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AN ALGORITHM FOR SOLVING A SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

A number of problems in science and engineering are modeled in terms of a system of ordinary differential equations. In this paper, an algorithm for solving a system of linear ordinary differential equations (ODE) has been presented, which converts a system of linear ODE to a system of linear algebraic equations. Some illustrative examples have been presented to illustrate the implementation of the algorithm and to see efficiency of the presented approach.

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1. Introduction

A number of problems in chemistry, physics and engineering are modeled in terms of a system of ordinary differential equations (ODE). Solving a system of ODE is a difficult problem. Various methods have been developed to solve systems of ODE. Biazar et al. [2] have employed the Adomian decomposition method to solve a system of ODE. Daftardar-Gejji and Jafari [4] have solved a system of fractional differential equations using Adomian decomposition. Taylor polynomial method is widely used in literature to solve ODE [3, 5, 6, 8]. In this paper, we present an algorithm for solving a system of ODE using Taylor polynomials. We convert a system of ODE to a system of linear algebraic equations and solve. Some illustrative examples are presented to illustrate the method.

The paper has been organized as follows. In Section 2, an algorithm is developed for solving a system of linear ordinary differential equations. Some illustrative examples are given in Section 3 followed by the discussion and conclusions presented in Section 4.

2. A Method for Solving a System of Linear ODEs

In the present paper, we consider the following system of linear differential equations:

$$\sum_{i=1}^{n}\sum_{j=0}^{m} a_{ijk}(x) y_i^{(j)}(x) = f_k(x), \quad k = 1, 2, \ldots, n, \tag{1}$$

where $a_{ijk}(x) (j = 1, 2, \ldots, m,\ i = 1, 2, \ldots, n)$ and $f_k(x) (k = 1, 2, \ldots, n)$ are C^∞ functions.

We look for a solution of (1), which is a Taylor polynomial of degree N:

$$y_i(x) = \sum_{s=0}^{N} \frac{1}{s!} y_i^{(s)}(\xi)(x-\xi)^s, \quad x, \xi \in I, \quad N > m, \tag{2}$$

where $y_i^{(s)}(\xi) (s = 0, 1, \ldots, N)$ are the coefficients to be determined.

Differentiating (1), N times with respect to x, we get

$$\left[\sum_{i=1}^{n}\sum_{j=0}^{m} a_{ijk}(x)y_i^{(j)}(x)\right]^{(l)} = f_k^{(l)}(x), \quad l = 0, 1, ..., N, \quad k = 1, 2, ..., n. \tag{3}$$

Using Leibnitz's rule, we express $f_k^{(l)}(x)$ as

$$f_k^{(l)}(x) = \left[\sum_{i=1}^{n}\sum_{j=0}^{m} a_{ijk}(x)y_i^{(j)}(x)\right]^{(l)}$$

$$= \left[\sum_{j=0}^{m} a_{1jk}(x)y_1^{(j)}(x)\right]^{(l)} + \cdots + \left[\sum_{j=0}^{m} a_{njk}(x)y_n^{(j)}(x)\right]^{(l)}$$

$$= \sum_{i=1}^{n}\sum_{j=0}^{m}\sum_{p=0}^{l}\binom{l}{p} a_{ijk}^{(l-p)}(x)y_i^{(p+j)}(x),$$

$$l = 1, 2, ..., N, \quad k = 1, 2, ..., n. \tag{4}$$

The system (4) can be written in the matrix form as

$$F = WY, \tag{5}$$

where $Y = [y_1^{(0)}, y_1^{(1)}, ..., y_1^{(N)}, y_2^{(0)}, ..., y_2^{(N)}, ..., y_n^{(0)}, ..., y_n^{(N)}]^T$ and $F = [f_1^{(0)}, f_1^{(1)}, ..., f_1^{(N)}, f_2^{(0)}, ..., f_2^{(N)}, ..., f_n^{(0)}, ..., f_n^{(N)}]^T$. Note that

$$W = [W_{ik}], \quad i, k = 1, 2, ..., n, \tag{6}$$

is a matrix, where each W_{ik} is again a matrix having lpth entry as w_{ik}^{lp}:

$$w_{ik}^{lp} = \sum_{s=0}^{m}\binom{l}{p-m+s} a_{im-sk}^{(l-p+m-s)}(x), \quad l, p = 0, 1, ..., N. \tag{7}$$

Note. For $r < 0$, $a_{ijk}^{(r)} = 0$ and for $j < 0$ and $j > i$, $\binom{i}{j} = 0$, where i, j and r are integers. So we can convert (5) into an algebraic equation with variables $y_{i0}, y_{i1}, ..., y_{iN}$, $i = 1, 2, ..., n$. After determining variables y_{il} $(i = 1, 2, ..., n, l = 1, 2, ..., N)$, i.e., the unknown Taylor coefficient, we can get the Taylor polynomials solutions of the system (1). In particular, if we choose $\xi = 0$, then the solution of the system (1) becomes

$$y_i(x) = \sum_{s=0}^{N} \frac{1}{s!} y_{is}(0) x^s. \tag{8}$$

It is clear that for large values of N, the complexity of solving the system (5) can be very high in practice. So we can use mechanization algorithm [7] or mathematics software such as Mathematica or Maple.

