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PERTURBATIVE NONLINEAR SCHRÖDINGER EQUATIONS UNDER VARIABLE GROUP VELOCITY DISSIPATION

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Abstract

In this paper, a nonlinear Schrödinger equation is solved approximately under variable group velocity dissipation in a limited time interval. Complex initial conditions and zero Neumann conditions are considered. The perturbation method together with the eigenfunction expansion and variation of parameters method are used to introduce an approximate solution for the perturbative nonlinear case for which a power series solution is proved to exist. Using Mathematica, the solution algorithm is tested through computing the only possible first order approximation for some variations of the variable group velocity. The method of solution is illustrated through case studies and figures.

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1. Introduction

In the last two decades, the nonlinear Schrödinger equation (NLS) got the interests of many scientists in engineering, applied and theoretical sciences, see [1-4, 9] for examples. There are a lot of NLS problems depending on additive or multiplicative noise in the random case [5, 6] or a lot of solution methodologies in the deterministic case [14-18]. By using coupled amplitude phase formulation, Parsezian and Kalithasan [11] constructed the quartic anharmonic oscillator equation from the coupled higher order NLS. Two-dimensional grey solitons to the NLS were numerically analyzed by Sakaguchi and Higashiuchi [13]. The generalized derivative NLS was studied by Huang et al. [10] introducing a new auxiliary equation expansion method. El-Tawil and El-Hazmy [7] used the perturbation method to introduce an approximate solution to a perturbative cubic NLS equation.

In this paper, a straightforward solution algorithm is introduced using the transformation from a complex solution to a coupled equations in two real solutions, eliminating one of the solutions to get separate independent and higher order equations, and finally introducing a perturbative approximate solution to the system.

2. The NLS Equation

Consider the homogeneous non-linear Schrödinger equation:

$$i\frac{\partial u(t,z)}{\partial z}+\beta(t)\frac{\partial^2 u(t,z)}{\partial t^2}+\varepsilon|u(t,z)|^2u(t,z)+i\gamma u(t,z)=0,\ (t,z)\in(0,T)\times(0,\infty), \quad (1)$$

where $u(t, z)$ is a complex valued function which is subjected to:

I.C. : $u(t, 0) = f_1(t) + if_2(t),$ a complex valued function,

B.C. : $u_z(0, z) = 0,\ \ u_z(T, z) = 0,$ and $\beta(t)$ is a variable group velocity dissipation.

Lemma. *The solution of equation* (1) *with its associated constraints is a power series in* ε, *if exists.*

Proof. At $\varepsilon = 0,$ the following linear equation is got:

$$i\frac{\partial u_0(t,\, z)}{\partial z} + \beta(t)\frac{\partial^2 u_0(t,\, z)}{\partial t^2} + i\gamma u(t,\, z) = 0, \quad (t,\, z) \in (0,\, T) \times (0,\, \infty)$$

which has the solution

$$u_0(t,\, z) = (\psi_0(t,\, z) + i\phi_0(t,\, z)),$$

where, Appendix-A,

$$\psi_0(t,\, z) = e^{-\gamma z}\sum_{n=0}^{\infty} A_n \cos(\Gamma_n z)\sin\left(\frac{n\pi}{T}\right)t,$$

$$\phi_0(t,\, z) = e^{-\gamma z}\sum_{n=0}^{\infty} B_n \cos(\Gamma_n z)\sin\left(\frac{n\pi}{T}\right)t,$$

in which

$$A_n = \frac{2}{T}\int_0^T f_1(t)\sin\left(\frac{n\pi}{T}\right)tdt,$$

$$B_n = \frac{2}{T}\int_0^T f_2(t)\sin\left(\frac{n\pi}{T}\right)tdt,$$

$$\Gamma_n = \left(\frac{n\pi}{T}\right)^2\left[\left(\frac{n\pi}{T}\right)^2\beta^2 - \beta\frac{\partial^2\beta}{\partial t^2}\right] \geq 0.$$

