Far East Journal of Mathematical Education
Volume 1, Number 1, 2007, Pages 1-32
Published Online: October 13, 2007
This paper is available online at http://www.pphmj.com

A LENGTHY PROCESS FOR THE ESTABLISHMENT OF THE CONCEPT OF LIMIT STARTING FROM PUPILS' PRE-CONCEPTIONS

And so any human knowledge begins from intuition, from there it goes on with shaping of concepts and it ends with ideas - KANT

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Abstract

The experience of a group of teacher-researchers, starting from the analysis of teaching practice, has allowed to spotlight some difficulties in learning the concept of limit, but also highlight the presence of pre-conceptions in very young pupils. The theoretical reflection which followed focused, on the one hand, on a historical-epistemological analysis; on the other hand, on planning research activities aimed firstly at progressively establishing the concept of limit in students and secondly at outlining the basics of teacher's classroom activities on this subject. Having taken into account the complexity and importance of such a concept, we believe that a gradual approach is appropriate for pupils, trying to develop in them an understanding from the first years at school, by taking the pre-conceptions of primary school children as a starting point. In particular, we wish to emphasize the role and importance of approximation in this itinerary, and, more in general, the need to modify, through training courses, the practice of the teachers of mathematics.

2000 Mathematics Subject Classification: 00A35, 97C70, 97D70.

Keywords and phrases: mathematics, limit, approximation, measure, pre-conception.

This work is done in the framework of the Local Research Unit in Mathematics Education, University of Parma – Italy.

Received January 7, 2007

Introduction

This paper relates the experience of a group of teacher-researchers coordinated by some researchers in Mathematics Education and involved in an in-service training project responsible to the Local Research Unit in Education of the University of Parma. The results of the research are used also for pre-service teacher education courses.

The main characteristic of our research group is that all stages in schooling (primary – 6 to 10 years; middle school – 11 to 13 years, and high school – 14 to 19 years) are represented, both the teachers within the group itself and as regards the pupils involved. This vertical cross-section is a resource which is as rare as it is precious. It allows us, indeed, to propose similar activities at different levels, check the evolution of the ideas, techniques and errors of the schoolchildren over time, and observe the effect of the teaching methods on them. From a statistical point of view different pupils in different school classes at the same time (a priori) could not be considered as the same population sample. But we assume a hypothesis that the average pupil's improvement in knowledge and competence is the result of schooling and pupils' growth. Under this hypothesis our experimental synchronic activities are interpreted as a diachronic research.

We consider this paper as a specimen of how a research style can be applied on a particular subject. We chose the concept of limit because of its learning-teaching difficulties and its relevance in school curricula (and in mathematics) (Sierpinska [38], Cornu [12], Tall [39], Dimarakis and Gagatsis [15], Mamona-Downs [33]). The experiences were originated from school practice; by the analysis of the pupils' attitudes and beliefs, this research also indirectly gave us indications on the teacher's actions practices which produced or favored such behaviors. This experience urged us to stress, in teachers' training courses, the importance of a careful recognition of students' beliefs systems. The examples drawn from this research show what can be done.

In a second stage, our research allowed us to interpret some of pupils' difficulties in light of the epistemological obstacles which are intrinsic to the concept itself.

Finally, with the support of research in mathematics education, we chose some issues, in particular measurement and approximation, which, in our opinion, can favour the gradual early development of the concept of limit. The activities we propose were tested in some sample classes and we discussed them in teachers' training lectures.

Moreover, our experience sets itself as self-training activity for the teachers who took part in it. In addition, thanks to the diffusion of publications (articles, books, online material), it was used for the pre- and in-service training of other teachers.

The issue of pre- and in-service teacher training is particularly relevant in Italy due to the recent innovative choice to entrust such education to the University. This kind of training was not institutionalized in the past (until 1999) and was left to each teacher's responsibility, but it has become necessary due to the crucial and growing complexity of being a teacher. Indeed more and more often teachers must take on different roles and have knowledge of different fields, related to contents (of their own and other subjects), pedagogy, psychology and didactics (inherent in planning, assessing and organizing a curriculum) (Zan [45]).

In particular, the pre- and in-service training proposed by the University of Parma (Maffini et al. [32]) is based on some fundamental assumptions which are linked to the theoretical paradigm of social constructivism (von Glasersfeld [42, 43], Ernest [18], Cobb [10, 11], Grugnetti [25]) and can be summarized in the need for:

- a definition of contents from a *historical-epistemological* point of view, paying special attention to difficulties and obstacles existing in the past;

- a shift of attention from *teaching* to *learning*, leading to the centrality of the pupil and the importance of the social dimension of knowledge (Grugnetti [26]);

- a critical and conscious choice of methodologies and tools suitable to assist learning (Brousseau [6], Arsac et al. [4]), which become particularly significant if linked to history and, in some cases, proposed with the support of information technology.

