We say that G and F satisfy the Im-Dom condition if $\mathrm{Im}(F_0) \subseteq \mathrm{Dom}(G_0)$ and $\mathrm{Im}(G_0) \subseteq \mathrm{Dom}(F_0)$.

Let us observe that if $G : \mathbf{A} \rightharpoonup \mathbf{X}$ and $F : \mathbf{X} \rightharpoonup \mathbf{A}$ satisfy the Im-Dom condition, then $\mathrm{Im}(F_1) \subseteq \mathrm{Dom}(G_1)$ and $\mathrm{Im}(G_1) \subseteq \mathrm{Dom}(F_1)$.

Proposition 2.49. *Let $G : \mathbf{A} \rightharpoonup \mathbf{X}$ and $F : \mathbf{X} \rightharpoonup \mathbf{A}$ be partial supersaturated functors such that G and F satisfy the* Im-Dom *condition. Then we have that $\mathrm{Dom}(G) = \mathrm{Dom}(F \circ G)$ and $\mathrm{Dom}(F) = \mathrm{Dom}(G \circ F)$. Therefore all functors $G, F \circ G, G \circ F \circ G, \ldots,$ have as domain of definition $\mathrm{Dom}(G)$, and all functors $F, G \circ F, F \circ G \circ F, \ldots,$ have as domain of definition $\mathrm{Dom}(F)$.*

Definition 2.50. Let $\mathbf{A}$ and $\mathbf{X}$ be categories. A *partial* adjunction from $\mathbf{X}$ to $\mathbf{A}$ is a quadruple $(F, G, \eta, \varepsilon)$, where F and G are partial supersaturated functors as in

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift left, "G"] & \mathbf{X} \arrow[l, shift left, "F"] \end{tikzcd}$$

which satisfy the Im-Dom condition, while η and ε are partial natural transformations

$$\eta : \mathrm{In}_{\mathrm{Dom}(F)} \rightharpoonup G \circ F, \quad \varepsilon : F \circ G \rightharpoonup \mathrm{In}_{\mathrm{Dom}(G)},$$

such that the following diagrams commute

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}_F * \eta"] \arrow[dr, "\mathrm{id}_F"'] & F \circ G \circ F \arrow[d, "\varepsilon * \mathrm{id}_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta * \mathrm{id}_G"] \arrow[dr, "\mathrm{id}_G"'] & G \circ F \circ G \arrow[d, "\mathrm{id}_G * \varepsilon"] \\ & G \end{tikzcd}$$

As for ordinary adjunctions we also have that for a partial supersaturated functor $G : \mathbf{A} \rightharpoonup \mathbf{X}$ and a subset O of $\mathrm{Ob}(\mathbf{X})$ if, for every $x \in O$, $\downarrow x$, $\uparrow x \subseteq O$, $\mathrm{Im}(G_0) \subseteq O$ and, for every $x \in O$, there is a universal arrow (t, a) from x to G, with $a \in \mathrm{Dom}(G_0)$ and $t : x \to F_0(a)$, then there is a partial supersaturated functor $F : \mathbf{X} \rightharpoonup \mathbf{A}$ such that F

and G satisfy the Im-Dom condition, $\mathrm{Dom}(F_0) = O$ and there are partial natural transformations $\eta : \mathrm{In}_{\mathrm{Dom}(F)} \rightharpoonup G \circ F$, $\varepsilon : F \circ G \rightharpoonup \mathrm{In}_{\mathrm{Dom}(G)}$ such that the quadruple $(F, G, \eta, \varepsilon)$ is a partial adjunction from $\mathbf{X}$ to $\mathbf{A}$. In this case we say that F is a *partial left adjoint* to G.

Example 2.51. As an example of the concept of partial adjunction we have that, for a sentential logic $\mathcal{S}$, the partial supersaturated functor of reduction $\mathrm{Rd}_{\mathcal{S}}$ from $\mathbf{M}(\mathcal{S})$ to $\mathbf{M}_{\mathrm{rf}}(\mathcal{S})$, which, we recall, has as domain of definition precisely $\mathbf{M}_{\mathrm{f}}(\mathcal{S})$, is a partial left adjoint to the inclusion of $\mathbf{M}_{\mathrm{rf}}(\mathcal{S})$ into $\mathbf{M}(\mathcal{S})$. Moreover, $\mathrm{Rd}_{\mathcal{S}}$ cannot be extended to $\mathbf{M}(\mathcal{S})$. Therefore we can think about $\mathbf{M}_{\mathrm{f}}(\mathcal{S})$ as representing a type of maximal domain beyond which are obstructed some natural constructions arising in algebraic logic.