Following Pickard approximation, equation (1) can be rewritten as

$$i\frac{\partial u_n(t,\, z)}{\partial z} + \beta(t)\frac{\partial^2 u_n(t,\, z)}{\partial t^2} + i\gamma u(t,\, z) = -\varepsilon|\, u_{n-1}(t,\, z)\,|^2 u_{n-1}(t,\, z), \quad n \geq 1.$$

At $n = 1$, the iterative equation takes the following form:

$$i\frac{\partial u_1(t,\, z)}{\partial z} + \beta(t)\frac{\partial^2 u_1(t,\, z)}{\partial t^2} + i\gamma u(t,\, z) = -\varepsilon|\, u_0(t,\, z)\,|^2 u_0(t,\, z) = \varepsilon h_1(t,\, z),$$

which can be solved as a linear case with zero initial and boundary conditions. The following general solution can be obtained:

$$\psi_1(t,\, z) = e^{-\gamma z}\sum_{n=0}^{\infty}(T_{0n} + \varepsilon T_{1n})\sin\left(\frac{n\pi}{T}\right)t,$$

$$\phi_1(t, z) = e^{-\gamma z}\sum_{n=0}^{\infty}(\tau_{0n} + \varepsilon\tau_{1n})\sin\left(\frac{n\pi}{T}\right)t,$$

$$u_1(t, z) = (\psi_1(t, z) + i\phi_1(t, z))$$
$$= u_1^{(0)} + \varepsilon u_1^{(1)}.$$

At $n = 2$, the following equation is obtained:

$$i\frac{\partial u_2(t, z)}{\partial z} + \beta(t)\frac{\partial^2 u_2(t, z)}{\partial t^2} + i\gamma u(t, z) = -\varepsilon|u_1(t, z)|^2 u_1(t, z) = \varepsilon h_2(t, z),$$

which can be solved as a linear case with zero initial and boundary conditions. The following general solution can be obtained:

$$u_2(t, z) = u_2^{(0)} + \varepsilon u_2^{(1)} + \varepsilon^2 u_2^{(2)} + \varepsilon^3 u_2^{(3)} + \varepsilon^4 u_2^{(4)}.$$

Continuing like this, one can get

$$u_n(t, z) = u_n^{(0)} + \varepsilon u_n^{(1)} + \varepsilon^2 u_n^{(2)} + \varepsilon^3 u_n^{(3)} + \cdots + \varepsilon^{(n+m)} u_n^{(n+m)}.$$

As $n \to \infty$, the solution (if exists) can be reached as $u(t, z) = \lim_{n\to\infty} u_n(t, z)$.

Accordingly, the solution is a power series in ε. □

According to the previous Lemma, one can assume the solution of equation (1) as the following:

$$u(t, z) = \sum_{n=0}^{\infty}\varepsilon^n u_n. \tag{2}$$

Let $u(t, z) = \psi(t, z) + i\phi(t, z)$, ψ, ϕ: real valued functions. The following coupled equations are got:

$$\frac{\partial\phi(t, z)}{\partial z} = \beta(t)\frac{\partial^2\psi(t, z)}{\partial t^2} + \varepsilon(\psi^2 + \phi^2)\psi - \gamma\phi, \tag{3}$$

$$\frac{\partial\psi(t, z)}{\partial z} = -\beta(t)\frac{\partial^2\phi(t, z)}{\partial t^2} - \varepsilon(\psi^2 + \phi^2)\phi - \gamma\psi, \tag{4}$$

where $\psi(t, 0) = f_1(t)$, $\phi(t, 0) = f_2(t)$, and all corresponding other I.C. and B.C. are zeros.

As a perturbation solution, one can assume that

$$\psi(t, z) = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \cdots, \tag{5}$$

$$\phi(t, z) = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots, \tag{6}$$

where $\psi_0(t, 0) = f_1(t)$, $\phi_0(t, 0) = f_2(t)$, and all corresponding other I.C. and B.C. are zeros.