The approach outlined above is innovative in the Italian panorama and can serve as an example of renewal in teacher education.

From another point of view, we maintain that there is a need to improve teacher's critical thinking and our approach should be useful in this direction for its strong component of epistemological nature. Nowadays school mathematics appears as a monolithic building of truths; the distance of everyday life with its troubles from the false solidity of school mathematics averts people from mathematics. Epistemology and history of mathematics lead to reconsider the foundation of this building. The twentieth century crisis of Science has shown that elementary topics have deep complexity, but this crisis had passed without consequences for the school. If the teacher is aware of these aspects, s/he can choose the level of rigour of a subject, adapting it to the age and knowledge of her/his students. On the contrary the teacher is hostage of the choices made by others, e.g., books, authors and curriculum makers.

Students' Pre-conceptions, Images and Intuitions

There is widespread agreement among researchers in the field of mathematics education that, if the teaching of a concept is going to be effective, the pupil's competence acquired in an out-of-school context or in previous studies must not be discounted. Such competence may indeed be of determining importance in relation to facilitating or impeding learning the concept itself. On the one hand, it is a matter of intuitive knowledge, mainly deriving from either common sense or experience acquired outside the school; on the other hand, it involves knowledge, which is more specifically mathematical and institutionalized through school practice, and therefore may have produced deep-rooted convictions. The meta-cognitive aspects related to attitudes and emotions which affect the management of one's own knowledge must not be neglected either (Zan [44]). This complex of pre-conceptions[1] is the basis for the pupil's own

[1] By this word we intend the totality of spontaneous conceptions (i.e., ideas, intuitions, images, knowledge) which come from daily experience and which the student has already developed before any teaching (Cornu [12]).

image of the concept[2]: learning will be efficient only if such an image is coherent with the definition of the concept itself (Tall and Vinner [40]). Instead, these images often turn out to be incoherent with, or even contradict, concepts, and learning remains superficial and is not interiorized. Fischbein [20, 21] distinguishes different levels of intuitions, and emphasizes the importance of sustaining and reinforcing «primary intuitions» in such a way as to allow them to evolve into the «secondary intuitions» stage so that they can act as favorable ground for the acquisition of a concept.

All these appear to be particularly important when dealing with the more complex concepts of mathematics, such as the concept of limit and the concepts connected to it, i.e., infinitesimal, infinity and continuity. The activities we propose here were devised by our research group in order to bring out the 'submerged world' of pre-conceptions related to these issues. This 'speleology' is aimed at stemming the risks linked to the formation of distorted or misleading images, and, at the same time, at bringing out positive intuitions. In such a case, the teacher's action should start from these intuitions, by emphasizing them and trying to make them develop. In this way they constitute a good foundation for the acquisition and understanding of the concept involved.

Our Research: Diagnostics

Here follows the description of part of an empirical research whose aim was to investigate what kinds of pre-conceptions are present in the students of different school levels and how the teaching process can support or hinder their development. In particular, we pointed out the presence of favorable intuitions about approximation, which are often neglected by didactical practice.

The experimental recording of presences in the whole school itinerary of such complex concepts is an invitation to teachers and future teachers

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[2] Tall and Vinner [40], define "*the image of the concept*" as the complex of verbal, visual, vocal, emotional associations evoked in the mind when a *notion* (name of a *concept*) is proposed to someone.

to take into account students' pre-conceptions in light of a high-profile (high-quality) and long-term objective.

Linguistic aspects

Some researches (e.g., Pluvinage [36]) have highlighted the fact that linguistic considerations have a considerable part in teaching mathematics. As emphasised by Iacomella et al. [30], in order to be able to establish a link with the schoolchildren's empirically based background understanding, it is necessary to obtain information on what certain terms may evoke in students' minds. This should help to avoid the creation of that type of barrier to communication between teacher and pupil deriving from the subject itself, one of the worst of its kind. In particular, in the case of the concept of limit, the natural language register[3] seems to be unhelpful in the transition to the mathematical register. Monaghan [35] points out that there are some expressions, used in connection with limit, equivalent in mathematical language, but such that students interpret them in different ways. On the contrary the same word 'limit' assumes in the natural language (for example in Italian) different meanings more or less close to the mathematical one.

