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EXISTENCE THEOREMS AND NECESSARY CONDITIONS FOR GENERAL FORMULATION OF LINEAR BOUNDED PHASE CO-ORDINATE CONTROL PROBLEMS

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Abstract

In this paper a generalized bounded phase co-ordinate control problem (involving certain unbounded operators) is considered by functional analytic approach. Existence and uniqueness of the solution is given in terms of certain closed linear operators. The optimal solutions are characterised in terms of adjoint operators. These results are applicable to linear minimum effort problems, constrained variational problems, optimal control of distributive systems and certain illposed variational problems.

1. Introduction

Let X, Y, Z be real normed linear spaces and $T : X \to Z$, $S : X \to Y$ be linear transformations. Let $\Omega \subseteq X$ be a convex set containing zero element θ, and J be a real valued convex continuous functional defined on X such that

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(1) $J(x) \geq 0, \forall x \in X$,

(2) $J(\theta) = 0$,

(3) $J(x) \to +\infty$ as $\| x \| \to \infty$.

Problem (P)

With $\xi \in Y$ and $\eta \in T[\text{core}_S \Omega \cap D(T)]$, find an element, called an *optimal solution* (if one exists) on the set

$$\{x \in [\Omega \cap D(T) \cap D(S)] : \eta = Tu, \text{ and } x \in B(\xi, \varepsilon)\}$$

which minimizes $J(u)$, where $B(\xi, \varepsilon) = \{x \in X : \| \xi - Sx \| \leq \varepsilon\}$.

The important particular cases of the above problem are the following:

Problem (P_1).

$\min_{u \in X} \| u \|$ subject to $\eta = Tu$, $\| \xi - Su \| \leq \varepsilon$ with T onto and S into.

Problem (P_2).

$$\min_{\| u \| \leq \rho} \| \eta - Tu \| \text{ subject to } \| \xi - Su \| \leq \varepsilon \ (0 < \rho < +\infty).$$

Our main objective is to discuss the existence and uniqueness of the solution of the problems through functional analytic approach.

Minimum effort control problem was considered by various authors in [1, 8, 9]. It was further generalized by Minamide and Nakamura [5] in an abstract Banach space which includes [8], [9] as special cases, using tools of functional analysis. Linear bounded phase co-ordinate control problems were considered by Minamide and Nakamura [6].

The principal objective of the present paper is to obtain the existence theorems for a generalized bounded phase co-ordinate control problem involving certain unbounded linear operators. Many of the established results may be obtained as a particular case.

Modern control problems and classical variational problems include very often differential operators, so it may be worthwhile to consider problems involving unbounded operators.

2. Some Preliminaries

Here we shall use some terminologies that agree with Minamide and Nakamura [5], and Burns [1]. Throughout the paper we shall assume, without any loss of generality, that the scalars are always real numbers. The interior, boundary and closure of a set K will be denoted respectively by $\operatorname{int} K$, ∂K and $\operatorname{cl} K$. Let X and Z be normed linear spaces and $T : D(T) \to Z$ be a linear operator with $D(T) \subseteq X$. A sequence $\{x_n\}$ in $D(T)$ is called a *semi-T convergent* if there is some element $z \in Z$ with $Tx_n \to z$. A sequence $\{x_n\}$ in $D(T)$ is called *T convergent* if there are $x \in X$ and $z \in Z$ with $x_n \to x$ and $Tx_n \to z$. The linear operator T is said to be *co-compact* (*weakly co-compact*) if each bounded semi-T convergent sequence $\{x_n\}$ with $Tx_n \to z$ has a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to x$ ($z_{n_k} \to x$ weakly), $x \in D(T)$ and $Tx = z$.

If K is convex set in a normed linear space X, then a point $x_0 \in K$ is said to be a *core point* of K, if for each $x \in X$ with $x \neq x_0$ there is an r satisfying $0 < r < 1$ such that if $0 \leq \lambda \leq r$, then $\lambda x + (1 - \lambda)x_0 \in K$. The set of all core points of K is called *core* of K and will be denoted by core K. The set of all core points of $K \cap D(T)$ relative to $D(T)$ will be denoted by $\operatorname{core}_T K$.

A point $\xi \in \operatorname{cl} K$ is said to be a *support point* of K if there exists a nonzero linear functional ϕ, not necessarily continuous, such that ϕ satisfies

$$\langle \xi, \phi \rangle = \sup\{\langle f, \phi \rangle : f \in K\}. \tag{1}$$

The set of all support points of K will be denoted by $\operatorname{supp} K$. A nonzero linear functional ϕ satisfying (1) is called a *support functional* of K at point $\xi \in \operatorname{cl} K$. If ξ is support point of K and there exists a nonzero continuous linear functional x' satisfying (1), then ξ will be called a *strong support point*. The set of all strong support points of K will be denoted by $\operatorname{Supp} *K$.

