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ESTIMATION OF SURVIVAL PROBABILITY WITH SPECIFIED NUMBER OF SHOCKS

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Abstract

A component is subjected to a sequence of shocks. Shocks are of two types, namely effective shocks and non-effective shocks. Damages due to effective shocks are exponential random variables. If the accumulated damage exceeds the threshold of the component, the component fails. The threshold of the component is also an exponential random variable. The survival probability of the component with k_0 (known) shocks is derived. The maximum likelihood estimator (MLE) and the uniformly minimum variance unbiased estimator (UMVUE) of the survival probability are obtained.

1. Introduction

Problems in reliability theory are widely varied in nature. Consider the problems in which, components (or organs) of various types of functioning without failure are desired. But, at the same time, we cannot

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be sure of components not being exposed to shocks or accidents during the period of operation. So, it is necessary to develop the mathematical methods to answer the questions like *with what probability the component can survive with* k_0 *shocks*, where k_0 is a pre-specified positive integer. Esary et al. [5] studied some models of the life distribution of a component, subjected to random shocks occurring according to a Poisson process. Abdel-Hameed and Proschan [1] considered models with the arrival rate of shocks increasing in time. Barlow and Proschan [2] have considered shock models yielding bivariate distributions. Chikkagoudar and Palaniappan [4] have obtained the UMVUE of reliability in shock models with cumulative damages and fixed threshold model, where the shocks are occurring as events in Poisson process. Shantikumar and Sumita [9] studied a general shock model associated with a correlated pair (X_n, Y_n), where X_n is the magnitude of the n-th shock and Y_n is the time interval between two consecutive shocks. Posner and Zuckerman [8] studied the optimal replacement strategy for a semi-Markov shock model with additive damage. Kunchur and Munoli [6] have obtained the estimators of reliability in shock models with cumulative damages, fixed threshold for two-component parallel system. Skoulakis [10] modeled reliability of a parallel system subject to shocks generated by a renewal point process. In the present study, we propose an approach that will enable us to find the probability of survival with specified number of shocks or continued functioning of a component (or organ) under overloading, say *once*, *twice*, *thrice*, etc., rather than finding the failure free operation for certain time period. In Section 2, the model is discussed and the probability that the component survives with k_0 shocks is obtained. The life testing experiment is considered in Section 3. The MLEs of parameters and complete sufficient statistics for the family of distribution are also obtained in this section. Section 4 deals with MLE and UMVUE of survival function.

2. Survival Function

Suppose a component is subjected to a sequence of shocks. Shocks are of two types, effective shocks and non-effective shocks. Let p denote the probability of a shock being effective and $q = 1 - p$ be the probability that the shock is non-effective. The effective shock damages the component and non-effective shock does not damage the component. The component survives if the accumulated damage due to effective shocks is less than threshold of the component. Let $X_1, X_2, \ldots, X_s$ be random damages due to s effective shocks to the component out of k_0 shocks, $s = 0, 1, 2, \ldots, k_0$. It is assumed that X_i's are independently, identically distributed exponential random variables with parameter θ ($\exp(\theta)$), $\theta > 0$. Let V be threshold of the component, which is also exponential random variable with parameter μ ($\exp(\mu)$), $\mu > 0$. The probability that the component survives with k_0 (pre specified number) shocks out of which s shocks are effective shocks is

$$R(k_0) = \sum_{s=0}^{k_0} \binom{k_0}{s} p^s q^{k_0 - s} P(X_1 + \cdots + X_s < V) \tag{2.1}$$

$$= \sum_{s=0}^{k_0} \binom{k_0}{s} \left(\frac{p\theta}{\theta + \mu} \right)^s q^{k_0 - s}$$

$$= \left(q + \frac{p\theta}{\theta + \mu} \right)^{k_0} \tag{2.2}$$

and $R(0) = 1$, $R(\infty) = 0$; $R(k)$ is non-increasing function of k.

Considering accidents and heart attacks as shocks, an automobile and heart patient serve as examples to this model.

