Far East Journal of Dynamical Systems
Volume 9, Issue 3, 2007, Pages 407-413
This paper is available online at http://www.pphmj.com

EXISTENCE OF POSITIVE SOLUTIONS TO SOME NONLINEAR BOUNDARY VALUE PROBLEMS

G. A. AFROUZI and S. KHADEMLOO

Department of Mathematics
Faculty of Basic Sciences
Mazandaran University
Babolsar, Iran
e-mail: afrouzi@umz.ac.ir

Abstract

In this paper we study the role played by the indefinite weight function $a(x)$ and the parameter λ on the existence of positive solutions to the problem

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) = \lambda a(x) F(u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\Re^N$, $1 < p < N$, λ is real parameter and $a(x)$ changes sign.

We prove that in the case $F(u) = u|u|^{p-2}(1 \pm |u|^{\gamma})$ for $\gamma > 0$, the problem has a positive solution.

1. Introduction

In this paper we study the existence of positive solutions of the Neumann boundary value problem

2000 Mathematics Subject Classification: 35J60, 35B30, 35B40.

Keywords and phrases: the p-Laplacian, indefinite weight function, variational method, Neumann boundary value problem.

Received May 8, 2007

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2}(1 - |u|^{\gamma}), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

on a bounded domain $\Omega \subseteq \Re^N$, with smooth boundary $\partial\Omega$, where $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian operator with $p > 1$ and $a(x)$ is a smooth weight function which changes sign on Ω. Here we say a function $a(x)$ changes sign if the measure of the sets $\{x \in \Omega; a(x) > 0\}$ and $\{x \in \Omega; a(x) < 0\}$ are both positive.

A host of literature exists for this type of problem when $p = 2$; see, e.g., [1], [5], [6] and the references therein. Li and Zhen [8] studied (1) with $p \geq 2$ and obtained some interesting results.

In this paper we consider a special type of function $F(u)$ and study the influence of the function $a(x)$ on the structure of the set of positive solutions of (1) in the cases where $F(u) = u|u|^{p-2}(1 - |u|^{\gamma})$ and $F(u) = u|u|^{p-2}(1 + |u|^{\gamma})$, $\gamma > 0$.

If $p = 2$, $a(x) = 1$ and $F(u) = u(1 - |u|^{\gamma})$, then (1) becomes

$$\begin{cases} -\Delta u = \lambda u(1 - |u|^{\gamma}), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

It is well known that positive solutions of this problem must satisfy $0 < u < 1$, and precisely arise out of bifurcation from the zero solution; moreover, the equation has no positive solutions if $\lambda < \lambda_1^+$, where λ_1^+ denotes the least eigenvalue of the Laplacian.

We shall show, however, that, when $a(x)$ changes sign, the variational method proves the existence of a positive solution for a special range of λ in the case p-Laplacian.

Our method relies on the eigencurve theory developed in [3, 4]. It turns out that the sign of the integral $\int_\Omega a$ plays an important role for the range of λ for which (1) has a positive solution.

In the next section we consider the map $\lambda \to \mu(\lambda)$, where $\mu(\lambda)$ denotes the principal eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2} + \mu u |u|^{p-2}, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

and we notice how eigencurves can be used to generate an equivalent norm on $W^{1,p}(\Omega)$, then we prove the existence of variational solutions by using this equivalent norm.

2. Main Results

We first introduce some notations and recall some results. Consider the eigenvalue problem (2) where we treat the eigenvalue μ associated with a positive eigenfunction, as a function of λ.

$$\mu(\lambda) = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega a(x)|u|^p dx}{\int_\Omega |u|^p dx}.$$

From whence it follows that (see, e.g., [3, 4, 7]).