Let $\alpha \in R$, where R denotes the set of real numbers and let $J(\alpha) = \{(x, y) \in X \times Y : J(x, y) \leq \alpha\}$. Suppose $D(T) \subseteq X$ is dense in X. Let $D(T')$ be the set of all $h' \in Y'$ such that there exists an element $f' \in X'$ satisfying $\langle Tf, h' \rangle = \langle f, f' \rangle$, $\forall f \in D(T)$. Since $D(T)$ is dense in X, the element f' is determined uniquely by h' and we define $T' : D(T') \subset Z'$ by $T'h' = f'$. The operator T' is called the *adjoint* of T.

Consequently, $\langle Tf, h' \rangle = \langle f, T'h' \rangle$ for all $f \in D(T)$ and $T' \in D(T')$.

If $K \subseteq X$ is a convex set and $f_0 \in \text{Supp} *K$, then the set of all continuous support functionals corresponding to f_0 will be denoted by $E(K, f_0')$. If f_0' is a nonzero continuous support functional to K, then $E(K, f_0')$ will denote the set of $f_0 \in \text{cl}K$ such that f_0 satisfies the condition $\langle f_0, f_0' \rangle = \sup\{\langle f, f_0' \rangle : f \in K\}$. Evidently, $E(K, f_0') = [f_0' : K]$ (see [1], [5]).

We now state the following hypothesis which will be used throughout the paper $(H_0) : \theta \in \text{core}_S(\Omega)$, $D(T) \subseteq D(S)$ and T is an onto mapping.

3. Solution of the Problem (P)

Let $\hat{S}$ be a linear mapping of X into $Y \times Z$ defined by $\hat{S} : u \to (Su, Tu)$, where $Y \times Z$ denotes a product space with the usual product topology. Let $\hat{S}(\Omega)$ denote the image of Ω under $\hat{S}$.

Consider the set $C_\varepsilon(\alpha, 0) = \{\hat{S}(J(\alpha) \cap (\Omega \cap D(T) \cap D(S)) + (\varepsilon U_Y \times \{0\}))\}$ for $\alpha > 0$, U_Y, being the unit sphere in Y.

Let us introduce the following definition:

Definition. A pair (ξ, η) will be called *regular* if there exists at least one element $u \in [J(\alpha) \cap (\Omega \cap D(T) \cap D(S))]$ satisfying the constraint $\eta = Tu$, and $u \in B^0(\xi, \eta)$, where $B^0(\xi, \eta) = \{u : \| \xi - Su \| < \varepsilon\}$.

Lemma 3.1. *If* (H_0) *holds and* $\alpha > 0$ *is given and assume that* Ω *is closed convex set with* $\theta \in \text{Int}(\Omega)$, J *is continuous*, $Y \times Z$ *is of the second* *category as a subset of itself and* $\hat{S}$ *is weakly co-compact, then* $C_\varepsilon(\alpha, 0)$ *is a closed convex body.*

Proof. Evidently, $\theta \in \operatorname{Int} C_\varepsilon(\alpha, 0)$ (by the construction and the nature of S and T). Thus $C_\varepsilon(\alpha, 0)$ is a convex body. Again as $\hat{S}$ is weakly co-compact, hence $C_\varepsilon(\alpha, 0)$ is a closed convex body [1, Theorems 2.5, 2.6].

Lemma 3.2. *Let* (H_0) *be satisfied. Suppose that* $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$ *is a regular pair. Then any hyperplane* $(\phi_1, \phi_2)(\neq 0) \in (Y \times Z)'$ *of support of* $C_\varepsilon(\alpha, 0)$ *at* (ξ, η) *satisfies*

(i) $\langle\langle(\xi, \eta), (\phi_1, \phi_2)\rangle\rangle \geq \langle C_\varepsilon(\alpha, 0)(S'\phi_1 + T'\phi_2; \phi_1)\rangle$, *where* S', T' *denote the conjugate of* S, T, *respectively.*

(ii) $S'\phi_1 + T'\phi_2 \neq 0$.