3. Life Testing Experiment

Suppose r identical components with thresholds $V_1, V_2, \ldots, V_r$ are subjected to r independent sequences of shocks and $V_i \sim \exp(\mu)$, $\mu > 0$, $i = 1, 2, \ldots, r$. The experiment is continued till all the r components fail. Let the i-th component fail due to s_i number of effective shocks and the shock arrival pattern be $1, 2, \ldots, k_{i1}; 1, 2, \ldots, k_{i2}; \ldots, 1, 2, \ldots, k_{i, s_i}$, that is,

$(k_{ij} - 1)$ is the number of non-effective shocks that the component experiences to have j-th effective shock, $j = 1, 2, \ldots, s_i$, $i = 1, 2, \ldots, r$ and the component fails at k_i-th shock, where $k_i = \sum_{j=1}^{s_i} k_{ij}$. That is out of $k_i = \sum_{j=1}^{s_i} k_{ij}$ shocks, s_i shocks will be effective shocks and k_{i1}-th, k_{i2}-th, ..., k_{i,s_i}-th shocks are effective shocks. Let p be the probability that the shock is effective shock and q be the probability that the shock is non-effective. Let $X_{i1}, X_{i2}, \ldots, X_{is_i}$ denote the damages due to effective shocks to the i-th component, $i = 1, 2, \ldots, r$ and are $\exp(\theta)$ random variables, $\theta > 0$. $\sum_{j=1}^{s_i} X_{ij}$ exceeds the threshold V_i of the component and it is assumed that the damage due to a shock at which the i-th component fails (fatal shock) is not observable but is known to exceed $\left(V_i - \sum_{j=1}^{s_i - 1} X_{ij}\right)$. It is also assumed that V_i is observable. The joint probability density function of the random variables $k_{i1}, k_{i2}, \ldots, k_{i,s_i}, s_i$ $X_{i1}, X_{i2}, \ldots, X_{i,s_i-1}, V_i$ is

$$p^{s_i} q^{\sum_{j=1}^{s_i} k_{ij} - s_i} \theta^{s_i - 1} e^{-\theta \sum_{j=1}^{s_i-1} x_{ij}} e^{-\theta\left(v_i - \sum_{j=1}^{s_i-1} x_{ij}\right)} \mu e^{-\mu v_i} = p^{s_i} q^{k_i - s_i} \theta^{s_i - 1} \mu e^{-(\mu+\theta) v_i} \tag{3.1}$$

with $0 \le s_i \le k_i < \infty$; $x_{ij} \ge 0$ for $j = 1, 2, \ldots, s_i - 1$; $\sum_{j=1}^{s_i - 1} x_{ij} < v_i$, $v_i \ge 0$, $k_i = 1, 2, \ldots$. Thus, the joint pdf of the random variables $k_{i1}, k_{i2}, \ldots, k_{i,s_i}$, s_i, $X_{i1}, \ldots, X_{i,s_i-1}$, V_i for all the r components is given by

$$p^{s_\bullet} q^{k_\bullet - s_\bullet} \theta^{s_\bullet - r} \mu^r e^{-(\mu+\theta) v_\bullet} \tag{3.2}$$

with

$$s_\bullet = \sum_{i=1}^{r} s_i, \; k_\bullet = \sum_{i=1}^{r} k_i \;\text{ and }\; v_\bullet = \sum_{i=1}^{r} v_i.$$

Using (3.2), the MLEs of p, μ, θ are obtained as

$$\hat{p} = \frac{s_\bullet}{k_\bullet}, \; \hat{\mu} = \frac{r}{v_\bullet}, \; \hat{\theta} = \frac{m_\bullet - r}{v_\bullet}. \tag{3.3}$$

The joint pdf (3.2) can be rewritten as

$$\exp\lfloor s_\bullet ln(p\theta/q) + k_\bullet lnq + rln(\mu/\theta) - (\mu + \theta)v_\bullet \rfloor. \tag{3.4}$$

This density belongs to three parameter exponential family. So, by Theorem 1 of Lehmann [7, p. 132], $(s_\bullet, k_\bullet, v_\bullet)$ is complete sufficient statistic for the family of densities.

4. MLE and UMVUE of $R(k_0)$

Substituting the MLEs of parameters from (3.3), in the expression for $R(k_0)$ (2.2), the MLE $\hat{R}(k_0)$ of $R(k_0)$ is obtained.

In order to obtain the UMVUE of $R(k_0)$, define

$$\varphi_{k_0}(k_1) = \begin{cases} 1, & \text{if } k_1 > k_0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

then

$$E(\varphi_{k_0}(k_1)) = P(k_1 > k_0)$$

$$= \sum_{k_1 = k_0 + 1}^{\infty} \sum_{s_1 = 1}^{k_1} \int_0^\infty P(k_1 \mid s_1, v_1) P(s_1, v_1) dv_1, \tag{4.2}$$

where $k_1 \sim NB(s_1, p)$ (for given s_1 and v_1).