3. Order and Additivity

Our first goal in this section is to show, for two categories $\mathbf{C}$ and $\mathbf{D}$, that the set of all partial supersaturated functors from $\mathbf{C}$ into $\mathbf{D}$, when ordered by extension, is an algebraic and coherent complete partial order, exactly as for sets and partial functions. However, we will prove that the concepts of finiteness and compactness, indistinguishable when applied to partial mappings, are not equivalent for partial supersaturated functors.

For completeness we begin by recalling the concept of complete partial order, and also those of algebraic and coherent complete partial order.

Definition 3.1. (1) A *complete* partial order is a partial order $(A, \leq)$ which has a minimum and is such that every nonempty directed subset of A has a join in $(A, \leq)$.

(2) If $(A, \leq)$ is a complete partial order, an element $a \in A$ is *compact* in $(A, \leq)$ if for every nonempty directed subset X of A, if $a \leq \bigvee X$, then $a \leq x$ for some $x \in X$.

(3) An *algebraic* complete partial order is a complete partial order $(A, \leq)$ such that every element of A is the join of a nonempty directed subset of compacts in $(A, \leq)$.

(4) A *coherent* complete partial order is a complete partial order $(A, \leq)$ such that every consistent subset of A has a join in $(A, \leq)$, where a subset X of A is *consistent* if every finite nonempty subset of X has an upper bound in $(A, \leq)$.

Definition 3.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Then the extension ordering $\leq$ on $\mathrm{P}(\mathbf{C}, \mathbf{D})$ is the relation defined as

$$F \leq G \text{ if, and only if, } F_0 \leq G_0 \text{ and } F_1 \leq G_1.$$

Proposition 3.3. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Then* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ *is a complete partial order. Moreover, given the situation described by the diagram*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "H"] & \mathbf{C} \arrow[r, shift left, "F"] \arrow[r, shift right, "G"'] & \mathbf{D} \arrow[r, "K"] & \mathbf{B} \end{tikzcd}$$

if $F \leq G$, *then* $F \circ H \leq G \circ H$ *and* $K \circ F \leq K \circ G$, *hence* $(\mathrm{P}(\mathbf{C}, \mathbf{C}), \circ, \mathrm{Id}_{\mathbf{C}}, \leq)$ *is a partially ordered monoid with zero (the nowhere defined partial supersaturated endofunctor of* $\mathbf{C}$, *denoted by* $\theta_{\mathbf{D}}$*). Moreover, there is a one-to-one correspondence between the predecessors of* $\mathrm{Id}_{\mathbf{C}}$ *and the supersaturated subcategories of* $\mathbf{C}$, *and, for a supersaturated subcategory* $\mathbf{X}$ *of* $\mathbf{C}$, *we agree to denote by* $\mathrm{In}_{\mathbf{X}}$ *the corresponding predecessor of* $\mathrm{Id}_{\mathbf{C}}$.

Proposition 3.4. *If* $F, G : \mathbf{C} \rightharpoonup \mathbf{D}$, *then F and G have an upper bound in the complete partial order* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ *if, and only if, the restrictions of F and G to* $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$ *are identical.*

Proposition 3.5. *Let* $(F^i)_{i \in I}$ *be a consistent family of partial supersaturated functors from* $\mathbf{C}$ *to* $\mathbf{D}$. *Then there exists the join of* $(F^i)_{i \in I}$ *in* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$. *Hence* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ *is a coherent complete partial order.*

If $(F^i)_{i \in I}$ is a nonempty directed family in $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$, then, for every $j,\ k \in I$, the restrictions of F^j and F^k to $\mathrm{Dom}(F^j) \cap \mathrm{Dom}(F^k)$ are identical, i.e., $(F^i)_{i \in I}$ is consistent; therefore the existence of the join for nonempty directed subsets of $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ follows from the existence of the join for consistent subsets of the same complete partial order.

