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EXACT TRAVELLING WAVE SOLUTIONS FOR CATTANEO'S REACTION DIFFUSION FOR PRISONER'S DILEMMA GAME

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Abstract

In this work, we use factorization method to find explicit exact particular travelling wave solutions for Cattaneo's reaction diffusion for Prisoner's dilemma (PD) game. Using the particular solutions for these equations we find the two-parameter solutions for the equation.

1. Introduction

It is known now that diffusion in many branches of sciences is better modelled by Cattaneo's equation (Telegraph reaction diffusion equation) where memory effects are included [1-4, 6]. E. Ahmed and S. Z. Hassan,

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have found an implicit travelling wave solutions for HD and PD Cattaneo's reaction diffusion equations by transforming the equations into first order equations and use a special choice for the wave speed.

Recently, factorization of second-order linear differential equations became a well established technique to find solutions in an algebraic manner [5, 7, 8]. Rosu and Cornejo find one particular solution once the nonlinear equation is factorized with the use of two first order differential operators [5, 8]. They use the method for equations of types:

$$u'' + \gamma u' + f(u) = 0, \tag{1}$$

where γ is a constant and

$$u'' + g(u)u' + f(u) = 0, \tag{2}$$

where $'$ means the derivative $D = \frac{d}{dz}$, $g(u)$, $f(u)$ are polynomials in u.

Now, equation (2) can be factorized as

$$[D - \varphi_2(u)][D - \varphi_1(u)]u = 0, \tag{3}$$

which leads to the equation

$$u'' - \frac{d\varphi_1}{du} uu' - \varphi_1 u' - \varphi_2 u' + \varphi_1\varphi_2 u = 0, \tag{4}$$

or

$$u'' - \left(\varphi_1 + \varphi_2 + \frac{d\varphi_1}{du} u\right)u' + \varphi_1\varphi_2 u = 0. \tag{5}$$

Comparing (5) and (2) we find

$$g(u) = -\left(\varphi_1 + \varphi_2 + \frac{d\varphi_1}{du} u\right) \text{ and } f(u) = \varphi_1\varphi_2 u. \tag{6}$$

If equation (2) can be factorized as in equation (3), then a first particular solution can be easily found by solving

$$[D - \varphi_1(u)]u = 0. \tag{7}$$

2. Explicit Exact Travelling Wave Solutions for Prisoner's Dilemma Game

Prisoner's dilemma is a 2×2 symmetric game in which two possible strategies cooperate (A) or defect (B), with payoff matrix $\begin{bmatrix} & A & B \\ A & R & S \\ B & T & U \end{bmatrix}$.

Cattaneo's reaction diffusion equation for Prisoner's dilemma game is given as [4]:

$$\tau \frac{\partial^2 P}{\partial t^2} + \left(1 - \tau \frac{df}{dP}\right) \frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2} + f(P), \quad f(P) = -P(1-P)(\gamma - \alpha P), \tag{8}$$

where $\alpha = R + U - S - T$, $\gamma = U - S$ with $T > R > U > S$, $2R > T + S$.

Using the coordinate transformation $z = x - ct$ (c is the propagation speed) in equation (8) we obtain the following nonlinear ordinary differential equation

$$P'' + g(P)P' + F(P) = 0, \tag{9}$$

where

$$g(P) = \frac{c(1-\tau\gamma)}{(\tau c^2 - 1)} - \frac{2\tau c(\alpha + \gamma)}{(1 - \tau c^2)} P + \frac{3\alpha}{(1 - \tau c^2)} P^2,$$

$$F(P) = \frac{1}{(1 - \tau c^2)} P(1-P)(\alpha P - \gamma), \quad 1 > \tau c^2.$$

Using operator notation, equation (9) takes the form

$$\left[D^2 + g(P)D + \frac{F(P)}{P}\right] P = 0. \tag{10}$$

The factorization of (10) leads to

$$[D - \Psi_2(P)][D - \Psi_1(P)]P = 0, \tag{11}$$

and then

$$P'' - \left[\Psi_2 + \Psi_1 + \frac{d\Psi_1}{dP} P\right] P' + \Psi_2 \Psi_1 P = 0. \tag{12}$$

Comparing (12) and (9) we obtain the conditions on Ψ_1 and Ψ_2 as:

$$-\left[\Psi_2 + \Psi_1 + \frac{d\Psi_1}{dP}P\right] = \frac{c(1-\tau\gamma)}{(1-\tau c^2)} - \frac{2\tau c(\alpha+\gamma)}{(1-\tau c^2)}P + \frac{3\alpha}{(1-\tau c^2)}P^2, \tag{13}$$

$$\Psi_2\Psi_1 = \frac{1}{(1-\tau c^2)}(1-P)(\alpha P - \gamma). \tag{14}$$

Now, we choose Ψ_1 and Ψ_2 such that

$$\Psi_1(P) = \frac{a}{\sqrt{(1-\tau c^2)}}(1-P), \quad \Psi_2(P) = \frac{1}{a\sqrt{(1-\tau c^2)}}(\alpha P - \gamma), \quad a \neq 0. \tag{15}$$

From (15) and (13), we get

$$\sqrt{(1-\tau c^2)}a^2 + c(1-\tau\gamma)a + \gamma\sqrt{(1-\tau c^2)} = 0 \tag{16}$$

which implies that

$$a = \frac{-c(1-\tau\gamma) \pm \sqrt{c^2(1-\tau\gamma)^2 - 4\gamma(1-\tau c^2)}}{2\sqrt{(1-\tau c^2)}}, \tag{17}$$

and

$$\left[D - \frac{1}{a\sqrt{(1-\tau c^2)}}(\alpha P - \gamma)\right]\left[D - \frac{a}{\sqrt{(1-\tau c^2)}}(1-P)\right]P = 0, \tag{18}$$

and the compatible first order differential equation is

$$P' - \delta(1-P)P = 0, \quad \delta = \frac{a}{\sqrt{(1-\tau c^2)}}. \tag{19}$$

By direct integration we get

$$P_1(z) = \frac{1}{1 \pm \exp[\delta(z-z_0)]}. \tag{20}$$

Now if we choose Ψ_1 and Ψ_2 such that

$$\Psi_1(P) = \frac{r}{\sqrt{(1-\tau c^2)}}(\alpha P - \gamma), \quad \Psi_2(P) = \frac{1}{r\sqrt{(1-\tau c^2)}}(1-P), \quad r \neq 0, \tag{21}$$

then we can find that

$$r = \frac{-c(1-\tau\gamma) + \sqrt{c^2(1-\tau\gamma)^2 + 4\gamma(1-\tau c^2)}}{2\gamma\sqrt{(1-\tau c^2)}} \tag{22}$$

and

$$\left[D - \frac{1}{r\sqrt{(1-\tau c^2)}}(1-P)\right]\left[D - \frac{r}{\sqrt{(1-\tau c^2)}}(\alpha P - \gamma)\right]P = 0, \tag{23}$$

the compatible first order differential equation is

$$P' - \eta(\alpha P - \gamma)P = 0, \quad \eta = \frac{r}{\sqrt{(1-\tau c^2)}}, \tag{24}$$

then

$$P_1(z) = \frac{\alpha}{\gamma \pm \exp[\eta(z - z_0)]}. \tag{25}$$

3. Conclusions

In this paper, the efficient factorization method, proposed by H. C. Rosu and O. Cornejo-Péerez [5, 8], has been applied to Cattaneo's Reaction Diffusion for Hawk-Dove Games. Exact particular solutions have been obtained.

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