Substituting from equations (5) and (6) into equations (3) and (4) and then equating the equal powers of ε, one can get the following set of coupled equations:

$$\frac{\partial\phi_0(t, z)}{\partial z} = \beta(t)\frac{\partial^2\psi_0(t, z)}{\partial t^2} - \gamma\phi_0, \tag{7}$$

$$\frac{\partial\psi_0(t, z)}{\partial z} = -\beta(t)\frac{\partial^2\phi_0(t, z)}{\partial t^2} - \gamma\psi_0, \tag{8}$$

$$\frac{\partial\phi_1(t, z)}{\partial z} = \beta(t)\frac{\partial^2\psi_1(t, z)}{\partial t^2} - \gamma\phi_1 + (\psi_0^3 + \psi_0\phi_0^2), \tag{9}$$

$$\frac{\partial\psi_1(t, z)}{\partial z} = -\beta(t)\frac{\partial^2\phi_1(t, z)}{\partial t^2} - \gamma\psi_1 - (\phi_0^3 + \phi_0\psi_0^2), \tag{10}$$

$$\frac{\partial\phi_2(t, z)}{\partial z} = \beta(t)\frac{\partial^2\psi_2(t, z)}{\partial t^2} - \gamma\phi_2 + (3\psi_0^2\psi_1 + 2\psi_0\phi_0\phi_1 + \psi_1\phi_0^2), \tag{11}$$

$$\frac{\partial\psi_2(t, z)}{\partial z} = -\beta(t)\frac{\partial^2\phi_2(t, z)}{\partial t^2} - \gamma\psi_2 - (3\phi_0^2\phi_1 + 2\phi_0\psi_0\psi_1 + \phi_1\psi_0^2), \tag{12}$$

and so on. The prototype equations to be solved are:

$$\frac{\partial\phi_i(t, z)}{\partial z} = \beta(t)\frac{\partial^2\psi_i(t, z)}{\partial t^2} + G_i^{(1)}, \quad i \geq 1, \tag{13}$$

$$\frac{\partial\psi_i(t, z)}{\partial z} = -\beta(t)\frac{\partial^2\phi_i(t, z)}{\partial t^2} + G_i^{(2)}, \quad i \geq 1, \tag{14}$$

where $\psi_i(t, 0) = \delta_{i,0}f_1(t)$, $\phi_i(t, 0) = \delta_{i,0}f_2(t)$ and all other corresponding conditions are zeros. $G_i^{(1)}$, $G_i^{(2)}$ are functions to be computed from previous steps.

Following the solution algorithm described in Appendix-A, the linear coupled equations can always be solved using eigenfunction expansion [8] and variable parameters method [12].

2.1. The zero order approximation

In this case,

$$u^{(0)}(t, z) = (\psi_0 + i\phi_0), \tag{15}$$

where

$$\psi_0(t, z) = \sum_{n=0}^{\infty} C\cos(\Gamma_n z)\sin\left(\frac{n\pi}{T}\right)t, \tag{16}$$

$$\phi_0(t, z) = \sum_{n=0}^{\infty} D\cos(\Gamma_n z)\sin\left(\frac{n\pi}{T}\right)t, \tag{17}$$

in which

$$\Gamma_n = \left(\frac{n\pi}{T}\right)^2\left[\left(\frac{n\pi}{T}\right)^2\beta^2 - \beta\frac{\partial^2\beta}{\partial t^2}\right], \tag{18}$$

$$C = \frac{2}{T}\int_0^T f_1(t)\sin\left(\frac{n\pi}{T}\right)tdt, \tag{19}$$

$$D = \frac{2}{T}\int_0^T f_2(t)\sin\left(\frac{n\pi}{T}\right)tdt. \tag{20}$$

The absolute value of the zero order approximation is

$$|u^{(0)}(t, z)|^2 = \psi_0^2 + \phi_0^2. \tag{21}$$

2.2. The first order approximation

$$u^{(1)}(t, z) = u^{(0)} + \varepsilon(\psi_1 + i\phi_1), \tag{22}$$

where

$$\psi_1(t, z) = \sum_{n=0}^{\infty} T_{1n}(z)\sin\left(\frac{n\pi}{T}\right)t, \tag{23}$$

