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A NECESSARY CONDITION FOR LOCAL ASYMPTOTIC STABILITY OF PERIODIC ORBITS OF NONLINEAR SYSTEMS WITH EXOGENOUS DISTURBANCE

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Abstract

In this paper, we derive a necessary condition for local asymptotic stability of periodic orbits of nonlinear systems with time-varying exogenous disturbance. This is an extension of our earlier work [7] providing a necessary condition for the local asymptotic stability of periodic orbits of nonlinear systems with constant parameters. We illustrate our result with examples.

1. Introduction

In this paper, we extend our earlier work [7] on the local asymptotic stability of periodic orbits of nonlinear systems with constant parameters. We derive a new necessary condition for local asymptotic stability of periodic orbits of nonlinear systems with time-varying exogenous disturbance.

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In this paper, we consider a nonlinear system described by

$$\dot{x} = f(x, \mu) \triangleq f_\mu(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the *state*, and $\mu \in \mathbb{R}^k$ the *parametric uncertainty* or *exogenous disturbance* for the nonlinear system (1). We assume that the vector field f is $\mathcal{C}^2$ in x, and jointly continuous in x and μ. We also assume that Γ is an isolated periodic orbit of the system dynamics

$$\dot{x} = f_{\mu^\star}(x). \tag{2}$$

The disturbance vector μ is assumed to satisfy the exosystem dynamics defined by

$$\dot{\mu} = s(\mu), \tag{3}$$

where s is $\mathcal{C}^1$ in μ in an open neighborhood V of $\mu^\star$.

We suppose also that the state x of the nonlinear system (1) takes values in X, where X is an open neighborhood of the periodic orbit Γ, and also that the parameter vector μ takes values in the open neighborhood V of $\mu^\star$.

We assume that the exosystem (3) is *neutrally stable* (recall that a system is called *neutrally stable* if it is Lyapunov stable in both forward and backward directions of time). Thus, the exogenous disturbance that we consider in this paper includes the important special cases of constant real-parametric uncertainty considered in our recent work [7] and periodic signals.

In this paper, we investigate the problem of finding a necessary condition for the asymptotic stability of the periodic orbits of nonlinear systems with time-varying exogenous disturbance of the form (1) for the value $\mu^\star$. Explicitly, we are interested in finding a necessary condition for the asymptotic stability of the periodic orbit Γ of the system (2).

The Poincaré map [5] for the periodic orbit Γ of the system (2) has the form

$$\chi(k+1) = P_{\mu^\star}(\chi(k)) = A_{\mu^\star}\chi(k) + \phi_{\mu^\star}(\chi(k)), \tag{4}$$

where χ is defined in an open neighborhood of the origin of $\mathbb{R}^{n-1}$, $A_{\mu^\star}$ is an $(n-1)\times(n-1)$ matrix, and $\phi_{\mu^\star}$ is a $\mathcal{C}^2$ function that vanishes at the origin of $\mathbb{R}^{n-1}$ together with all its first partial derivatives.

If the periodic orbit Γ of the dynamics (2) is locally exponentially stable, then it follows from Lyapunov stability theory for periodic orbits that $A_{\mu^\star}$ is *convergent* [3], i.e., that all the eigenvalues of $A_{\mu^\star}$ lie in the open unit disc of the complex plane. Then it follows immediately that the matrix $I - A_{\mu^\star}$ is nonsingular. Thus, by the Inverse Function Theorem [6], it follows that for all values of μ near $\mu^\star$, the equation

$$(I - P_\mu)(\chi) = \chi - P_\mu(\chi) = y \tag{5}$$

is solvable. In particular, taking the value $y = 0$ in (5), it follows that there exists a periodic orbit of the dynamics (1) for all values of μ near $\mu^\star$.

We contend that this is the case for locally asymptotically stable periodic orbits as well. That is, if Γ is a locally asymptotically stable periodic orbit of the dynamics (2), then we contend that for all values of μ near $\mu^\star$, there exists a periodic orbit for the system (1).

We illustrate our claim with some examples of nonlinear systems with time-varying neutrally stable exogenous disturbance.