3. Illustrative Examples

To demonstrate the effectiveness of the method, we consider some systems of linear ordinary differential equations.

(I) Consider the system of linear ordinary differential equations

$$\begin{aligned} &y_1 + y_1'' + y_2' - 3y_3 = x^3 + 3x - 8, \\ &y_1' + y_2 + y_3' = 4x^2 + 3, \\ &y_1'' - y_2' + y_3 + y_3' = 5x + 4. \end{aligned} \tag{9}$$

The matrix equations of ODE are as follows: $WY = F$, where

$$W = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -8 \\ 3 \\ 0 \\ 6 \\ 3 \\ 0 \\ 8 \\ 0 \\ 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}.$$

Then we get the following system of algebraic equations:

$$\begin{cases} y_{10} + y_{12} + y_{21} - 3y_{30} = -8, \quad y_{11} + y_{13} + y_{22} - 3y_{31} = 3, \\ y_{12} + y_{23} - 3y_{32} = 0, \quad y_{13} - 3y_{33} = 6, \\ y_{11} + y_{20} + y_{31} = 3, \quad y_{12} + y_{21} + y_{32} = 0, \\ y_{13} + y_{22} + y_{33} = 8, \quad y_{23} = 0, \\ y_{12} - y_{21} + y_{30} + y_{31} = 4, \quad y_{13} - y_{22} + y_{31} + y_{32} = 5, \\ -y_{23} + y_{32} + y_{33} = 0, \quad y_{33} = 0. \end{cases} \tag{10}$$

The solution of this system of algebraic equations is as follows:

$$Y = \{y_{10} = 1,\ y_{11} = -2,\ y_{12} = 0,$$
$$y_{13} = 6,\ y_{20} = 4,\ y_{21} = 0,$$
$$y_{22} = 2,\ y_{23} = 0,\ y_{30} = 3,$$
$$y_{31} = 1,\ y_{32} = 0,\ y_{33} = 0\}.$$

Then in view of (8), the solution of the system of linear ODE is

$$y_1(x) = x^3 - 2x + 1,$$
$$y_2(x) = x^2 + 4,$$
$$y_3(x) = x + 3.$$

Note that this is an exact solution.

(II) Consider the system of linear ordinary differential equations

$$xy_1 + x^2 y_1'' - y_2 + xy_2' = 3x^3 + 4x^2 + 2x - 1,$$
$$x^2 y_1' - xy_2 + y_2''' = -x^4 + 3x^3 - x + 6. \tag{11}$$

In view of (7), we get

$$W = \begin{bmatrix} x & 0 & x^2 & 0 & 0 & -1 & x & 0 & 0 & 0 \\ 1 & x & 2x & x^2 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 2 & x+2 & 4x & x^2 & 0 & 0 & 1 & x & 0 \\ 0 & 0 & 3 & x+6 & 6x & 0 & 0 & 0 & 2 & x \\ 0 & 0 & 0 & 9 & x+12 & 0 & 0 & 0 & 0 & 3 \\ 0 & x^2 & 0 & 0 & 0 & -x & 0 & 0 & 1 & 0 \\ 0 & 2x & x^2 & 0 & 0 & -1 & -x & 0 & 0 & 1 \\ 0 & 2 & 4x & x^2 & 0 & 0 & -2 & -x & 0 & 0 \\ 0 & 0 & 6 & 6x & x^2 & 0 & 0 & -3 & -x & 0 \\ 0 & 0 & 0 & 12 & 8x & 0 & 0 & 0 & -4 & -x \end{bmatrix},$$

and $F = [3x^3 + 4x^2 + 2x - 1, 9x^2 + 8x + 2, 18x + 8, 18, 0, -x^4 + 3x^3 - x + 6, -4x^3 + 9x^2 - 1, 18x - 12x^2, 18 - 24x, -24]^t$. Now we put $x = 0$ and solve following system of linear algebraic equations

$$\begin{cases} -y_{20} = -1,\ y_{10} = 2,\ 2y_{11} + 2y_{12} + y_{22} = 8, \\ 3y_{12} + 6y_{13} + 2y_{23} = 18,\ 9y_{13} + 12y_{14} + 3y_{24} = 0, \\ y_{23} = 6,\ -y_{20} + y_{24} = -1,\ 2y_{11} - 2y_{21} = 0, \\ 6y_{12} - 3y_{22} = 18,\ 12y_{13} - 4y_{23} = -24. \end{cases} \tag{12}$$