In order to test this point of view, we decided to conduct a survey on the linguistic meaning given to the term limit and even to a number of commonly used expressions in the everyday Italian language (Andriani et al. [2], Grugnetti et al. [28]). We prepared a survey in form of a written interview in which the questions were open-ended in order not to influence the answers.

Explain what the term LIMIT means to you
Explain what the following expressions mean to you:
THAT IS THE LIMIT
LIMITLESS
WITHIN LIMITS
If you want you can add your remarks.

Figure 1. Interview card on the meaning of word limit.

[3] To use an expression employed by Duval [17] and Pluvinage [36].

The interviewed sample numbered a total of about 600 people, including pupils (ranging from 14 to 19 years of age from different types of secondary and high schools) and adults without specialist mathematical knowledge (10% of the total). Pupils from a school specialising in arts were also asked to provide a graphical representation of their ideas. This same question was then put to a number of third-year students from middle schools.

The open-ended nature of the questionnaire undoubtedly complicated the interpretation of the answers. Considering in particular, the question "explain what the term limit means to you" – (which incorporates the others to some extent), an attempt has been made to classify responses in accordance with the main idea expressed by the interviewee:

1. The idea of impediment, barrier, rule, restriction 44%

An example: *In my opinion the limit is something we cannot exceed* (2nd year at high school).

2. The idea of border, boundary, closing 30%

An example: *When we consider an object*, *the limit will be the border and beyond it the object no longer exists* (5th year at high school).

3. The idea of extremity, end… 19%

An example: *aim*, *arrival point* (an adult).

4. Other meanings (including the 'mathematical' meanings) 3%

An example: *Limit=human nature* (an adult).

5. No reply 4%

The interpretation of the answers obtained may not be objective in character and should be considered as being more an indicator than scientific in the strict sense. We present here the above frequency rates not for a statistical analysis but as a suggestion of guidelines for our research.

The analysis of the questionnaire emphasises how a large proportion of pupils associates the term limit with an idea of finite or finiteness, in time and space. In another questionnaire (submitted to the same people) we investigated the interplay between limit and infinity, for detecting more if the two words are connected in the everyday language. Actually, the presence of a kind of opposition between the ideas of limit and infinity appeared with great frequency in the response to the question «Explain what the word 'infinity' means to you» in the second questionnaire: *Infinity is something that has no limits*.

The answers to the set of questions about limit gave rise to the formulation of the conjecture according to which the strong idea of a limit as a barrier represents a hindrance to the process of understanding the concept of limit. It is a hindrance to the extent that it may give rise to at least two types of difficulties, i.e.,

(a) difficulty in relating 'limit' to an idea of iteration process (hence a process that continues to infinity);

(b) difficulty in accepting the possibility of an infinite limit.

Activities on measure

According to these remarks it becomes necessary to give pupils activities for developing intuitions which are closer to the mathematical meaning (for example that of limit as an extreme) and for correcting those which are only partially adequate (for example that of limit as a border).

It became important to further investigate the students' intuitions, moving from a linguistic to a mathematical context. We made the following hypotheses:

- the same kinds of intuitions referring to the concepts of limit (and infinity, infinitesimal and continuity) are present in every age level, especially in the younger students;
- during the school process no further development of these intuitions can be checked, on the contrary, in some cases a regression is possible.

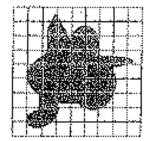
In this perspective our group carried out research (for more details see Dallanoce et al. [14], Alberti et al. [1]) in order to explore the intuitive knowledge of pupils aged 10 to 19 inclusive (300 pupils), relating to the concept of limit together with those connected to it.

We show here two examples of activities we proposed.

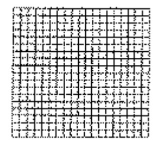
MARK

Mario and Giovanna must find the area of this picture

Mario suggests measuring the surface of the picture with a sheet of squared paper like the one you can see on your right:



Giovanna thinks of doing it with a sheet of squared paper like this:



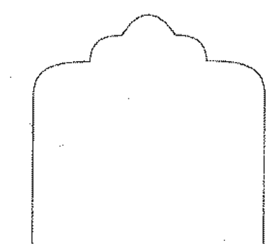
What about you? Any suggestion of yours? What do you think the surface measurement of the picture is? Explain how you find it.

Figure 2. The mark card.

PORTAL

A Challenge

You have to measure the outline of this portal:



Use only a ruler: you must measure in the most precise way. How can you do it? Explain how you do it.

Figure 3. The portal card.