Proposition 1. *Assume that* $a \in L^\infty(\Omega)$. *Then* $\mu(\lambda)$ *is continuous and concave and* $\mu(0) = 0$. *If* $a(x) > 0$, *then* $\mu(\lambda)$ *is decreasing, and if* $a(x) < 0$, *then* $\mu(\lambda)$ *is increasing. Assume now that* $a(x)$ *changes sign in* Ω. *If* $\int_\Omega a(x)dx < 0$, *then there exists a unique* $\lambda_1^+ > 0$ *such that* $\mu(\lambda_1^+) = 0$ *and* $\mu(\lambda) > 0$ *for* $\lambda \in (0, \lambda_1^+)$. *If* $\int_\Omega a(x)dx = 0$, *then* $\mu(0) = 0$ *and* $\mu(\lambda) < 0$ *for* $\lambda \neq \lambda_1^+$. *If* $\int_\Omega a(x)dx > 0$, *then there exists a unique* $\lambda_1^- > 0$ *such that* $\mu(\lambda_1^-) = 0$ *and* $\mu(\lambda) > 0$ *for* $\lambda \in (\lambda_1^-, 0)$.

It follows from this proposition that when $a(x)$ changes sign and $\int_\Omega a(x)dx < 0$, the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2}, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega \end{cases} \tag{3}$$

has a positive eigenvalue λ_1^+ associated with the positive principal eigenfunction u_1^+.

Thus we assume that there hold the following conditions:

$$(\mathcal{A}_1) \quad a(x) \in L^\infty(\Omega)$$

$$(\mathcal{A}_2) \quad \int_\Omega a(x) dx < 0.$$

With these constructions we have [2].

Proposition 2. *Assume* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$. *Then for every* $\lambda \in (0, \lambda_1^+)$

$$\| u \|_\lambda := \left(\int_\Omega (|\nabla u|^p - \lambda a(x) |u|^p) dx \right)^{\frac{1}{p}}$$

defines a norm in $W^{1,p}(\Omega)$ *which is equivalent to the short norm of* $W^{1,p}(\Omega)$.

Now we can state our main result:

Theorem 1. *Assume that* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *and* $0 < \gamma < p^*$, *where* p^* *is the critical Sobolev exponent. Then for any* $\lambda \in (0, \lambda_1^+)$, *the problem* (1) *has a positive solution.*

Proof. We introduce the functional I on the space $W^{1,p}(\Omega)$ by

$$I(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p - \lambda a |u|^p) dx + \frac{\lambda}{p+\gamma} \int_\Omega a |u|^{p+\gamma} dx.$$

It is easy to see that $I(u)$ is the Euler functional associated with equation (1) which is neither bounded from above nor from below on $W^{1,p}(\Omega)$.

Take the set $\Gamma = \{u \in W^{1,p}(\Omega); \frac{\lambda}{p+\gamma}\int_\Omega a|u|^{p+\gamma}dx = -1\}$. Since $a(x)$ is a sign changing function, there exists an open subset B of Ω such that $a(x) < 0$ on B. Then taking $u \in W^{1,p}(\Omega)$ with $\operatorname{Supp} u \subseteq B$, we get $\Gamma \neq \varnothing$.

Moreover, as $L^{p+\gamma}(\Omega)$ may be compactly embedded in $W^{1,p}(\Omega)$, Γ is weakly closed in $W^{1,p}(\Omega)$. Now using the homogeneity of (1), a solution of the equation (1) can be obtained by solving a constrained minimization problem for the functional

$$E(u) = \frac{1}{p}\int_\Omega (|\nabla u|^p - \lambda a|u|^p)dx = \frac{1}{p}\|u\|_\lambda^p$$

on the $W^{1,p}(\Omega)$, restricted to the set Γ.

We verify $E : \Gamma \to \mathbf{R}$ satisfies the hypotheses of Theorem 1.2 of [9].

Equivalence property of Proposition 2 leads us to get E is coercive. Moreover sequentially lower semicontinuity of E follows from weak lower semicontinuity of the equivalent norm. It follows that E attains its infimum at a point $\underline{u}$ in Γ. Remark that since $E(u) = E(|u|)$ we may assume that $\underline{u} \geq 0$.