Using the renewal process, the distribution of s_1 and v_1 is obtained

$$P(s_1, v_1) = P\left(\sum_{j=1}^{s_1 - 1} x_{1j} \le v_1 \le \sum_{j=1}^{s_1} x_{1j}\right)$$

$$= \frac{\mu e^{-\mu v_1} e^{-\theta v_1} (\theta v_1)^{s_1 - 1}}{(s_1 - 1)!}, \quad s_1 = 1, 2, \ldots; \; 0 \le v_1 < \infty. \tag{4.3}$$

Substituting these probabilities in (4.2) and simplifying, it is easily verified that $\varphi_{k_0}(k_1)$ is an unbiased estimator of $R(k_0)$.

Using Rao-Blackwell and Lehmann-Scheffe theorems, the UMVUE of $R(k_0)$ is given by

$$R^*(k_0) = E[\varphi_{k_0}(k_1) \mid k_\bullet, s_\bullet, v_\bullet].$$

Using (4.2) the above conditional expectation can be written as

$$R^*(k_0) = \sum_{k_1=k_0+1}^{\infty} \sum_{s_1=1}^{k_1} \int_0^{\infty} P(k_1, s_1, v_1 \mid k., s., v.) dv_1. \tag{4.4}$$

The conditional distribution of (k_1, s_1, v_1) given $(k., s., v.)$ is obtained following on the lines of Basu [3] as follows:

Letting

$$k_\bullet^1 = \sum_{i=2}^{r} k_i,\ s_\bullet^1 = \sum_{i=2}^{r} s_i \text{ and } v_\bullet^1 = \sum_{i=2}^{r} v_i,$$

we have

$$P(k_\bullet^1 \mid s_\bullet^1, v_\bullet^1) = \binom{k_\bullet^1 - 1}{s_\bullet^1 - 1} p^{s_\bullet^1} q^{k_\bullet^1 - s_\bullet^1},\ k_\bullet^1 = s_\bullet^1, s_\bullet^1 + 1, \ldots \tag{4.5}$$

and

$$P(s_\bullet^1, v_\bullet^1) = \frac{\mu^{r-1} (v_\bullet^1)^{r-2} (v_\bullet^1 \theta)^{s_\bullet^1 - (r-1)} e^{-(\mu+\theta) v_\bullet^1}}{(\Gamma(r-1))(s_\bullet^1 - (r-1))!} \tag{4.6}$$

which follow on the lines of (4.3).

In the joint distribution of (k_1, s_1, v_1) and $(k_\bullet^1, s_\bullet^1, v_\bullet^1)$ which are independent, making the transformations $k_\bullet = k_\bullet^1 + k_1$, $s_\bullet = s_1 + s_\bullet^1$, $v_\bullet = v_1 + v_\bullet^1$; $k_1 = k_1$, $s_1 = s_1$, $v_1 = v_1$, we get the joint distribution of (k_1, s_1, v_1) and $(k_\bullet, s_\bullet, v_\bullet)$, and dividing this joint distribution by the distribution of $(k_\bullet, s_\bullet, v_\bullet)$, we obtain the conditional distribution of

(k_1, s_1, v_1) given $(k_\bullet, s_\bullet, v_\bullet)$ as

$$P(k_1, s_1, v_1 \mid k_\bullet, s_\bullet, v_\bullet) = \frac{(r-1)\binom{k_1-1}{s_1-1}\binom{k_\bullet - k_1 - 1}{s_\bullet - s_1 - 1}\binom{s_\bullet - r}{s_1 - 1}}{\binom{k_\bullet - 1}{s_\bullet - 1}}$$

$$\cdot \frac{1}{v_\bullet}\left(\frac{v_1}{v_\bullet}\right)^{s_1 - 1}\left(1 - \frac{v_1}{v_\bullet}\right)^{s_\bullet - s_1 - 1} \tag{4.7}$$

with $s_1 = 1, 2, \ldots, \min(k_1, s_\bullet - r + 1)$; $k_1 = 1, 2, \ldots, (k_\bullet - r + 1)$ and $0 \le v_1 < v_\bullet$.

Finally the UMVUE of $R(k_0)$ is obtained by substituting the above conditional distribution in (4.3) and integrating over the ranges of v_1, and is given by

$$R^*(k_0) = \frac{(r-1)}{\binom{k_\bullet - 1}{s_\bullet - 1}} \sum_{k_1 = k_0 + 1}^{k_\bullet - r + 1} \sum_{s_1 = 1}^{s} \binom{k_1 - 1}{s_1 - 1}\binom{k_\bullet - k_1 - 1}{s_\bullet - s_1 - 1}\binom{s_\bullet - r}{s_1 - 1} B(s_1, s_\bullet - s_1) \tag{4.8}$$

with $s = \min(k_1, s_\bullet - r + 1)$.

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