Advances and Applications in Statistics
Volume 8, Number 1, 2008, Pages 101-107
Published online: November 23, 2007
This paper is available online at http://www.pphmj.com

BAYESIAN TESTING FOR INDEPENDENCE IN MARSHALL AND OLKIN'S BIVARIATE EXPONENTIAL MODEL

JANG SIK CHO[*], CHANG WAN KANG[**] and SEUNG BAE CHOI[**]

[*]Department of Informational Statistics
Kyungsung University
Busan, 608-736, Korea
e-mail: jscho@ks.ac.kr

[**]Department of Data Information Science
Dongeui University
Busan, 614-714, Korea

Abstract

In this paper, we propose a Bayesian testing procedure for independence in Marshall and Olkin's bivariate exponential model based on Bayes factor. We use a noninformative prior such as an improper prior for the parameters so that such prior is defined only up to arbitrary constant which affects the values of Bayes factors. So we compute the fractional Bayes factor (FBF) proposed by O'Hagan [6] to compensate for that arbitrariness. We compute FBF's and calculate the posterior probabilities for the hypotheses, respectively. We illustrate our procedure through a numerical example.

1. Introduction

Bayesian testing depends rather strongly on the prior distributions. But subjective elicitation can easily result in poor prior distribution and

2000 Mathematics Subject Classification: 62.

Keywords and phrases: Bayesian testing, bivariate exponential model, fractional Bayes factor, improper prior, Marshall and Olkin's model, posterior probability.

Received June 1, 2007

statistical analysis is often required to appear objective. So, the research on noninformative priors has grown enormously over recent years. However, noninformative priors are typically improper so that such priors are defined only up to arbitrary constants which affect the values of Bayes factors. So, San Martini and Spezzaferri [7] and O'Hagan [6] have made efforts to compensate for that arbitrariness. Also Berger and Pericchi [1] introduced a new model selection and hypotheses testing criterion, called the *Intrinsic Bayes Factors*. Cho and Cha [3] suggested Bayesian testing procedure for the ratio of the failure rates in two independent exponential model based on FBF. And Cho et al. [4] and Cho [2] proposed multiple comparisons procedure based on FBF for geometric and negative binomial populations, respectively.

In this paper, we propose a Bayesian testing procedure for independence of bivariate exponential model by Marshall and Olkin [5]. We use an improper prior for the parameters which are defined only up to arbitrary constants. Also, we obtain FBF to compensate for that arbitrariness. Further, we compute posterior probability for the hypotheses and select hypothesis which has the largest posterior probability. Finally, we illustrate our procedure through a numerical example.

2. Preliminaries

Let (X, Y) be random variables of Marshall and Olkin's bivariate exponential model with parameters $\theta = (\lambda_1, \lambda_2, \lambda_3)$. Then the joint probability density function is given as

$$f(x, y) = \begin{cases} \lambda_1(\lambda_2 + \lambda_3)\exp(-\lambda_1 x - (\lambda_2 + \lambda_3)y), \; y > x, \\ \lambda_2(\lambda_1 + \lambda_3)\exp(-(\lambda_1 + \lambda_3)x - \lambda_2 y), \; x > y, \\ \lambda_3 \exp(-\lambda x), \; x = y, \end{cases} \tag{1}$$

where $\lambda_1, \lambda_2, \lambda_3 > 0$. It is well known that $\lambda_3/(\lambda_1 + \lambda_2 + \lambda_3)$ is the correlation coefficient between X and Y and is also $P(X = Y)$, the probability of simultaneous failure of both the components. The parameter $\lambda_3 = 0$ is equivalent to independence of two components.

Suppose that there are n two components units under study and ith pair of the components has lifetime (x_i, y_i). Let $(x, y) = ((x_1, y_1), \ldots, (x_n, y_n))$ be observation of sample size n.

Let $\pi_i^N(\theta_i)$ be an improper prior distribution under H_i, $i = 1, 2$ usually written as $\pi_i^N(\theta_i) \propto h_i(\theta_i)$, where h_i is a function whose integral over the parameter space under H_i diverges. Formally, we can write $\pi_i^N(\theta_i) = c_i h_i(\theta_i)$, although the normalizing constant c_i does not exist, but treating it as an unspecified constant. The posterior probability that H_i is true is given as

$$P(H_i \mid x, y) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji}^N \right)^{-1}, \tag{2}$$

where p_i is the prior probability of H_i being true and B_{ji}^N, the Bayes factor of H_j to H_i, is defined by

$$B_{ji}^N = \frac{\int_{\Theta_j} f(x, y \mid \theta_j) \pi_j^N(\theta_j) d\theta_j}{\int_{\Theta_i} f(x, y \mid \theta_i) \pi_i^N(\theta_i) d\theta_i}, \tag{3}$$

where $f(x, y \mid \theta_i)$ is the density under H_i, $i = 1, 2$. The posterior probabilities in (2) are then used to select the most plausible hypothesis.