$$\phi_1(t, z) = \sum_{n=0}^{\infty} \tau_{1n}(z)\sin\left(\frac{n\pi}{T}\right)t, \tag{24}$$

in which

$$T_{1n}(z) = A_{11}(z)\sin(\Gamma_n z) + (C_{12} + B_{11}(z))\cos(\Gamma_n z), \tag{25}$$

$$\tau_{1n}(z) = A_{12}(z)\sin(\Gamma_n z) + (\widetilde{C}_{12} + B_{12}(z))\cos(\Gamma_n z), \tag{26}$$

where the constants and variables $A_{11}(z)$, C_{12}, $B_{11}(z)$, $A_{12}(z)$, $\widetilde{C}_{12}$ and $B_{12}(z)$ can be evaluated in a similar manner as the corresponding ones in the linear case, Appendix-A.

The absolute value of the first order approximation can be got using:

$$|u^{(1)}(t, z)|^2 = |u^{(0)}(t, z)|^2 + 2\varepsilon(\psi_0\psi_1 + \phi_0\phi_1) + \varepsilon^2(\psi_1^2 + \phi_1^2). \tag{27}$$

2.3. The second order approximation

$$u^{(2)}(t, z) = u^{(1)}(t, z) + \varepsilon^2(\psi_2 + i\phi_2), \tag{28}$$

where

$$\psi_2(t, z) = \sum_{n=0}^{\infty} T_{2n}(z)\sin\left(\frac{n\pi}{T}\right)t, \tag{29}$$

$$\phi_2(t, z) = \sum_{n=0}^{\infty} \tau_{2n}(z)\sin\left(\frac{n\pi}{T}\right)t, \tag{30}$$

in which

$$T_{2n}(z) = A_{21}(z)\sin(\Gamma_n z) + (C_{22} + B_{21}(z))\cos(\Gamma_n z), \tag{31}$$

$$\tau_{2n}(z) = A_{22}(z)\sin(\Gamma_n z) + (\widetilde{C}_{22} + B_{22}(z))\cos(\Gamma_n z), \tag{32}$$

where the constants and variables $A_{21}(z)$, C_{22}, $B_{21}(z)$, $A_{22}(z)$, $\widetilde{C}_{22}$ and $B_{22}(z)$ can be evaluated in a similar manner as the previous case.

The absolute value of the second order approximation can be got using:

$$\begin{aligned} &|u^{(2)}(t, z)|^2 \\ &= |u^{(1)}(t, z)|^2 + 2\varepsilon^2(\psi_0\psi_2 + \phi_0\phi_2) + 2\varepsilon^3(\psi_1\psi_2 + \phi_1\phi_2) + \varepsilon^4(\psi_2^2 + \phi_2^2). \end{aligned} \tag{33}$$

3. Case Studies

To examine the proposed solution algorithm, see Appendix-B, two case studies are illustrated.

3.1. Case study-1

Taking $f_1(t) = \rho_1$, $f_2(t) = \rho_2$, $\gamma = 0$, $\beta = 1$ and following the solution algorithm, the following selective results for the first order approximation are got:

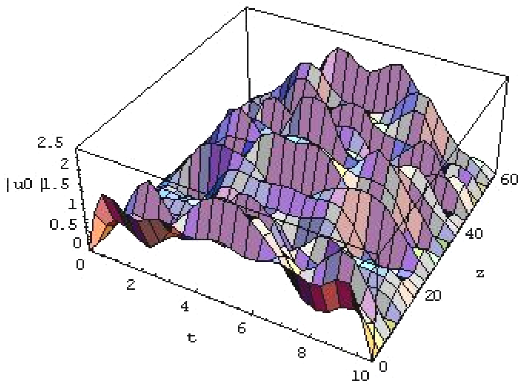

Figure 1. The first order approximation of $|u^{(1)}|$ at $\varepsilon = 0$, $\rho_1, \rho_2 = 1$, $T = 10$, $\gamma = 0$, $\beta = 1$ with considering only ten terms in the series $(M = 10)$.

One can notice that it is an identical result with that in [12, Figure 1].