Proof. Since $C_\varepsilon(\alpha, 0)$ is a convex body, there exists a supporting hyperplane $(\phi_1, \phi_2)(\neq 0) \in (Y \times Z)'$ such that

$$\langle\langle(\xi, \eta)(\phi_1, \phi_2)\rangle\rangle \geq \langle\langle(Su, Tu) + \varepsilon(y, 0), (\phi_1, \phi_2)\rangle\rangle,$$

$$\forall u \in [J(\alpha) \cap (\Omega \cap D(S) \cap (T))],\ y \in U_Y$$

$$= \langle\langle(Su + \varepsilon y, Tu), (\phi_1, \phi_2)\rangle\rangle = \langle\langle(u, \varepsilon y), (S'\phi_1 + T'\phi_2, \phi_1)\rangle\rangle.$$

Hence taking supremum of the R.H.S. $\forall u \in \Omega$, $y \in U_Y$, we get

$$\langle\langle(\xi, \eta), (\phi_1, \phi_2)\rangle\rangle \geq \langle C_\varepsilon(\alpha, 0)\,(\phi_1, \phi_2)\rangle.$$

(ii) Let us first note that the hyperplane (ϕ_1, ϕ_2) with $\phi_1 \neq 0$ supports $C_\varepsilon(\alpha, 0)$ at (ξ, η). Then $\langle\langle(\xi, \eta)\,(\phi_1, \phi_2)\rangle\rangle \geq \langle C_\varepsilon(\alpha, 0)\,(\phi_1, \phi_2)\rangle$.

Suppose $\phi_1 = 0$. Then $\phi_2 \neq 0$ and we have

$$\langle \eta, \phi_2 \rangle \geq \langle \varepsilon U_Y, \phi_1 \rangle \tag{A}$$

which contradicts the assumption that $\eta \in T[\operatorname{core}_S \Omega \cap D(T)]$.

Next we shall show that $S'\phi_1 + T'\phi_2 \neq 0$.

If possible, let $S'\phi_1 + T'\phi_2 \neq 0$. Then for all $u \in T^{-1}(\eta)$, we have

$$\begin{aligned}\langle \xi, \phi_1 \rangle + \langle \eta, \phi_2 \rangle &= \langle \xi - Su + Su, \phi_1 \rangle + \langle Tu, \phi_2 \rangle \\ &= \langle \xi - Su, \phi_1 \rangle \geq \langle \varepsilon U_Y, \phi_1 \rangle \quad \text{(by A)}\end{aligned}$$

which contradicts the fact that $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$ is a regular pair.

Hence $S'\phi_1 + T'\phi_2 \neq 0$.

Lemma 3.3. *Let* $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$ *be a regular pair. Then for all* $u \in \Omega[J(\alpha) \cap (\Omega \cap D(S) \cap D(T))]$ *satisfying* $\eta = Tu$ *and* $u \in B(\xi, \varepsilon)$, *we have* $J(u) \geq \alpha$.

Proof. Let $u_0 \in \Omega \cap D(T)$, $J(u_0) < \alpha (\neq 0)$.

Since $\eta \in T[\mathrm{Core}_S \Omega \cap D(T)]$, there exists an element $\overline{u} \in \mathrm{Int}(\Omega)$ such that $T\overline{u} = \eta$.

Put $u_\lambda = \lambda \overline{u} + (1 - \lambda) u_0$. Then for sufficiently small $\lambda > 0$, $u_\lambda \in \mathrm{Int}(\Omega)$ satisfying $Tu_\lambda = \eta$, $u_\lambda \in B(\xi, \varepsilon)$ and $J(u_\lambda) < \alpha$.

Since J is continuous, a neighbourhood U of the origin θ in X exists, such that $\{u_\lambda + U\} \subset J(\alpha) \cap (\Omega \cap D(T))$.

Applying $\hat{S}$ (which is an open mapping), one can show that $(\xi, \eta) \in \mathrm{Int} C_\varepsilon(\alpha, 0)$, which contradicts the hypothesis.

Lemma 3.4. *Suppose that* (ξ, η) *is regular and* $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$. *Then for all* $\hat{\alpha} > \alpha$, *we have* $(\xi, \eta) \in \mathrm{Int}\{C_\varepsilon(\hat{\alpha} > 0)\} \subset C_\varepsilon(\hat{\alpha}, 0)$.

Proof. Since $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$, we have

$$\begin{aligned}\langle (\xi, \eta), (\psi_1, \psi_2) \rangle \leq &\langle J(\alpha) \cap (\Omega \cap D(T)), (S'\psi_1 + T'\psi_2) \rangle \\ &+ \varepsilon \langle U_Y, \psi_1 \rangle, \quad \forall (\psi_1, \psi_2) \in (Y \times Z)'. \qquad \text{(B)}\end{aligned}$$

If possible, let there exists an $\alpha' > \alpha$ such that $(\xi, \eta) \notin \mathrm{Int}\{C_\varepsilon(\hat{\alpha}, 0)\}$. Then a separating hyperplane $(\phi_1, \phi_2)(\neq 0) \in (Y \times Z)'$ exists such that

$$\langle (\xi, \eta), (\phi_1, \phi_2) \rangle \geq \langle J(\alpha) \cap (\Omega \cap D(T)), (S'\phi_1 + T'\phi_2) \rangle + \varepsilon \langle U_Y, \phi_1 \rangle,$$

which contradicts (B) by Lemma 3.2.