Hence, the corresponding Bayes factor B_{ji}^N is indeterminate. To solve this problem, O'Hagan [6] proposed the FBF for Bayesian testing problem as follow. The FBF of model H_j to model H_i is

$$B_{ji}^F = \frac{q_j(b, x, y)}{q_i(b, x, y)}, \tag{4}$$

where $q_i(b, x, y) = \dfrac{\int_{\Theta} f(x, y \mid \theta_i) \pi_i^N(\theta_i) d\theta_i}{\int_{\Theta_i} f^b(x, y \mid \theta_i) \pi_i^N(\theta_i) d\theta_i}$ and b specifies a fraction of likelihood which is to be used as a prior density.

3. Bayesian Hypothesis Test

The goal here is to propose a Bayesian testing procedure for independence based on FBF. In this paper, we assume $\lambda_1 = \lambda_2 (\equiv \lambda_o)$ so that the lifetimes of two components are equal failure rates. We set the hypothesis $H_1 : \lambda_3 = 0$ v.s. H_2 : not H_1. Here, let $\theta_1 = \lambda_o$ and $\theta_2 = (\lambda_o, \lambda_3)$.

In this paper, we set the noninformative priors for $H_1 : \lambda_3 = 0$ v.s. H_2 : not H_1 as follows:

$$\pi_1^N(\theta_1) = \frac{1}{\lambda_o} \tag{5}$$

and

$$\pi_2^N(\theta_2) = \frac{1}{\lambda_o \lambda_3}. \tag{6}$$

To test the hypothesis of independence based on FBF, we need to compute (4). The likelihood function under $H_1 : \lambda_3 = 0$ is

$$L(\Theta_1) = \lambda_o^{2(n_1+n_2)} \cdot \exp\left(-\lambda_o \sum_{i=1}^{n} (x_i + y_i)\right). \tag{7}$$

Then $q_1(b, x, y)$ under $H_1 : \lambda_3 = 0$ is given by

$$q_1(b, x, y) = \frac{\int_{\Theta_1} f(x, y \mid \theta_1) \pi_1^N(\theta_1) d\theta_1}{\int_{\Theta_1} f^b(x, y \mid \theta_1) \pi_1^N(\theta_1) d\theta_1}$$

$$= \frac{\Gamma(2(n_1 + n_2)) \cdot \left[b \sum_{i=1}^{n} (x_i + y_i) \right]^{2b(n_1+n_2)}}{\Gamma(2b(n_1 + n_2)) \cdot \left[\sum_{i=1}^{n} (x_i + y_i) \right]^{2(n_1+n_2)}}. \tag{8}$$

On the other side, the likelihood function under H_2 : not H_1 is

$$L(\Theta_2) = \lambda_o^{n_1+n_2}\lambda_3^{n_3}(\lambda_o+\lambda_3)^{n_1+n_2}\exp\left(-\lambda_o\sum_{i=1}^{n}(x_i+y_i)-\lambda_3\sum_{i=1}^{n}\max(x_i, y_i)\right), \quad (9)$$

where $n_1 = \sum_{i=1}^{n} I(x_i < y_i)$, $n_2 = \sum_{i=1}^{n} I(x_i > y_i)$, $n_3 = \sum_{i=1}^{n} I(x_i = y_i)$. Then $q_2(b, x, y)$ under H_2 : not H_1 is given by

$$q_2(b, x, y) = \frac{\int_{\Theta_2} f(x, y\,|\,\theta_2)\pi_2^N(\theta_2)d\theta_2}{\int_{\Theta_2} f^b(x, y\,|\,\theta_2)\pi_2^N(\theta_2)d\theta_2} = \frac{A_1}{A_2}, \quad (10)$$