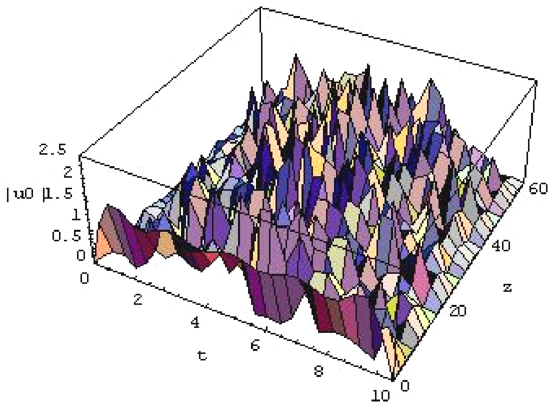

Figure 2. The first order approximation of $|u^{(1)}|$ at $\varepsilon = 0$, $\rho_1, \rho_2 = 1$, $T = 10$, $\gamma = 0$, $\beta = t$ with considering only ten terms in the series $(M = 10)$.

One can notice the increase of oscillation due to the variability of β.

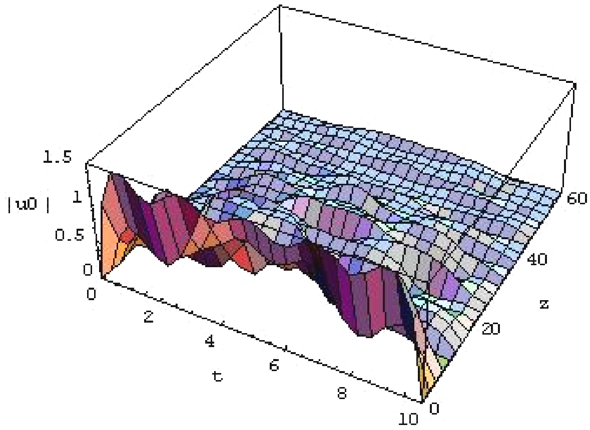

Figure 3. The first order approximation of $|u^{(1)}|$ at $\varepsilon = 0$, $\rho_1, \rho_2 = 1$, $T = 10$, $\gamma = .05$, $\beta = t$ with considering only ten terms in the series ($M = 10$).

One can notice the high regression effect of γ.

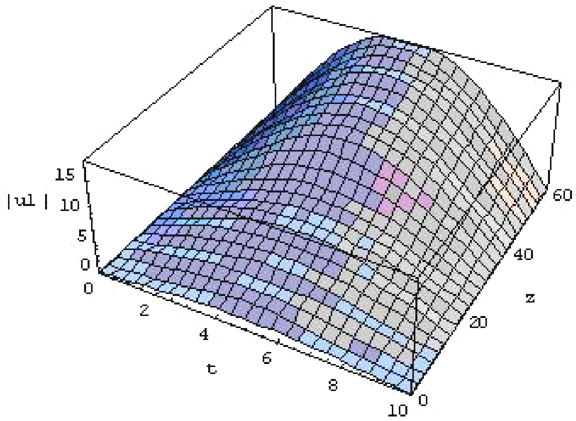

Figure 4. The first order approximation of $|u^{(1)}|$ at $\varepsilon = .2$ and $\rho_1, \rho_2 = 1$, $T = 10$, $\gamma = 0$, $\beta = 1$ with considering only one term in the series ($M = 1$).

One can notice that the results are identical with the case $\gamma = 0$ in [12, Figure 2].

3.2. Case study-2

Taking the case of $f_1(t) = \rho_1, \; f_2(t) = \rho_2 \sin\left(\frac{m\pi}{T}\right)t,$ the following final results for the first order approximation are obtained:

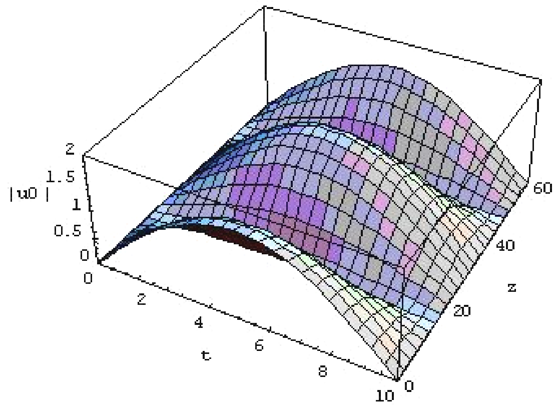

Figure 5. The first order approximation of $|u^{(1)}|$ at $\varepsilon = 0, \; \rho_1, \rho_2 = 1, \; T = 10, \; \gamma = 0, \; \beta = 1$ with considering only one term in the series $(M = 1)$.