So considering Lemmas 3.3 and 3.4 one can conclude the following theorem:

Theorem 3.1. *Problem* (P) *has a solution for each regular pair* $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$ *iff* $(\xi, \eta) \in C_\varepsilon(\alpha, 0)$.

In view of Theorem 3.1 one can conclude that the existence of solution of Problem (P) depends on whether $C_\varepsilon(\alpha, 0)$ is closed or not in $Y \times Z$.

But Lemma 3.1 gives some sufficient condition for the existence of the solution of Problem (P).

Lemma 3.5. *A pair* (ξ, η) *is regular iff* $\langle(\xi, \eta), (\phi_1, \phi_2)\rangle < \langle \varepsilon U_Y, \phi_1\rangle$ *holds for all* $(\phi_1, \phi_2)(\neq 0) \in (Y \times Z)'$ *satisfying* $S'\phi_1 + T'\phi_2 = 0$.

Proof. Let (ξ, η) be a regular pair. Then there exists an element $u \in J(\alpha) \cap (\Omega \cap D(T))$ such that $u \in B^0(\xi, \varepsilon)$ and $\eta = Tu$.

Consequently, from Lemma 3.4, it follows for $\alpha > J(u)$, $(\xi, \eta) \in \mathrm{Int}\{C_\varepsilon(\alpha, 0)\}$.

Hence for all $(\psi_1, \psi_2)(\neq 0) \in (Y \times Z)'$, we have

$$\langle(\xi, \eta), (\psi_1, \psi_2)\rangle < \langle J(\alpha) \cap (\Omega \cap D(T)), (S'\psi_1 + T'\psi_2)\rangle + \langle \varepsilon U_Y, \psi_1\rangle.$$

Hence $\langle(\xi, \eta), (\psi_1, \psi_2)\rangle < \langle \varepsilon U_Y, \psi_1\rangle$ $(\because S'\psi_1 + T'\psi_2 = 0)$.

Conversely, suppose that (ξ, η) is not a regular pair, i.e., for all u, $\eta = Tu$ but $u \notin B^0(\xi, \varepsilon)$.

Consequently, $(\xi, \eta) \notin \mathrm{Int}\{C_\varepsilon(\alpha, 0)\}$.

Hence there exists a separating hyperplane $(\phi_1, \phi_2) \in (Y \times Z)'$ such that

$$\langle(\xi, \eta), (\phi_1, \phi_2)\rangle \geq \langle J(\alpha) \cap (\Omega \cap D(T)), (S'\phi_1 + T'\phi_2)\rangle + \langle \varepsilon U_Y, \phi_1\rangle, \ \forall u \in X$$

which, in turn, implies $S'\phi_1 + T'\phi_2 = 0$ and $\langle(\xi, \eta), (\phi_1, \phi_2)\rangle \geq \langle \varepsilon U_Y, \phi_1\rangle$.

But this contradicts the hypothesis.

Theorem 3.2. *Suppose that* (ξ, η) *is a regular pair and condition of Lemma* 3.1 *holds. Then an optimal solution of Problem* (P) *exists and is of the form*:

$$u_0 = [S'\phi_1 + T'\phi_2 : J(\alpha_0) \cap (\Omega \cap D(T))], \tag{1}$$

where α_0 *and* (ϕ_1, ϕ_2) *with norm* 1 *satisfy any of the following conditions*:

$$\xi = [S'\phi_1 + T'\phi_2 : J(\alpha_0) \cap (\Omega \cap D(T))] + [\phi_1 : \varepsilon U_Y] \tag{2}$$

$$\eta = T[S'\phi_1 + T'\phi_2 : J(\alpha_0) \cap (\Omega \cap D(T))]$$

$$\max_{S'\phi_1 + T'\phi_2 \neq 0} \frac{\langle(\xi, \eta), (\phi_1, \phi_2)\rangle - \langle \varepsilon U_Y, \phi_1 \rangle}{\langle J(\alpha_0) \cap (\Omega \cap D(T)), S'\phi_1 + T'\phi_2 \rangle} = \alpha_0. \tag{3}$$

Conversely, if (ϕ_1, ϕ_2) *having norm* 1 *solves either of the above conditions, then* $u_0 \in [S'\phi_1 + T'\phi_2 : J(\alpha_0) \cap (\Omega \cap D(T))]$ *is optimal.*

Furthermore if X is rotund, then the solution is unique.

Proof. Let $u_0 (\neq 0)$ be an optimal solution. Then we shall show (1)-(3).