where

$$A_1 = \sum_{i=0}^{n_1+n_2}\frac{(n_1+n_2)!}{i!(n_1+n_2-i)!}\cdot\frac{\Gamma(n_1+n_2+i)}{\left[\sum_{i=1}^{n}(x_i+y_i)\right]^{n_1+n_2+i}}\cdot\frac{\Gamma(n_1+n_2+n_3-i)}{\left[\sum_{i=1}^{n}\max(x_i+y_i)\right]^{n_1+n_2+n_3-i}}$$

and

$$A_2 = \int_0^\infty\int_0^\infty \lambda_o^{b(n_1+n_2)-1}\lambda_3^{bn_3-1}(\lambda_o+\lambda_3)^{b(n_1+n_2)}$$

$$\cdot\exp\left(-\lambda_o b\sum_{i=1}^{n}(x_i+y_i)-\lambda_3 b\sum_{i=1}^{n}\max(x_i, y_i)\right)d\lambda_o d\lambda_3.$$

Therefore, the FBF of H_2 to H_1 is given by

$$B_{21}^F = \frac{\Gamma(2bn)\cdot\left[\sum_{i=1}^{n}(x_i+y_i)\right]^{2n}\cdot A_1}{\Gamma(2n)\cdot\left[b\sum_{i=1}^{n}(x_i+y_i)\right]^{2bn}\cdot A_2}. \quad (11)$$

Using FBF by (8) and (10), the posterior probability for hypothesis H_i, $i = 1, 2$ is given by $P(H_i\,|\,x, y) = \left(\sum_{j=1}^{2}\frac{p_j}{p_i}B_{ji}^F\right)^{-1}$. Thus, we can select the hypothesis with the highest posterior probability based on FBF by (2).

4. A Numerical Example

In this section, we present a numerical example to illustrate for the proposed Bayesian procedure for independence testing $H_1 : \lambda_3 = 0$ v.s. H_2 : not H_1 based on FBF. We take the prior probability of H_i being true, $p_i = 0.5$, $i = 1, 2$.

The samples of size 15 are simulated from Marshall and Olkin's bivariate exponential model with the parameters $\lambda_o = 3.0$ and $\lambda_3 = 2.0$. Then we note that the true hypothesis may be H_2.

The generated Marshall and Olkin's bivariate exponential data is given as: (0.2875, 0.1651), (0.0026, 0.0026), (0.0975, 0.0194), (0.1924, 0.1924), (0.0256, 0.0256), (0.4248, 0.1275), (0.1421, 0.5548), (0.0289, 0.0474), (0.3159, 0.1303), (0.1506, 0.1800), (0.0603, 0.0603), (0.0053, 0.0053), (0.0283, 0.3573), (0.0091, 0.0400), (0.0297, 0.0297).

For above generated data, we can compute the FBF $B_{21}^F = 9.2619$ by (11). Also we can obtain the posterior probability $P(H_2 \mid x, y) = 0.9025$ by (2). That is, there is strong evidence for H_2 in terms of the posterior probabilities based on FBF.

Until now, we have considered the problem of developing a Bayesian testing for independence of Marshall and Olkin's bivariate exponential model based on FBF. In conclusion, FBF methodology can also be applied in general when the samples come from any distribution. An extension of the method to Bayesian testing problems for the another models would be accomplished straightforwardly. The research topics pertaining to the examination of its performance are worthy to study and are left as a future subject of research.

References

[1] J. O. Berger and L. R. Pericchi, The intrinsic Bayes factor for model selection and prediction, J. Amer. Statist. Assoc. 91 (1996), 109-122.

[2] J. S. Cho, Multiple comparisons of the proportions for the negative binomial populations using fractional Bayes factor, J. Korean Data Anal. Soc. 8(4) (2006), 1361-1368.

[3] J. S. Cho and Y. J. Cha, Bayesian testing for the ratio of the failure rates in two components system, J. Korean Data Anal. Soc. 8(3) (2006), 913-918.

[4] J. S. Cho, Y. J. Cha and J. M. Lee, Bayesian multiple comparisons in geometric populations, J. Korean Data Anal. Soc. 8(3) (2006), 919-926.

[5] A. W. Marshall and I. Olkin, A multivariate exponential distribution, J. Amer. Statist. Assoc. 62 (1967), 30-44.

[6] A. O'Hagan, Fractional Bayes factors for model comparison (with discussion), J. Roy. Statist. Soc. 56 (1995), 99-118.

[7] A. San Martini and F. Spezzaferri, A predictive model selection criterion, J. Roy. Statist. Soc. 46 (1984), 296-303.