One can notice that the results are identical with the case $\gamma = 0$ in [12, Figure 6].

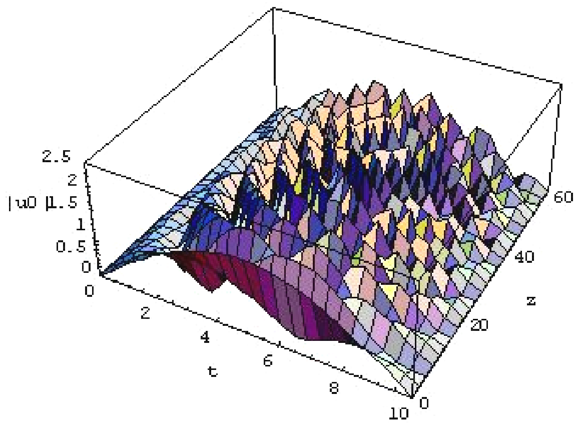

Figure 6. The first order approximation of $|u^{(1)}|$ at $\varepsilon = 0, \; \rho_1, \rho_2 = 1, \; T = 10, \; \gamma = 0, \; \beta = t$ with considering only one term in the series $(M = 1)$.

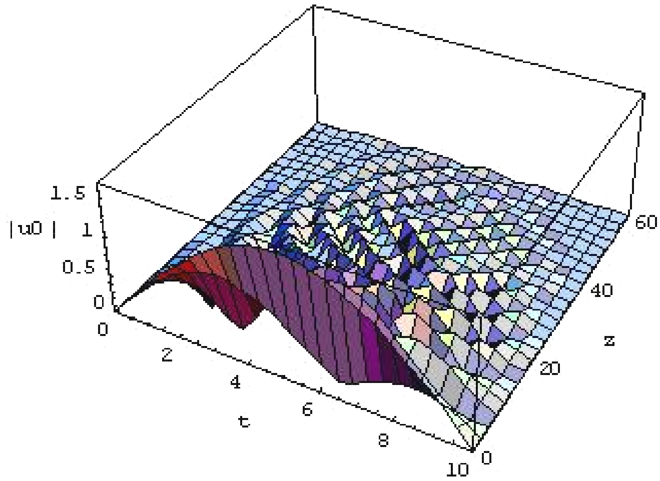

Figure 7. The first order approximation of $|u^{(1)}|$ at $\varepsilon = 0$, $\rho_1, \rho_2 = 1$, $T = 10$, $\gamma = .05$, $\beta = t$ with considering only ten terms in the series $(M = 10)$.

One can notice the high regression effect of γ.

Conclusions

The perturbation technique together with the eigenfunction expansion and variation of parameters method introduce an approximate solution to the cubic NLS equation under variable group velocity dissipation for a finite time interval. The difficult and huge first order computations were achieved using Mathematica-5. To get more improved orders, it is expected to face a problem of computation. In general, the solution oscillations are increased with the variability of β. The increase of γ causes a high regression in the solution level.