Example 1. Consider the planar system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1(1 - r^2)[\mu_1 - (r^2 - 1)^2] \\ \dot{x}_2 &= x_1 - x_2(1 - r^2)[\mu_1 - (r^2 - 1)^2], \end{aligned} \tag{6}$$

where $r^2 = x_1^2 + x_2^2$ and $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ is the exogenous disturbance satisfying the dynamics

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= -\mu_1. \end{aligned} \tag{7}$$

The exosystem dynamics (7) is the vector form of the *simple pendulum*, and it is clearly neutrally stable. In fact, the general solution of the exosystem (7) is given by

$$\mu_1(t) = \mu_1^0 \cos t + \mu_2^0 \sin t \quad \text{and} \quad \mu_2(t) = -\mu_1^0 \sin t + \mu_2^0 \cos t.$$

In polar coordinates, the plant equation (7) takes the form

$$\begin{aligned} \dot{r} &= -r(1 - r^2)[\mu_1 - (r^2 - 1)^2] \\ \dot{\theta} &= 1. \end{aligned} \tag{8}$$

When $\mu^\star = 0$, the plant equation in (8) reduces to

$$\begin{aligned} \dot{r} &= r(1 - r^2)^3 \\ \dot{\theta} &= 1. \end{aligned} \tag{9}$$

It is easy to see that the zero-parameter plant dynamics (9) has a periodic orbit Γ represented by

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

From the dynamics (9), it is evident that the periodic orbit Γ is asymptotically stable.

We note that for all values of $\mu \in \mathbb{R}^2$, the system (8) has a one-parameter family of periodic orbits represented by

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

This example is a generalized form of a *pitchfork bifurcation* [5] at a non-hyperbolic periodic orbit. □

Example 2. Consider the planar system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1[\mu_1 - (r^2 - 1)^2] \\ \dot{x}_2 &= x_1 - x_2[\mu_1 - (r^2 - 1)^2], \end{aligned} \tag{10}$$

where $r^2 = x_1^2 + x_2^2$ and $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ is the exogenous disturbance satisfying the neutrally stable dynamics

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= -\mu_1. \end{aligned} \tag{11}$$

In polar coordinates, the plant equation (10) takes the form

$$\begin{aligned} \dot{r} &= -r[\mu_1 - (r^2 - 1)^2] \\ \dot{\theta} &= 1. \end{aligned} \tag{12}$$

When $\mu^\star = 0$, the plant dynamics (12) reduces to

$$\begin{aligned} \dot{r} &= r(r^2 - 1)^2 \\ \dot{\theta} &= 1. \end{aligned} \tag{13}$$

Note that the zero-parameter plant dynamics (13) has a periodic orbit Γ represented by

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

It is easy to see that the periodic orbit Γ is unstable for the zero-parameter dynamics (13).

Note that for any value of μ for which $\mu_1 < 0$, there is no periodic orbit for the system (12).

This example is a generalized form of a *saddle-node bifurcation* [5] at a non-hyperbolic periodic orbit. $\square$

2. Main Results

In this section, using degree theory, we derive a necessary condition for Γ to be a locally asymptotically stable periodic orbit of the system

$$\dot{x} = f_{\mu^\star}(x).$$

Our main result is a generalization of our earlier work [7] giving a necessary condition for local asymptotic stability of a periodic orbits of nonlinear systems with constant real parametric uncertainty. This new result is also similar to the necessary condition [2] obtained for local asymptotic stability of equilibria of nonlinear systems with constant real parametric uncertainty. Our result asserts that any asymptotically stable periodic orbit of a $\mathcal{C}^2$ dynamical system persists as a periodic orbit in a robust way.

Theorem 1. *Consider a nonlinear system described by*

$$\dot{x} = f(x, \mu) \triangleq f_\mu(x) \quad [x \in \mathbb{R}^n, \mu \in \mathbb{R}^k], \tag{14}$$

where the state x is defined in an open neighborhood of the periodic orbit Γ *of the dynamics*

$$\dot{x} = f_{\mu^\star}(x). \tag{15}$$

Suppose that the vector field f is $\mathcal{C}^2$ *in x, and jointly continuous in x and* μ. *The disturbance vector* μ *is assumed to satisfy the exosystem dynamics given by*

$$\dot{\mu} = s(\mu), \tag{16}$$

where s is $\mathcal{C}^1$ *in* μ *in an open neighborhood V of* $\mu^\star$ *and* $s(\mu^\star) = 0$. *A necessary condition for* Γ *to be a locally asymptotically stable periodic orbit of the system* (15) *is that for all values of* μ *near* $\mu^\star$, *there exists a periodic orbit* Γ *of the dynamics* (14).