Evidently, $u_0 \in B(\xi, \varepsilon)$ and $\eta = Tu_0$, and $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$, where we put $J(u_0) = \alpha_0$. Let (ϕ_1, ϕ_2) be a hyperplane of support of $C_\varepsilon(\alpha, 0)$ at (ξ, η). Then by Lemma 3.2, $S'\phi_1 + T'\phi_2 \neq 0$ and

$$\langle(\xi, \eta), (\phi_1, \phi_2)\rangle \geq \langle J(\alpha_0) \cap (\Omega \cap D(T)), S'\phi_1 + T'\phi_2 \rangle + \langle \varepsilon U_Y, \phi_1 \rangle.$$

On the other hand,

$$\begin{aligned} \langle(\xi, \eta), (\phi_1, \phi_2)\rangle &= \langle \xi - Su_0, \phi_1 \rangle + \langle u_0, S'\phi_1 + T'\phi_2 \rangle \\ &\leq \sup_{u \in J(\alpha) \cap (\Omega \cap D(T))} \langle u, S'\phi_1 + T'\phi_2 \rangle + \sup_{u \in U_Y} \langle \xi - Su, \phi_1 \rangle \\ &= \langle J(\alpha) \cap (\Omega \cap D(T)), S'\phi_1 + T'\phi_2 \rangle + \langle \varepsilon U_Y, \phi_1 \rangle. \end{aligned}$$

Hence combining we have

$$u_0 = [S'\phi_1 + T'\phi_2 : J(\alpha) \cap (\Omega \cap D(T))], \quad \xi - Su_0 = [\phi_1 : \varepsilon U_Y].$$

Again

$$\xi = S[S'\phi_1 + T'\phi_2 : J(\alpha) \cap (\Omega \cap D(T))] + [\phi_1 : \varepsilon U_Y]$$

$$\eta = T(u) = T[S'\phi_1 + T'\phi_2 : J(\alpha) \cap (\Omega \cap D(T))].$$

To show (3), note that, $(\xi, \eta) \in \delta C_\varepsilon(\alpha_0, 0) \cap C_\varepsilon(\alpha_0, 0)$.

Hence

$$\langle(\xi, \eta)(\psi_1, \psi_2)\rangle \le \langle J(\alpha) \cap (\Omega \cap D(T)), (S'\phi_1 + T'\phi_2)\rangle + \langle \varepsilon U_Y, \psi_1\rangle, \quad \forall(\psi_1, \psi_2) \in (Y \times Z)'.$$

Now using Kuhn Tucker's theorem in a locally convex linear topological space [1] and the hypothesis $J(0, 0) = 0$, we have

$$\langle J(\alpha) \cap (\Omega \cap D(T)), S'\psi_1 + T'\psi_2\rangle + \langle \varepsilon U_Y, \psi_1\rangle$$

$$= \sup_{u \in A(\alpha_0)} \{\langle u, S'\psi_1 + T'\psi_2\rangle\} + \varepsilon \sup_{y \in U_Y} \langle y, \psi_1\rangle$$

$$= \sup_{u \in A(\alpha_0)} \{\langle u, S'\psi_1 + T'\psi_2\rangle + \varepsilon \sup_{y \in U_Y} \langle y, \psi_1\rangle - \lambda(J(u) - \alpha_0)\}, \quad \lambda > 0,$$

where $A(\alpha) = J(\alpha) \cap (\Omega \cap D(T))$. Consequently,

$$\langle(\xi, \eta)(\psi_1, \psi_2)\rangle \le \sup_{u \in A(\alpha_0)} \{\langle u, S'\psi_1 + T'\psi_2\rangle + \varepsilon \sup_{y \in U_Y} \langle y, \psi_1\rangle - \lambda(J(u) - \alpha_0)\}.$$

Hence

$$\alpha_0 \ge \langle(\xi, \eta)(\psi_1, \psi_2)\rangle - \sup_{u \in A(\alpha_0)} \{\langle u, S'\psi_1 + T'\psi_2\rangle + \varepsilon \sup_{y \in U_Y} \langle y, \psi_1\rangle - \lambda(J(u) - \alpha_0)\}$$

$$= \langle(\xi, \eta)(\psi_1, \psi_2)\rangle + \min_{u \in A(\alpha_0)} \min_{y \in U_Y} \{J(u) - \langle u, S'\psi_1 + T'\psi_2\rangle - \varepsilon\langle y, \psi_1\rangle\}.$$