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Appendix-A. The Homogeneous Linear Case

Consider the nonhomogeneous linear Schrödinger equation:

$$i\frac{\partial u(t, z)}{\partial z} + \beta(t)\frac{\partial^2 u(t, z)}{\partial t^2} + i\gamma u(t, z) = 0,\ (t, z) \in (0, T) \times (0, \infty), \quad \text{(A-1)}$$

where $u(t, z)$ is a complex valued function which is subjected to:

$$\text{I.C.} : u(t, 0) = f_1(t) + if_2(t), \text{ a complex valued function,} \quad \text{(A-2)}$$

$$\text{B.C.} : u_z(0, z) = 0,\ u_z(T, z) = 0. \quad \text{(A-3)}$$

Let $u(t, z) = \psi(t, z) + i\phi(t, z)$, ψ, ϕ: real valued functions. The following coupled equations are got as follows:

$$\frac{\partial \phi(t, z)}{\partial z} = \beta(t)\frac{\partial^2 \psi(t, z)}{\partial t^2} - \gamma\phi, \quad \text{(A-4)}$$

$$\frac{\partial \psi(t, z)}{\partial z} = -\beta(t)\frac{\partial^2 \phi(t, z)}{\partial t^2} - \gamma\psi, \quad \text{(A-5)}$$

where $\psi(t, 0) = f_1(t)$, $\phi(t, 0) = f_2(t)$, and all corresponding other I.C. and B.C. are zeros.

Let the loss terms in equations (4, 5), $\psi = e^{-\gamma z}w$, $\phi = e^{-\gamma z}\nu$, be eliminated.

Eliminating one of the variables in the resultant equations, one can get the following independent equations:

$$\frac{\partial^2 w(t, z)}{\partial z^2} + \beta^2\frac{\partial^4 w(t, z)}{\partial t^4} + \beta\beta''\frac{\partial^2 w(t, z)}{\partial t^2} = 0, \quad \text{(A-6)}$$

$$\frac{\partial^2 w(t, z)}{\partial z^2} + \beta^2\frac{\partial^4 w(t, z)}{\partial t^4} + \beta\beta''\frac{\partial^2 w(t, z)}{\partial t^2} = 0. \quad \text{(A-7)}$$

Using the eigenfunction expansion technique [8], the following solution expressions are obtained:

$$\psi(t, z) = e^{-\gamma z}\sum_{n=0}^{\infty} T_n(t, z)\sin\left(\frac{n\pi}{T}\right)t, \quad \text{(A-8)}$$

$$\phi(t, z) = e^{-\gamma z} \sum_{n=0}^{\infty} \tau_n(t, z) \sin\left(\frac{n\pi}{T}\right) t, \tag{A-9}$$

where $T_n(t, z)$ and $\tau_n(t, z)$ can be got through the applications of initial conditions and then solving the resultant second order differential equations. The final expressions can be got as the following:

$$T_n(z) = C_1 \cos(\sqrt{\Gamma_n}\, z), \tag{A-10}$$

$$\tau_n(z) = C_2 \cos(\sqrt{\Gamma_n}\, z), \tag{A-11}$$

where

$$\Gamma_n = \left(\frac{n\pi}{T}\right)^2 \left[\left(\frac{n\pi}{T}\right)^2 \beta^2 - \beta \frac{\partial^2 \beta}{\partial t^2}\right] \geq 0, \tag{A-12}$$

$$C_1 = \frac{2}{T} \int_0^T f_1(t) \sin\left(\frac{n\pi}{T} t\right) dt, \tag{A-13}$$

$$C_2 = \frac{2}{T} \int_0^T f_2(t) \sin\left(\frac{n\pi}{T} t\right) dt. \tag{A-14}$$

Appendix-B. The Non-homogeneous Linear Case

Consider the non-homogeneous couple equations;

$$\frac{\partial \phi(t, z)}{\partial z} = \beta(t) \frac{\partial^2 \psi(t, z)}{\partial t^2} - \gamma \phi + g_1, \tag{B-1}$$

$$\frac{\partial \psi(t, z)}{\partial z} = -\beta(t) \frac{\partial^2 \phi(t, z)}{\partial t^2} - \gamma \psi + g_2, \tag{B-2}$$

where $\psi(t, 0) = 0$, $\phi(t, 0) = 0$, and all corresponding other I.C. and B.C. are also zeros.

Let the loss terms in equations (B-1, B-2), $\psi = e^{-\gamma z} w$, $\phi = e^{-\gamma z} \nu$, be eliminated.