Note that E is continuously Frechet differentiable in $W^{1,p}(\Omega)$ with

$$(E'(u), v) = \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v - \lambda a|u|^{p-2}uv)dx.$$

Moreover, letting

$$K(u) = \frac{\lambda}{p+\gamma}\int_\Omega a|u|^{p+\gamma}dx + 1$$

$K(u) : W^{1,p}(\Omega) \to \mathbf{R}$ is continuously Frechet differentiable with

$$(K'(u), v) = \lambda\int_\Omega a|u|^{p+\gamma-2}uvdx.$$

In particular at any point $u \in \Gamma$

$$(K'(u), u) = \lambda \int_\Omega a|u|^{p+\gamma} dx = -(p+\gamma) \neq 0,$$

and by the implicit function theorem, the set $\Gamma = K^{-1}(0)$ is a C^1-submanifold of $W^{1,p}(\Omega)$. Now the Lagrange multiplier rule follows that there exists a parameter $\mu \in \mathbf{R}$ such that

$$(E'(\underline{u}) - \mu K'(\underline{u}), v) = 0 \quad \text{for all } v \in W^{1,p}(\Omega).$$

Setting $v = \underline{u}$, above gives

$$\int_\Omega (|\nabla \underline{u}|^p - \lambda a|\underline{u}|^p) dx + \mu\lambda \int_\Omega a|\underline{u}|^{p+\gamma} dx = 0,$$

i.e.,

$$\|\underline{u}\|_\lambda^p = -\mu\lambda \int_\Omega a|\underline{u}|^{p+\gamma} dx = \mu(p+\gamma).$$

Since $\underline{u} \in \Gamma$ cannot vanish identically, $\|\underline{u}\|_\lambda > 0$ and so $\mu > 0$. Scaling with a suitable power of μ, we obtain a weak solution $u = \mu^{\gamma^{-1}} \underline{u} \in W^{1,p}(\Omega)$ of (1) in the sense that

$$\mu^{-\frac{p-1}{\gamma}} \left[\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla v - \lambda a|u|^{p-2} uv) dx \right]$$

$$+ \mu\mu^{-\frac{p+\gamma-1}{\gamma}} \lambda \int_\Omega a|u|^{p+\gamma-2} uv dx = 0,$$

i.e.,

$$\int_\Omega (|\nabla u|^{p-2} \nabla u \nabla v - \lambda a|u|^{p-2} uv + \lambda a|u|^{p+\gamma-2} uv) dx = 0$$

for all $v \in W^{1,p}(\Omega)$.

It follows from standard regularity arguments that $u \in C^2(\Omega)$ is a classical solution satisfying the appropriate boundary condition, and finally by the maximum principle $u > 0$ on Ω. □

References

[1] D. Arcoya and S. Villegas, Nontrivial solutions for a Neumann problem with a nonlinear term asymptotically linear at $-\infty$ and superlinear at $+\infty$, Math. Z. 219 (1995), 499-513.

[2] P. A. Binding, P. Drabek and Y. X. Huang, On Neumann boundary value problems for some quasilinear elliptic equations, Electron. J. Differential Equations 1997 (1997), 1-11.

[3] P. A. Binding and Y. X. Huang, The principal eigencurve for the p-Laplacian, Differential Integral Equations 8 (1995), 405-414.

[4] P. A. Binding and Y. X. Huang, Two parameter problems for the p-Laplacian, Proc. First Int. Cong. Nonl. Analysts, 1996, pp. 891-900.

[5] J. P. Gossez and P. Omari, A necessary and sufficient condition of nonresonance for a semilinear Neumann problem, Proc. Amer. Math. Soc. 114 (1992), 433-442.

[6] J. P. Gossez and P. Omari, On a semilinear elliptic Neumann problem with asymmetric nonlinearities, Trans. Amer. Math. Soc. 347 (1995), 2553-2562.

[7] Y. X. Huang, On eigenvalue problem for the p-Laplacian with Neumann boundary conditions, Proc. Amer. Math. Soc. 109 (1990), 177-184.

[8] W. Li and H. Zhen, The applications for sums of ranges of accretive operators to nonlinear equations involving the p-Laplacian operator, Nonlinear Anal. 24 (1995), 185-193.

[9] M. Struwe, Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer, Heidelberg, 1996.