Again

$$\langle(\xi, \eta)(\psi_1, \psi_2)\rangle \le \sup_{u \in A(\alpha_0)} \{\langle u, S'\psi_1 + T'\psi_2\rangle + \varepsilon \sup_{y \in U_Y} \langle y, \psi_1\rangle\} - \lambda(J(u) - \alpha_0)\}$$

$$\alpha_0 = \langle(\xi, \eta)(\psi_1, \psi_2)\rangle - \sup_{u \in A(\alpha_0)} \{\langle u, S'\psi_1 + T'\psi_2\rangle + \varepsilon \sup_{y \in U_Y} \langle y, \psi_1\rangle\},$$

i.e.,

$$\alpha_0 = \langle(\xi, \eta)(\phi_1, \phi_2)\rangle + \min_{u \in A(\alpha_0)} \min_{y \in U_Y} \{J(u) - \langle x, S'\psi_1 + T'\psi_2\rangle - \varepsilon\langle y, \psi_1\rangle\}.$$

Hence combining

$$\alpha_0 = \max_{(\psi_1, \psi_2) \in (Y \times Z)'} [\langle(\xi, \eta)(\psi_1, \psi_2)\rangle$$

$$+ \min_{u \in A(\alpha_0)} \min_{y \in U_Y} \{J(u) - \langle u, S'\psi_1 + T'\psi_2 \rangle - \varepsilon \langle y, \psi_1 \rangle\}]$$

$$= \langle \xi, \phi_1 \rangle + \langle \eta, \phi_2 \rangle + \min_{u \in A(\alpha_0)} \min_{y \in U_Y} \{J(u) - \langle u, S'\psi_1 + T'\psi_2 \rangle - \varepsilon \langle y, \psi_1 \rangle\}.$$

Conversely, suppose that (ξ, η) is regular and $\alpha_0, (\phi_1, \phi_2)$ satisfy anyone of the following conditions (1), (2), (3). Then $x_0 \in [S'\phi_1 + T'\phi_2 : A(\alpha_0)]$ is optimal.

Moreover, if $A(\alpha_0)$ is rotund, then the solution is unique.

To show the converse part one should note that assuming anyone of the above three conditions, one can deduce the other two, following the above argument in the reverse order.

Thus, if $\{\alpha_0, (\phi_1, \phi_2)\}$ satisfies any of the conditions (1)-(3), then in any case $(\xi, \eta) \in \delta C_\varepsilon(\alpha_0, 0) \cap C_\varepsilon(\alpha, 0)$ and (ϕ_1, ϕ_2) is the supporting hyperplane to $C_\varepsilon(\alpha_0, 0)$ at (ξ, η) and $x_0 \in [S'\phi_1 + T'\phi_2 : A(\alpha_0)]$ is an optimal solution. To show the uniqueness of x_0, let x_0, x_1 be any two solutions. Then we have

$$\langle (\xi, \eta), (\phi_1, \phi_2) \rangle \geq \langle A(\alpha_0), S'\phi_1 + T'\phi_2 \rangle + \langle \varepsilon U_Y, \phi_1 \rangle$$

and

$$\langle (\xi, \eta), (\phi_1, \phi_2) \rangle \leq \langle \varepsilon U_Y, \phi_1 \rangle + \langle A(\alpha_0), S'\phi_1 + T'\phi_2 \rangle.$$

Hence we have

$$\langle (x_0, \eta - Tx_0)(S'\phi_1 + T'\phi_2, \phi_1) \rangle = \langle (x_1, \eta - Tx_1)(S'\phi_1 + T'\phi_2, \phi_2) \rangle$$
$$= [S'\phi_1 + T'\phi_2 : A(\alpha_0)] + \varepsilon[\phi_1 : U_Y].$$

Consequently, the hyperplane $(S'\phi_1 + T'\phi_2) \neq 0$ supports $A(\alpha_0)$ at $x_0 = x_1$ as $A(\alpha_0)$ is rotund.

Minimization problem with norm criteria

We have seen how a minimum cost control problem with convex functional criteria in normed linear spaces having unbounded operators can be solved by function space approach. But if, in particular, the functional J and the constraint set $A(\alpha)$ are specified in term of norms, one can obtain an explicit characterization of the solution.

Solution of the problem $(\mathbf{P}_1)$

Here we identify $J(u) = \| u \|$. So the problem is to minimize $\| u \|$ over $u \in X$, satisfying the constraints $\eta = Tu$ and $\| \xi - su \| \leq \varepsilon$ $(\varepsilon > 0)$, where $S : X \xrightarrow{\text{into}} Y$, $T : X \xrightarrow{\text{onto}} Z$, and X, Y, Z are normed linear spaces over R, and S, T are linear operators.