Eliminating one of the variables in the resultant equations, one can get the following independent equations:

$$\frac{\partial^2 w(t, z)}{\partial z^2} + \beta^2 \frac{\partial^4 w(t, z)}{\partial t^4} + \beta\beta'' \frac{\partial^2 w(t, z)}{\partial t^2} = \widetilde{\psi}_1, \tag{B-3}$$

$$\frac{\partial^2 w(t, z)}{\partial z^2} + \beta^2 \frac{\partial^4 w(t, z)}{\partial t^4} + \beta\beta'' \frac{\partial^2 w(t, z)}{\partial t^2} = \widetilde{\psi}_2, \tag{B-4}$$

where

$$\widetilde{\psi}_1(t, z) = \frac{\partial G_2}{\partial z} - \beta \frac{\partial^2 G_1}{\partial t^2}, \tag{B-5}$$

$$\widetilde{\psi}_2(t, z) = \beta \frac{\partial G_2}{\partial t^2} + \frac{\partial G_1}{\partial z}, \tag{B-6}$$

in which

$$G_1 = e^{\gamma z} g_1, \; G_2 = e^{\gamma z} g_2.$$

Using the eigenfunction expansion technique [8], the following solution expressions are obtained:

$$\psi(t, z) = e^{-\gamma z} \sum_{n=0}^{\infty} T_n(t, z) \sin\left(\frac{n\pi}{T}\right) t, \tag{B-7}$$

$$\phi(t, z) = e^{-\gamma z} \sum_{n=0}^{\infty} \tau_n(t, z) \sin\left(\frac{n\pi}{T}\right) t, \tag{B-8}$$

where $T_n(t, z)$ and $\tau_n(t, z)$ can be got through the applications of initial conditions and then solving the resultant second order differential equations using the method of the variational parameter [12]. The final expressions can be got as the following:

$$T_n(z) = A_1(z) \sin(\sqrt{\Gamma_n}\, z) + (C_2 + B_1(z)) \cos(\sqrt{\Gamma_n}\, z), \tag{B-9}$$

$$\tau_n(z) = A_2(z) \sin(\sqrt{\Gamma_n}\, z) + (C_3 + B_2(z)) \cos(\sqrt{\Gamma_n}\, z), \tag{B-10}$$

where

$$\Gamma_n = \left(\frac{n\pi}{T}\right)^2 \left[\left(\frac{n\pi}{T}\right)^2 \beta^2 - \beta \frac{\partial^2 \beta}{\partial t^2}\right] \geq 0, \tag{B-11}$$

$$A_1(z) = \frac{1}{\sqrt{\Gamma_n}} \int \tilde{\psi}_{1n}(z;\ n)\cos(\sqrt{\Gamma_n}\, z)dz, \tag{B-12}$$

$$B_1(z) = \frac{-1}{\sqrt{\Gamma_n}} \int \tilde{\psi}_{1n}(z;\ n)\sin(\sqrt{\Gamma_n}\, z)dz, \tag{B-13}$$

$$A_2(z) = \frac{1}{\sqrt{\Gamma_n}} \int \tilde{\psi}_{2n}(z;\ n)\cos(\sqrt{\Gamma_n}\, z)dz, \tag{B-14}$$

$$B_2(z) = \frac{-1}{\sqrt{\Gamma_n}} \int \tilde{\psi}_{2n}(z;\ n)\sin(\sqrt{\Gamma_n}\, z)dz, \tag{B-15}$$

in which

$$\tilde{\psi}_{1n}(z;\ n) = \frac{2}{T} \int_0^T \tilde{\psi}_1(t,\ z)\sin\left(\frac{n\pi}{T}\right)dt, \tag{B-16}$$

$$\tilde{\psi}_{2n}(z;\ n) = \frac{2}{T} \int_0^T \tilde{\psi}_2(t,\ z)\sin\left(\frac{n\pi}{T}\right)dt. \tag{B-17}$$

The following conditions should also be satisfied:

$$C_2 = -B_1(0), \tag{B-18}$$

$$C_3 = -B_2(0). \tag{B-19}$$