Proposition 3.6. *A partial supersaturated functor* $F : \mathbf{C} \rightharpoonup \mathbf{D}$ *is compact in* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ *if, and only if, there is a finite subset* $O \subseteq \mathrm{Ob}(\mathbf{C})$ *such that the supersaturated subcategory of* $\mathbf{C}$ *generated by* $(O, \bigcup_{(a, b) \in O^2} \mathrm{Hom}_{\mathbf{C}}(a, b))$ *is* $\mathrm{Dom}(F)$.

Proof. If there is a finite subset O of $\mathrm{Ob}(\mathbf{C})$ such that the supersaturated subcategory of $\mathbf{C}$ generated by $(O, \bigcup_{(a, b) \in O^2} \mathrm{Hom}_{\mathbf{C}}(a, b))$ is $\mathrm{Dom}(F)$, and $\mathbf{G}$ is a nonempty directed subset of $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ such that $F \leq \bigvee \mathbf{G}$, then $O \subseteq \bigcup_{G \in \mathbf{G}} \mathrm{Dom}(G_0)$. Hence, for every $x \in O$ there is a $G^x \in \mathbf{G}$ such that $x \in \mathrm{Dom}(G_0^x)$. Now, because $\mathbf{G}$ is a directed subset, let $G \in \mathbf{G}$ be an upper bound for the family $(G^x \mid x \in O)$. It is obvious that $F \leq G$, therefore F is compact.

Reciprocally, if F is such that for every finite subset O of $\mathrm{Ob}(\mathbf{C})$, the supersaturated subcategory of $\mathbf{C}$ generated by $(O, \bigcup_{(a, b) \in O^2} \mathrm{Hom}_{\mathbf{C}}(a, b))$ is different of $\mathrm{Dom}(F)$, then given a finite subset K of $\mathrm{Dom}(F_0)$, let G^K be the partial supersaturated functor from $\mathbf{C}$ to $\mathbf{D}$ whose domain of definition is the supersaturated subcategory of $\mathbf{C}$ generated by $(K, \bigcup_{a, b \in K} \mathrm{Hom}_{\mathbf{C}}(a, b))$ and is such that, for $x \in \mathrm{Dom}(G_0^K)$, $G_0^K(x) = F_0(x)$, and for $f \in \mathrm{Dom}(G_1^K)$, $G_1^K(f) = F_1(f)$. In this way we have obtained a nonempty directed subset $\mathbf{G} = \{G^K \mid K \in \mathrm{Sub}_{\mathrm{fin}}(\mathrm{Dom}(F_0))\}$ of $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$, such that $F \leq \bigvee \mathbf{G}$ and for every $K \in \mathrm{Sub}_{\mathrm{fin}}(\mathrm{Dom}(F_0))$, $F \not\leq G^K$, therefore F is not compact.

Proposition 3.7. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Then* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$ *is an algebraic complete partial order.*

Proof. For $F : \mathbf{C} \rightharpoonup \mathbf{D}$ we have that F is the join of the nonempty directed subset of compacts $\{G^K \mid K \in \mathrm{Sub}_{\mathrm{fin}}(\mathrm{Dom}(F_0))\}$ in $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$.

If, as for partial mappings, we say that a partial supersaturated functor F from $\mathbf{C}$ to $\mathbf{D}$ is finite precisely when $\mathrm{Dom}(F)$ is finite, then, obviously, every finite partial supersaturated is compact. However, for the category determined by the additive monoid of the natural numbers, the unique functor from such a category to the final category, is compact but not finite.

Our next task is to state the concept of summability for families of partial supersaturated functors that will allows us to show that $\mathbf{Cat}_{\mathrm{p}}$ is an ordered partially additive category.

Definition 3.8. Let $(F^i)_{i \in I}$ be a family of partial supersaturated functors from $\mathbf{C}$ to $\mathbf{D}$. Then we say that $(F^i)_{i \in I}$ is *summable* in $\mathrm{P}(\mathbf{C}, \mathbf{D})$ if $\mathrm{Dom}(F^j)$ and $\mathrm{Dom}(F^k)$ are disjoint for $j \neq k$.

Proposition 3.9. *Let* $(F^i)_{i \in I}$ *be a summable family in* $\mathrm{P}(\mathbf{C}, \mathbf{D})$. *Then there exists the join of* $(F^i)_{i \in I}$ *in* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \leq)$, *denoted, in this context by* $\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I}$.