For this particular case we choose

$$A(\alpha) = \alpha U_X \cap D(T), \text{ for any real } \alpha > 0$$

$$C_\varepsilon(\alpha, 0) = \hat{S}\{(\alpha U_X \cap D(T)) + \varepsilon U_Y \times \{0\}\},$$

where U_X, U_Y are unit balls in X and Y, respectively. Evidently, the hypothesis (H_0) is satisfied for the problem. Now, if $(\xi, \eta) \in C_\varepsilon(\alpha, 0)$, then

$$(\xi, \eta) = (Sx, Tx) + (\varepsilon y, \theta_z) = (\varepsilon y + Sx, Tx).$$

So, for $(\xi, \eta) \in \delta C_\varepsilon(\alpha, 0)$ and $(\phi_1, \phi_2) \in (Y \times Z)'$ defining a supporting hyperplane to $C_\varepsilon(\alpha, 0)$ at (ξ, η), we have

$$\begin{aligned} \langle\langle(\xi, \eta)(\phi_1, \phi_2)\rangle\rangle &\geq \langle\langle(\xi', \eta'), (\phi_1, \phi_2)\rangle\rangle = \varepsilon\langle y, \phi_1\rangle + \langle x, S'\phi_1 + T'\phi_2\rangle \\ &\geq \varepsilon\| \phi_1 \| + \alpha\| S'\phi_1 + T'\phi_2\|, \ \forall x \in U_X, \ y \in U_Y. \qquad (1) \end{aligned}$$

Also

$$\begin{aligned} \langle\langle(\xi, \eta), (\phi_1, \phi_2)\rangle\rangle &= \langle \xi - Sx, \phi_1\rangle + \langle x, S'\phi_1, T'\phi_2\rangle = \langle \varepsilon y, \phi_1\rangle + \langle x, S'\phi_1, T'\phi_2\rangle \\ &\leq \varepsilon\| \phi_1 \| + \alpha\| S'\phi_1 + T'\phi_2\|, \ \forall(\phi_1, \phi_2) \in (Y \times Z)'. \qquad (2) \end{aligned}$$

Then from (1) and (2), we have

$$\langle\langle(\xi, \eta), (\phi_1, \phi_2)\rangle\rangle = \varepsilon\| \phi_1 \| + \alpha\| S'\phi_1 + T'\phi_2 \|.$$

Thus $\langle x_0, S'\phi_1 + T'\phi_2\rangle = \alpha\| S'\phi_1 + T'\phi_2\|$, and $\varepsilon\langle y_0, \phi_1\rangle = \varepsilon\| \phi_1 \|$, where x_0, y_0 being the points where the supremum are attained.

Then $x_0 = \alpha\overline{S'\phi_1 + T'\phi_2}$, $y_0 = \varepsilon\bar{\phi}_1$.

Consequently, $\xi = S\overline{(\alpha S'\phi_1 + T'\phi_2)} + \varepsilon\bar{\phi}_1$, $\eta = Tx_0 = T\overline{(\alpha S'\phi_1 + T'\phi_2)}$ and from (2)

$$\alpha_0 = \max_{\substack{(\psi_1, \psi_2) \in (Y \times Z)', \\ \| S'\psi_1, T'\psi_2 \| \neq 0}} \frac{\langle (\xi, \eta), (\psi_1, \psi_2) \rangle - \varepsilon \| \psi_1 \|}{\| S'\psi_1 + T'\psi_2 \|}.$$

Solution of problem ($\mathbf{P_2}$)

Here we specify $J(u) = \| \eta - Tu \|$.

So the problem is to minimize $\| \eta - Tu \|$, under the constraint $\| \xi - Su \| \leq \varepsilon$, $\{u \in \rho U_X = u : \| u \| \leq \rho, u \in X\}$, where $S : X \xrightarrow{\text{into}} Y$, $T : X \xrightarrow{\text{into}} Z$ are linear transformations and X, Y, Z are real normed linear spaces. We further assume that η is (ε, ρ) normal with respect to $(\hat{S}, \xi)$, i.e.,

$$\inf_{\| \xi - Su \| \leq \varepsilon, \| u \| \leq \rho} \| \eta - Tu \| > \inf_{\| \xi - Su \| \leq \varepsilon, u \in X} \| \eta - Tu \|.$$

For this particular case let us choose

$$A(\alpha) = J(\alpha) \cap (\Omega \cap D(T)) = \rho U_X,$$

$$C_\varepsilon(\alpha, \rho) = \{\rho \hat{S}(U_X) + (\varepsilon U_Y \times \alpha U_Z)\}, \; \varepsilon > 0, \; \alpha > 0.$$

Then from any $(\xi, \eta) \in C_\varepsilon(\alpha, \rho)$ and $(\psi_1, \psi_2) \in (Y \times Z)'$,

$$(\xi, \eta) = (\varepsilon y + Sx, Tx + z)$$

and

$$\langle (\xi, \eta), (\psi_1, \psi_2) \rangle \leq \varepsilon \| \psi_1 \| + \rho \| S'\psi_1 + T'\psi_2 \| + \alpha \| \psi_1 \|.$$