In these activities that are referred to as 'mark' and 'portal' cards, the investigated subject is the approach of students to problems of measure (of either areas or lengths) that cannot be solved by using formulas and methods usually performed at school, in a simple way. The results show deep differences: in our opinion an unusual question about area mobilizes the pupil's pre-conceptions more than a (mathematically) similar question about the length.

Tables 1 and 2 summarize the frequencies of answers to the mark and portal cards.

Table 1. Frequencies of answers to mark card

Answer types	Primary school 5th grade	Middle school 6th-8th grade	High school 9th-13th grade
0	25%	18%	18%
1a	12%	0%	5%
1b	5%	3%	7%
2a	0%	10%	8%
2b	3%	10%	8%
3a	0%	17%	24%
3b	0%	30%	10%
4	55%	12%	12%
5	0%	0%	1%
0	**No strategy**		
1	**Pupil does not acknowledge that measure can be considered as the computation of squares**		
1a	Pupil tries to decompose the mark using geometrical well-known shapes		
1b	Pupil uses other strategies		
2	**Pupil stops to the first division into squares**		
2a	Pupil shows awareness of approximated measure		
2b	Pupil seems to be unaware of the approximation of the measure		
3	**Pupil stops to the second division into squares**		
3a	Pupil shows awareness of approximated measure		
3b	Pupil seems to be unaware of the approximation of the measure		
4	**Pupil guesses that the procedure can be repeated with division into smaller squares**		
5	**Pupil guesses that the solution exists only as a limit of iteration of approximations**		

Table 2. Frequencies of answers to portal card

Answer types	Primary school 5th grade	Middle school 6th-8th grade	High school 9th-13th grade
0	8%	13%	12%
1	36%	16%	23%
2a	1%	5%	19%
2b	49%	64%	33%
3a	6%	2%	6%
3b	0%	0%	5%
4	0%	0%	2%
0	**No strategy**		
1	**Pupil uses a string or similar things (arcs of circumferences) avoiding approximation**		
2	**Pupil stops at a first approximation**		
2a	Pupil shows awareness of approximated measure		
2b	Pupil seems to be unaware of the approximation of the measure		
3	**Pupil tries to improve the previous result**		
3a	Pupil stops at the second attempt		
3b	Pupil produces further attempts		
4	**Pupil guesses that the solution exists only as a limit of iteration of approximations**		

The mark problem seems more difficult than the portal one, in the sense that in the second one there is a greater frequency of attempts showing the use of a strategy. It is noteworthy that there is a substantial stability of frequency rate of pupils which do not display any strategy passing from primary to high school. We can assess the answers of types 2a, 3a, 4 and 5 in mark card, as the answers of pupils conscious (at different degrees) of the role of approximation. The frequencies are 55% for Primary school, 39% for Middle school and 45% for High school. We assess the intuition of improvement of the procedure by iteration as a proof of a pre-conception of limit (answers of types 4 and 5), 55%, 12% and 13% respectively. In the portal card the role of answers of types 4 and 5 of the mark card is played by the answers of types 3b and 4. In the second card the intuition of limit seems blocked by the possibility to reduce the question to a linear measure made by means of a string or similar thing.

Remark that the task in portal card mentions the ruler explicitly, as the unique allowable tool. In this card the refusal of approximation has higher rates: the sum of types 1 and 2b gives: 85% for Primary school, 80% for Middle school and 56% for High school.

Analysing the students' work, we could infer that intuitions connected with the idea of approximation and/or limit are present in any school level, including the primary school. We did not see, however, that schooling promotes any evolution of these ideas, neither in a quantitative way (because intuitions are always present in a similar percentage) nor a qualitative way (because in any case we observe only primary intuitions which are not developed in secondary ones). Therefore, even if our sample cannot be considered as representative, we can confirm our hypothesis that these intuitions, in the didactic practice, do not find the chance to develop and become consolidated and even blocked by the insistence of school requirements of exact formulas. On the basis of our results in teachers' training courses we warn teachers against teaching methods that are too much based on automatisms, since they can obstruct the production of 'natural', good intuitions.

For example, in the 'mark card' and the 'portal card' some of the youngest students took care to underline the approximation of the result: they used expressions like *the measure is about*..., *approximately*, etc. On the contrary, the oldest students often gave a number as a solution and are convinced of the exactness of their result on the basis of computation or formulas deduced from theorems. In particular, Jacqueline (5th grade) realized that it is possible to find the length of the portal drawing inner and outer lines to approximate the portal (Figure 4). Another pupil of the same grade states: «I could take a transparent paper, I could superpose it on the figure tracing the border, then I could put it on the graph paper (millimetre paper) and so I could understand that the area is about 7 cm^2. I have found the area by counting roughly the cm^2 which could stay in the figure[4].» (Figure 5):

[4] "Prenderei un foglio di carta da lucido, lo appoggerei sulla figura ricalcando il contorno, poi lo metterei sulla carta millimetrata e così capirei che l'area è di circa 7 cm^2. Ho trovato l'area contando pressappoco i cm^2 che ci potevano stare nella figura."