The solution of this system of algebraic equations is as follows:

$$Y = \{y_{10} = 2,\ y_{11} = 3,\ y_{12} = 2,\ y_{13} = 0,\ y_{14} = 0,\ y_{20} = 1,$$
$$y_{21} = 3,\ y_{22} = -2,\ y_{23} = 6,\ y_{24} = 0\}.$$

Using (8) the solution of the system of linear ODE turns out to be

$$y_1(x) = x^2 + 3x + 2,$$

$$y_2(x) = x^3 - x^2 + 3x + 1,$$

which is an exact solution.

(III) Consider the system of linear ordinary differential equations

$$y_1 + y_2' = x,$$

$$y_1' \cos x - y_2' \sin x = 1 + \cos x,$$

$$y_2(0) = 1. \tag{13}$$

In view of (7), we get

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & 0 & 0 & 0 & 0 & 0 & 0 & -S & 0 & 0 & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 & 0 & 0 & -C & -S & 0 & 0 & 0 & 0 \\ 0 & -C & -2S & C & 0 & 0 & 0 & 0 & S & -2C & -S & 0 & 0 & 0 \\ 0 & S & -3C & -3S & C & 0 & 0 & 0 & C & 3S & -3C & -S & 0 & 0 \\ 0 & C & 4S & -6C & -4S & C & 0 & 0 & -S & 4C & 6S & -4C & -S & 0 \\ 0 & -S & 5C & 10S & -10C & -5S & C & 0 & -C & -5S & 10C & 10S & -5C & -S \end{bmatrix},$$

and $F = [x, 1, 0, 0, 0, 0, 0, 1, 1 + C, -S, -C, S, C, -S]^T$, where S and C denote $\sin x$ and $\cos x$, respectively. Corresponding to (5), we get the following system of algebraic equations at $x = 0$,

$$\begin{cases} y_{10} + y_{21} = 0,\ y_{11} + y_{22} = 1, \\ y_{12} + y_{23} = 0,\ y_{13} + y_{24} = 0, \\ y_{14} + y_{25} = 0,\ y_{15} + y_{26} = 0,\ y_{16} = 0, \\ y_{20} = 1,\ y_{11} = 2,\ y_{12} - y_{21} = 0, \\ -y_{11} + y_{13} - 2y_{22} = -1, \\ -3y_{12} + y_{14} + y_{21} - 3y_{23} = 0, \\ y_{11} - 6y_{13} + y_{15} + 4y_{22} - 4y_{24} = 1, \\ 5y_{12} - 10y_{14} - y_{21} + 10y_{23} - 5y_{25} = 0. \end{cases} \tag{14}$$

The solution of this equation is: $Y = \{y_{10} = 0,\ y_{11} = 2,\ y_{12} = 0,\ y_{13} = -1,\ y_{14} = 0,\ y_{15} = 1,\ y_{16} = 0,\ y_{20} = 1,\ y_{21} = 0,\ y_{22} = -1,\ y_{23} = 0,\ y_{24} = 1,\ y_{25} = 0,\ y_{26} = -1\}$. Using (8), we get

$$y_1(x) = 2x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

$$y_2(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}. \tag{15}$$

In fact, the exact solution of above system is $y_1 = x + \sin x$, $y_2 = \cos x$, (15) gives approximate solutions which are the Taylor polynomials of $\cos x$ and $\sin x$ of order 5 and 6, respectively.

Comment. If $y_i^{(j)}$, $j = 0, 1, \ldots, k(< m)$ are absent for some i, in the system of ODE, then we need to supply the initial conditions $y_i^{(j)}(0) = c_j^i$, $j = 0, 1, \ldots, k(< m)$.

4. Discussion and Conclusions

In this paper, a method is described for solving linear systems of ordinary differential equations, which converts a system of linear ODE into a system of linear algebraic equations, solving which we obtain an approximate solution of the system of linear ODE. The illustrative examples explain the procedure. The method is simple and can provide an approximate solution of desired accuracy.

Mathematica has been used for computations involved in the illustrative examples.

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