Proof. It is given that Γ is a locally asymptotically stable periodic orbit of the dynamics (15). Hence, by a necessary condition due to Krasnoselski and Zabreiko [4], it follows that

$$\kappa_{\mu^\star}(\Gamma) = \text{index}(I - P_{\mu^\star}, 0) = 1,$$

where $P_{\mu^\star}$ is the *Poincaré map* for the dynamics (15) and $\kappa_{\mu^\star})$ is the *index* of the periodic orbit Γ for the dynamics (15).

Since the exosystem dynamics (16) is neutrally stable, for any given neighborhood V of $\mu^\star$, we can choose a small neighborhood $V^\star \subset V$ so that all solutions $\mu(t)$ with $\mu(0) = \mu_0 \in V^\star$ stay inside V for all values of $t \geq 0$. We note also that the index operator κ is robust with respect to small variations in the parameter μ. Hence, it follows that for all values of μ near $\mu^\star$, we have

$$\text{index}(I - P_\mu, 0) \neq 0.$$

Now, we can apply the degree theory [1] to conclude that the map $I - P_\mu$ is locally onto, i.e., the equation

$$\chi - P_\mu(\chi) = y \tag{17}$$

is locally solvable. In particular, taking $y = 0$ in (17), we conclude that for all values of μ near $\mu^\star$, there exists a periodic orbit of the dynamics (14). This completes the proof. □

The following example shows that the converse of Theorem 1 is not true.

Example 3. Consider the planar system

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1(1 - r^2)(1 + \mu_1 - r^2) \\ \dot{x}_2 &= x_1 - x_2(1 - r^2)(1 + \mu_1 - r^2), \end{aligned} \tag{18}$$

where $r^2 = x_1^2 + x_2^2$ and $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ is the exogenous disturbance satisfying the neutrally stable dynamics

$$\begin{aligned} \dot{\mu}_1 &= \mu_2 \\ \dot{\mu}_2 &= -\mu_1. \end{aligned} \tag{19}$$

In polar coordinates, the plant equation (18) takes the form

$$\begin{aligned} \dot{r} &= -r(1 - r^2)(1 + \mu_1 - r^2) \\ \dot{\theta} &= 1. \end{aligned} \tag{20}$$

Hence, it is clear that for all values of $\mu \in \mathbb{R}^2$, the system (20) has a one-parameter family of periodic orbits represented by

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Setting $\mu^\star = 0$, in the dynamics (20), we obtain the zero-parameter dynamics as

$$\begin{aligned} \dot{r} &= -r(1 - r^2)^2 \\ \dot{\theta} &= 1 \end{aligned} \tag{21}$$

which has the periodic orbit Γ represented by the equation $r = 1$ and described by the solution $\gamma(t)$. It is easy to see that the periodic orbit Γ is unstable with respect to the dynamics (21).

This example is a generalized form of a *transcritical bifurcation* [5] at a non-hyperbolic orbit, and demonstrates that the converse of Theorem 1 is not true. □

References

[1] M. Agoston, Algebraic Topology, Marcel Dekker, New York, 1976.

[2] C. I. Byrnes and V. Sundarapandian, Persistence of equilibria for locally asymptotically systems, Internat. J. Robust Nonlinear Control 11 (2001), 87-93.

[3] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[4] M. A. Krasnoselski and P. P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Springer, Berlin, 1984.

[5] L. Perko, Differential Equations and Dynamical Systems, Springer, New York, 1991.

[6] W. Rudin, Principles of Mathematical Analysis, McGraw Hill, 1964.

[7] V. Sundarapandian, A necessary condition for local asymptotic stability of periodic orbits of nonlinear systems with parameters, Indian J. Pure Appl. Math. 34 (2003), 241-246.