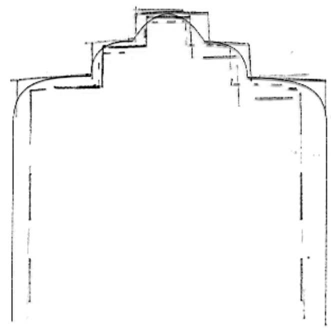


Figure 4. The Jacqueline portal protocol.

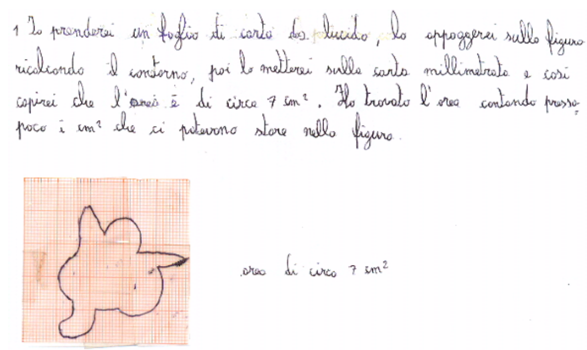


Figure 5. The mark protocol of an 11 year old student.

Some high school students seem to seek comfort in computations (Figure 7) and/or theorems avoiding approximation. A 17 old student states «It would be sufficing to look at what did in the figure and do pythagoras' theorems. My principle is that the hypotenuse of the found chord has the same measure of the found arc.[5]» (Figure 6):

[5] "Basta guardare ciò che ho fatto alla figure e fare i teoremi di pitagora. Il mio principio è che l'ipotenusa delle corde trovate abbia la misura eguale dell'arco trovato."

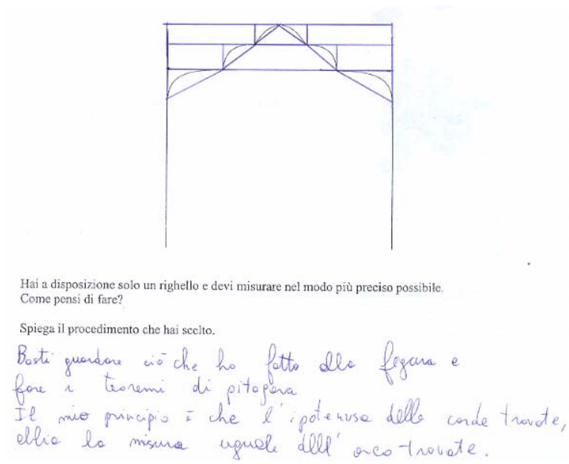


Figure 6. The portal protocol of a 17 year old student.

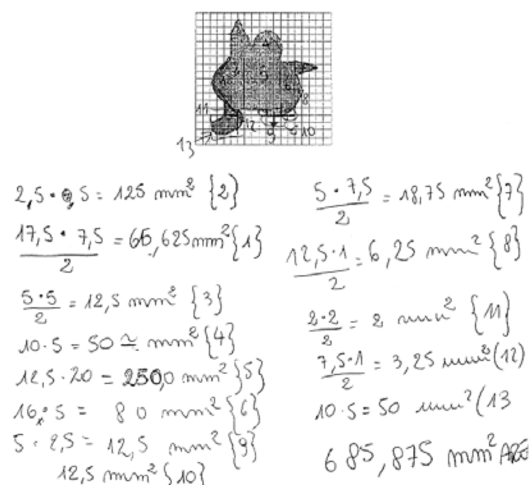


Figure 7. The mark protocol of a 16 year old student.

The teaching methods used, as we said above, frequently based on automatisms, appeared to have had mainly the effect of introducing reliance on automatic mechanisms and to have switched the pupils' attention to calculation and the search for the most appropriate formulas rather than concentrating on critical reasoning. In other words, not only does such an intuition appear not to evolve over the pupils' school career, but generally tends to degenerate, at least in our sample of students. We see here an obstacle of didactical nature.

Taking account of this risk of degeneracy, in pre-service training courses, we stressed the importance of a correct detection of pupils' and students' pre-conceptions. We emphasise the role of a teacher's heedful attention in order to catch each early appearance of main mathematical concepts even if teaching these subjects is out of the scope of the teacher's work programme for that school year.