Proposition 3.10. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Then* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \Sigma_{\mathbf{C}, \mathbf{D}})$ *is a generalized partially additive monoid, i.e., it satisfies the following conditions*:

(1) *If* $(F^i)_{i \in I}$ *is a family in* $\mathrm{P}(\mathbf{C}, \mathbf{D})$ *and* $(I_\lambda)_{\lambda \in \Lambda}$ *is a partition of* I (*where* $I_\lambda = \varnothing$ *is allowed for any number of* λ), *then* $(F^i)_{i \in I}$ *is summable if, and only if,* $(F^i)_{i \in I_\lambda}$ *is summable for every* $\lambda \in \Lambda$ *and* $(\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I_\lambda})_{\lambda \in \Lambda}$ *is summable, and then* $\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I} = \Sigma_{\mathbf{C}, \mathbf{D}} \ (\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I_\lambda})_{\lambda \in \Lambda}$. *This condition is called the Partition-Associativity Axiom*.

(2) *If* $(F^i)_{i \in I}$ *is a one-term family in* $\mathrm{P}(\mathbf{C}, \mathbf{D})$ *and* $I = \{i\}$, *then* $(F^i)_{i \in I}$ *is summable and* $\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I} = F^i$. *This condition is called the Unary Sum Axiom*.

(3) *If* $(F^i)_{i \in I}$ *is a family in* $\mathrm{P}(\mathbf{C}, \mathbf{D})$ *and if for every finite subset* J *of* I *the subfamily* $(F^j)_{j \in J}$ *is summable, then* $(F^i)_{i \in I}$ *is summable. This condition is called the Limit Axiom*.

Moreover, for all $G : \mathbf{A} \rightharpoonup \mathbf{C}$, $H : \mathbf{D} \rightharpoonup \mathbf{B}$ *and for all summable families* $(F^i)_{i \in I}$ *in* $\mathrm{P}(\mathbf{C}, \mathbf{D})$, $(F^i \circ G)_{i \in I}$ *and* $(H \circ F^i)_{i \in I}$ *are also summable and*

(1) $(\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I}) \circ G = \Sigma_{\mathbf{C}, \mathbf{D}}(F^i \circ G)_{i \in I}$.

(2) $H \circ (\Sigma_{\mathbf{C}, \mathbf{D}}(F^i)_{i \in I}) = \Sigma_{\mathbf{C}, \mathbf{D}}(H \circ F^i)_{i \in I}$.

Therefore the family $(\Sigma_{\mathbf{C}, \mathbf{D}})_{(\mathbf{C}, \mathbf{D}) \in \mathrm{Ob}(\mathbf{Cat}_{\mathrm{p}})}$, *of partial operators, is a generalized partially additive structure on* $\mathbf{Cat}_{\mathrm{p}}$.

For partial supersaturated functors, as for partial mappings, the Limit Axiom is true in a stronger form: If $(F^i)_{i \in I}$ is a family in $\mathrm{P}(\mathbf{C}, \mathbf{D})$ and if, for every $j, k \in I$ with $j \neq k$, $F^j + F^k$ exists, then $(F^i)_{i \in I}$ is summable.

We also point out that the restriction of $(\Sigma_{\mathbf{C}, \mathbf{D}})_{(\mathbf{C}, \mathbf{D}) \in \mathrm{Ob}(\mathbf{Cat}_{\mathrm{p}})}$ to countable families, can be used in order to consider constructs like those occurring in programming, such as:

(1) If $(\mathbf{X}_i)_{i \in n}$ is a nonempty n-indexed family of supersaturated subcategories of $\mathbf{C}$ such that $\mathbf{X}_j$ and $\mathbf{X}_k$ are disjoint, for $j \neq k$, and $(F^i)_{i \in n}$ is a nonempty n-indexed family of partial supersaturated functors from $\mathbf{C}$ into $\mathbf{D}$, then

$$\text{case } (\mathbf{X}_i)_{i \in n} \text{ of } (F^i)_{i \in n} = \Sigma_{\mathbf{C}, \mathbf{D}}(F^i \circ \mathrm{In}_{\mathbf{X}_i})_{i \in n}.$$

(2) If $\mathbf{X}$ is a supersaturated subcategory of $\mathbf{C}$ and F, $G : \mathbf{C} \rightharpoonup \mathbf{D}$, then

$$\text{if } \mathbf{X}, \text{ then } F \text{ else } G = F \circ \mathrm{In}_{\mathbf{X}} + G \circ \mathrm{In}_{\mathbf{C}-\mathbf{X}}.$$