Also, if $(\xi, \eta) \in \delta C_\varepsilon(\alpha, \rho) \cap C_\varepsilon(\alpha, \rho)$, $(\phi_1, \phi_2) \in (Y \times Z)'$ is a supporting hyperplane to $C_\varepsilon(\alpha, \rho)$ at (ξ, η), and

$$\langle (\xi, \eta), (\phi_1, \phi_2) \rangle \geq \varepsilon \| \phi_1 \| + \rho \| (S'\phi_1 + T'\phi_2) \| + \alpha \| \phi_2 \|.$$

Hence

$$\langle (\xi, \eta), (\phi_1, \phi_2) \rangle = \varepsilon \| \phi_1 \| + \rho \| (S'\phi_1 + T'\phi_2) \| + \alpha \| \phi_2 \|,$$

i.e.,

$$\langle (x_0, z_0), (S'\phi_1 + T'\phi_2, \phi_2) \rangle + \langle \varepsilon y_0, \phi_1 \rangle = \varepsilon \| \phi_1 \| + \rho \| (S'\phi_1 + T'\phi_2) \| + \alpha \| \phi_2 \|,$$

where (x_0, z_0) and y_0 are the points where the supremums are attained.

Consequently, $\langle x_0, S'\phi_1 + T'\phi_2 \rangle = \rho \| (S'\phi_1 + T'\phi_2) \|$, $\langle z_0, \phi_2 \rangle = \alpha \| \phi_2 \|$ and $\langle \varepsilon y_0, \phi_1 \rangle = \varepsilon \| \phi_1 \|$. Therefore,

$$x_0 = \rho \overline{(S'\phi_1 + T'\phi_2)}, \quad z_0 = \alpha \bar{\phi}_2, \quad y_0 = \varepsilon \bar{\phi}_1,$$

i.e., $\eta - Tx_0 = \alpha \bar{\phi}_2$, i.e., $\eta - T(\rho \overline{S'\phi_1 + T'\phi_2}) = \alpha \bar{\phi}_2$ and $\xi = S(\rho \overline{S'\phi_1 + T'\phi_2}) + \varepsilon \phi_1$. Also,

$$\alpha = \max_{\substack{(\psi_1, \psi_2) \neq (0, 0), \\ \| \psi_2 \| \neq 0}} \frac{\langle (\xi, \eta), (\psi_1, \psi_2) \rangle - \rho \| (S'\psi_1 + T'\psi_2) \| - \varepsilon \| \psi_1 \|}{\| \psi_2 \|}.$$

Note. The problems (P_1) and (P_2) are the same problems as in [4]. But the difference is that the underlying spaces are normed linear spaces instead of Banach spaces [4] and the corresponding transformations are only linear without being bounded as assumed in [4]. But the same results as in [4] are obtained.

Example. Consider the system, $[Lx](t) = u(t)$, $M\hat{x} = \eta$.

The problem is to find an admissible control vector $u(t)$ such that trajectories described by the system under $u(t)$ remain within an ε neighbourhood of the target state x^d, i.e., $\| x(t_1) - x^d \| \leq \varepsilon$ when $\| x \| = \operatorname*{ess\,sup}_{t_0 \leq t \leq t_1} \max_{1 \leq j \leq r} | x_j(t) |$ while minimizing the fuel functional

$$J(u) = \left[\left\{ \operatorname*{ess\,sup}_{t_0 \leq t \leq t_1} \max_{1 \leq j \leq r} | u_j(t) | \right\}^2 + \int_\zeta | \eta - T(u) | (t) dt^{\frac{1}{2}} \right]$$

$\tau = [t_0, t_1]$, t_0, t_1 being specified initial and final times, respectively.

Let $X = \overset{(r)}{B}_{\infty, \infty} = L_\infty(l_\infty(r), \tau)$, $Y = l_\infty(r)$, $Z = \overset{(r)}{B}_{1,1} = L_1(l_1(r), \tau)$ and T, S are linear operators defined by $D(S) = \{x \in X, Lx \in Y\}$ and $Sx = Lx$ while $D(S) = D(T)$ and $Tx = M\hat{x}$.

Let $\xi \in \overset{(r)}{B}_{\infty, \infty}$ and $\eta \in \overset{(r)}{B}_{1,1}$ and define $\Omega \subseteq X$ by

$$\Omega = \{x \in D(S) : \| Sx \| \leq 1\}.$$

Hence we have an example of Problem (P) for which hypothesis (H_0) is satisfied. Also Ω is closed and $\hat{S}$ is a weakly co-compact operator. For further detail see [3, pp. 349-350].

Note. L may represent partial differential operator and M defines boundary condition.

Conclusion

The minimum effort control problem discussed in this paper has the advantage that the linear operators T and S are not bounded. Moreover the spaces X, Y, Z are just normed linear spaces.

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