Our Research: From Practice to Theory

A theoretical reflection about teaching of limits

The concepts of limit, continuity, derivative and integral of real functions are generally introduced in a rather formal way either in the final year or the last two years of all Italian High schools. The approach most followed is enriched by technical details and formal proves. The application of such concepts, though, is often limited to routine exercises (calculations of limits, variation of functions, …). As a consequence, learning is somewhat mechanical and superficial (Impedovo [31]), in particular the concept of limit does not remain deeply rooted (Furinghetti and Paola [23]). The identification by many researchers of a variety of different difficulties and obstacles capable of hindering the process of construction of the concept of limit is well known; these are

- of an epistemological nature, due to reasons which are internal to mathematics itself (Sierpinska [38], Brousseau [7]);
- of a didactic nature, due to teaching methods which are not always effective (CREM [13], Dimarakis and Gagatsis [15], Artigue [5], Groupe AHA [24]);
- of a cognitive nature, due to the abstraction and conceptualisation processes involved (Cornu [12], Dubinsky [16], Sfard [37], Tall [39]);
- of a meta-cognitive nature, due to the overall attitude with which students generally tend to approach mathematics (Zan [44]).

Taking into account the underlined theoretical framework and the remarks related to the inquiries described in the above paragraph, our group is particularly convinced of the necessity of a long-term construction of the concept of limit starting from the pupils' prior experience and intuition.

On the basis of what we have seen above, it appears to us that the essential point is to identify teaching strategies and constructive activities capable of enriching the learning experience and stimulating evolution of pre-conceptions towards the theoretical concept of limit rather than concentrating on the formal aspects. It is hoped that pupils are able to use these tools to build on their own experience from their first years at school. We are aware however, that such evolution is far from easy or natural, and that there are often real conflicts between 'naive' ideas and intuitions and the mathematical concepts under construction with the consequent need, as far as the pupils are concerned, of a continuing re-organisation of their mental images (Mamona-Downs [33]). We believe that choices made by the teacher are of fundamental importance in sustaining or (involuntarily) impeding such a process. Another risk is that of undervaluing the pupils' convictions and attitudes towards themselves and their relations with mathematics, which particular teaching techniques may cause to become real obstacles to learning.

In particular, it seems crucial to us that the work of the teacher should be concentrated on three levels, as follows:

- Contents: in our opinion it needs a lot of work designed to familiarise pupils with the concept of limit starting from the 'naive' mental images, then continuing with successive levels of abstraction. The basic materials for this are already present in the curricula applying to the various Italian schooling levels. The teacher should not, however, be too concerned to keep the problem 'hidden' until the last year of school, when it will be formally defined.

- Methodology: if teacher just relies to calculation procedures and sets exercises applying special formulas or techniques only, then s/he focuses attention on the result (product) rather than on the solving strategy (process). On the contrary it is appropriate to give problems whose solution requires alternative strategies including approximate methods.

- Meta-cognition: we maintain that it is appropriate to explain and discuss the students' attitudes and convictions; a critical awareness of their 'implicit beliefs' appears to us to be an essential element in the conscious construction of their own knowledge and in their consequential ability to mobilise and use them again.

Therefore an effective pre-service teacher education becomes fundamental, in particular for what concerns the necessity of managing at the same time the three levels mentioned above.

Approximation as a teaching resource for the construction of the concept of limit by suitable activities

In previous paragraphs we saw that the subject of measure provides many starting points which can give rise to conflicts and stimulate pupils' strategies. We wish to underline the importance of using this subject from the point of view of approximation, as a way to move towards the concept of limit. As far as the first point (i.e., contents) is concerned, it is important to underline above all the fact that, when approaching the concept of limit, within recourse to infinity in act, it is appropriate to do so through an explanation of potential infinity first. In this sense, approximation as a possibility of an ever closer and testable approach to a theoretical limit, represents an excellent opportunity for such an explanation[6].

The activities we prepared within our research were designed to achieve the evolution of an intuitive idea of limit as a boundary or barrier towards the construction of a process which is capable of continuing improvement and can be controlled *a priori*.

As we emphasised above, as pupils progress through the school system, approximation appears, at least in Italy, to lose its legitimacy and become something extraneous, almost harmful, to mathematical activities. Teaching methods and a kind of implicit agreement seem to generate beliefs in pupils that become increasingly widespread and well-rooted as they grow older.