(3) If $\mathbf{X}$ is a supersaturated subcategory of $\mathbf{C}$ and $F : \mathbf{C} \rightharpoonup \mathbf{C}$, then

$$\text{while } \mathbf{X} \text{ do } F = \Sigma_{\mathbf{C},\mathbf{C}}(\mathrm{In}_{\mathbf{C}-\mathbf{X}} \circ (F \circ \mathrm{In}_{\mathbf{X}})^n)_{n\in\mathrm{N}}.$$

(4) If $\mathbf{X}$ is a supersaturated subcategory of $\mathbf{C}$ and $F : \mathbf{C} \rightharpoonup \mathbf{C}$, then

$$\text{repeat } F \text{ until } \mathbf{X} = (\Sigma_{\mathbf{C},\mathbf{C}}(\mathrm{In}_{\mathbf{X}} \circ (F \circ \mathrm{In}_{\mathbf{C}-\mathbf{X}})^n)_{n\in\mathrm{N}}) \circ F.$$

Now we state the connection between the extension ordering and the partial binary sums.

Proposition 3.11. *If* F, $G : \mathbf{C} \rightharpoonup \mathbf{D}$, *then* $F \leq G$ *if, and only if,* $G = F + H$ *for some* $H : \mathbf{C} \rightharpoonup \mathbf{D}$, *i.e., the sum-ordering on* $\mathrm{P}(\mathbf{C}, \mathbf{D})$ *is exactly the extension ordering on the same set.*

Corollary 3.12. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories. Then* $(\mathrm{P}(\mathbf{C}, \mathbf{D}), \Sigma_{\mathbf{C},\mathbf{D}})$ *is a generalized additive domain, i.e., it is a sum-ordered generalized partially additive monoid, and moreover for every summable family* $(F^i)_{i\in I}$ *and every* G *in* $\mathrm{P}(\mathbf{C}, \mathbf{D})$, *if for each finite subset* J *of* I, G *is an upper bound for* $\Sigma_{\mathbf{C},\mathbf{D}}(F^j)_{j\in J}$, *then* G *is also an upper bound for* $\Sigma_{\mathbf{C},\mathbf{D}}(F^i)_{i\in I}$.

Proposition 3.13. *The category* $\mathbf{Cat}_{\mathrm{p}}$ *has coproducts.*

Proposition 3.14. *Let* $F : \mathbf{C} \rightharpoonup \coprod_{i\in I}\mathbf{D}$ *be a partial supersaturated functor. Then the family* $(\mathrm{q}_i \circ F)_{i\in I}$ *is summable, where, for every* $i \in I$, *the quasi-projection* $\mathrm{q}_i : \coprod_{i\in I}\mathbf{D} \rightharpoonup \mathbf{D}$ *is the unique partial supersaturated functor from* $\coprod_{i\in I}\mathbf{D}$ *to* $\mathbf{D}$ *such that, for every* $j \in J$, *the following diagram commutes*

$$\begin{tikzcd}
\mathbf{D} \arrow[r, "\mathrm{In}_{\mathbf{D}}^{j}"] \arrow[dr, "\delta_i^j"'] & \coprod_{i \in I} \mathbf{D} \arrow[d, "q_i"] \\
& \mathbf{D}
\end{tikzcd}$$

where, for every $i, j \in I$, *the partial supersaturated endofunctor* δ_i^j *at* $\mathbf{D}$ *is defined as follows*

$$\delta_i^j = \begin{cases} \mathrm{Id}_{\mathbf{D}}, & \text{if } j = i; \\ \theta_{\mathbf{D}}, & \text{if } j \neq i. \end{cases}$$

Proposition 3.15. *If* $F, G : \mathbf{C} \rightharpoonup \mathbf{D}$ *are summable, then the partial supersaturated functors* $\mathrm{In}_{\mathbf{D}}^{0} \circ F$ *and* $\mathrm{In}_{\mathbf{D}}^{1} \circ F : \mathbf{C} \rightharpoonup \coprod_{i \in 2} \mathbf{D}$ *are summable.*

From this it follows immediately the following

Corollary 3.16. *The category* $\mathbf{Cat}_{\mathrm{p}}$ *is an ordered partially additive category. Moreover, it has an initial object and every partial supersaturated functor pulls back summands.*

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