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[6] Another subject where we have already conducted research and which we consider fertile ground in this context, is that of geometrical progressions (Grugnetti et al. [29]).

An example of such a belief is that mathematics must necessarily be used explicitly when solving any mathematical problem; in some way, the freshness of the younger schoolchildren is lost, together with their readiness to have recourse to common sense or 'empirical' solving strategies. Otherwise, students of Middle and High schools become more and more convinced that each mathematical problem:

(a) is solved (and therefore must be solved) through a formula;

(b) has a result in round numbers (a whole or decimal number up to two decimal places).

It seems to us that in the Italian school system there is not sufficient insistence on the fact that for many mathematical problems there is no immediate formula (and sometimes there never will be) and that often the result is not a round number. This does not mean that such problems are less interesting and significant. Their crux lies not so much in the calculation of a 'precise' result, it is rather the search for a testable and generalised approximation procedure.

Such 'approximation methods' are disdained by certain teaching practices, yet constitute an essential aspect of analysis. Calculus above all, modelling the real world, cannot avoid the problems relating to approximation, connected to every measuring operation.

From a teaching point of view then, it is precisely because of the empirical/experiential character of approximation that can constitute a valid bridge for the construction of thought processes which, by analogy, proceed from the concrete to the abstract.

In particular, the ε-δ definition of limit may be seen in terms of approximation, as the setting the degree of error that one is prepared to accept a priori (ε) and choosing, as a consequence, an appropriate strategy (δ) to be sure not to exceed such a tolerance threshold.

Our Research: From Theory to Practice for a Long-term Construction of the Concept of Limit

We present here only few excerpts of activities that in our opinion could contribute to the statement of a correct conception of limit. Other examples can be found in Andriani et al. [3]. The main difficulties in the choice of such activities are that these ones have to be suitable and non trivial for every school grade. Moreover these activities could display the right pupils' intuitions and to be «error triggering tasks» (i.e., tasks that are known to elicit incorrect responses) (Tsamir and Tirosh [41]) in order to produce a useful conflict, a first step to build a new knowledge.

The following examples are more complex and complete as to the previous ones, especially for the addition of *a priori* analysis. From pre-service teacher education point of view these examples could be used as activities to be performed in the classroom. Our *a posteriori* analysis gave important hints for the teacher's *a priori* analysis for these or other activities in the same topic. In this way the research could have a positive spin-off for teacher education.

The a priori *analysis*

The *a priori* analysis (Charnay [9]) is one of the teachers' professional tools to be used when taking decisions to anticipate pupils' reactions and, therefore, direct some teachers' choices.

In particular, the *a priori* analysis of a problematic situation is a work on hypotheses concerning, at least:

(a) procedures, strategies, reasoning, solutions which the pupil could use in the situation proposed taking into account the supposed knowledge: can s/he start to work out the problem?

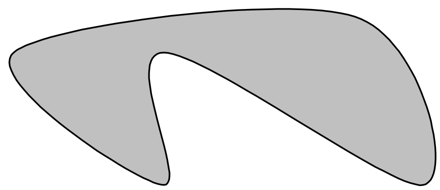
(b) difficulties s/he can have and errors s/he could make: in particular, does the situation allow the pupil to put into evidence his/her wrong conceptions?

We present in more details two activities, tried out in six classes at different levels within the school system (Falcade and Rizza [19]). These have been designed, on the one hand, to make pupils' convictions explicit concerning the points (a) and (b) (in the previous paragraph), and on the other hand, restoring approximation to its proper place in mathematics education.

The little lake card and its a priori *analysis*

The ‘little lake’ is a problem concerning the area of an irregular figure. These kinds of problems are rather unusual in Italian schools, where great attention is given to the search for the area of particular polygons and circles.

Imagine you are observing a little mountain lake from above and your draw the following figure:



Can we measure the area of the figure?

Figure 8. The little lake card.

Compared with the ‘mark card’ (Figure 2) this situation is given in a more open form in which instructions about the strategies needed are not given. Moreover, here there is the problem of *the existence* of the measurement of the area: actually, the aim is to verify whether it is connected with the presence of a formula. Finally, we wish to study the relationships students establish between the existence of a result and the possibility of finding it in a precise way or, if that is impossible, in an approximate way.

The aim of this problem is that of testing whether a student:

(1) accepts the possibility of calculating the area of an irregular figure,

(2) is willing to use empirical methods instead of rules and formulas,

(3) becomes aware of the necessity of approximation for the measure.

We think that this problem could well introduce the subject of approximation by provoking a conflict between the intuition that the area exists and the impossibility of calculating it exactly.

The little lake card and its a posteriori *analysis*

The problem was proposed to the pupils of 15 classes at different school levels.

Here we consider only the results concerning four classes, each belonging to different age groups.

Table 3 indicates the percentages of affirmative, negative and absent answers, to the first question involving the possibility of measuring the area:

Table 3

Class	Age	Number of pupils	yes	no	no reply
Cl 1 (grade 4)	9	19	0%	100%	0%
Cl 2 (grade 8)	13	22	9%	82%	9%
Cl 3 (grade 9)	14	22	59%	41%	0%
Cl 4 (grade 11)	16	12	67%	17%	17%

We are aware that percentages on such a small sample cannot have a statistical value, but in this form the comparison here and in the other tables is facilitated.

We can see the increase in positive responses with the age («We can measure the area of the figure») and therefore the previously stated aims (1) and (2) would be reached. The results seem to indicate a positive influence of the schooling on the construction of the concept of area. By the analysis of the responses to question 2 («If it is possible, in which way? If it is not possible, why?»), we will show that, even if the existence of the area is accepted, many pupils do not suggest any strategy or propose wrong constructions. The pupils' beliefs appear to be in opposition with approximation as a tool for solving the problem and for this reason the aim number (3) is not reached.

The negative answers have been classified on the basis of mathematical contents present in pupils' argumentations. The results are summarized in Table 4:

Table 4

Class	**N1**	**N2**	**N3**
Cl 1	68%	0%	32%
Cl 2	73%	0%	9%
Cl 3	32%	9%	0%
Cl 4	8%	9%	0%
N1: because of the impossibility of referring to known figures (qualitative/quantitative description)			
N2: because of the impossibility of referring to known formulas			
N3: other answers (non pertinent ones)			

As for the younger pupils, it can be noted that the qualitative and quantitative reasons (N1) are more frequent, whilst for the oldest ones the reason is «there are no formulas» (N2).

In the following Table 5 we present the percentage of affirmative answers, skipping Cl 1. The reasons for the affirmative answers have been classified according to the degree of applicability of the proposed strategy.

Table 5

Class	**Y1**	**Y2**	**Y3**	**Y4**
Cl 2	0%	4%	5%	0%
Cl 3	13%	9%	32%	5%
Cl 4	34%	0%	33%	0%
Y1: I do not know how, I do not have appropriate tools, there are no formulas…				
Y2: I can transform the figure into a polygon having the same area…				
Y3: I can cut the figure into polygons or I can use a grid…				
Y4: other strategies (e.g., *weighing it*)				

The first actual attempt for solving the problem is that of transforming the figure into an equivalent polygon (Y2), but with the ambiguity and conflict of perimeter and area. Finally, the correct responses are Y3, but even in these cases, the analysis of the explanations shows negative connotations: pupils use the word ‘approssimativo’ (rough) instead of ‘approssimato’ (approximate, in mathematical meaning).

The dynamical idea of iteration and of gradually approaching is not present in any answer.

The little lake card - interview and discussion in a class

In Cl 4 class (students aged 16) we interviewed students on their responses. During the discussion student-student, teacher-student, researcher-student, the whole class agreed on the grid strategy, as being the best one. We interpreted this fact as the overcoming of the conflict between intuition (the area exists) and computation difficulty of area.

However, some students expressed doubts about the accuracy or precision of this method. So, we asked them «What does the phrase “to measure in a precise way” mean?»

Two meanings arose:

1. to cover the whole figure,

2. to find a ‘good’ result, i.e., a round number.

On the contrary, for the students, the measure of the area of a rectangle is an example of a measure which is always ‘precise’.

«What happens whether we use a smaller grid?» the teacher suggested.

When they tried, students reached ‘another non precise result’, different from the previous one, but they were still not convinced of the precision/convergency of an iterative process. One of them, Federica said: «It is impossible to reach a result because we cannot stop the numbers».

The target game and its a priori *analysis*

The second activity (cf. Fischbein [22]) is a game based on approximation and it is possible to use it at whatever the school level is.

The teacher marks a point on an A4 sheet of paper and each pupil, after having seen it, has to mark on his/her own A4 sheet a point in such a way that, once this sheet superposed on the teacher’s sheet, the two points coincide (more details in Falcade and Rizza [19] and in Andriani et al. [3]). No measure tools are allowed.

The aim is introducing the subject of approximation as a resort when it is impossible or very unlikely to find an acceptable result: in effect, the teacher's point represents a precise value, that is very difficult to reproduce, while the pupils' points can be considered different approximations of this value.

Our hypothesis is that this activity leads to reflect on the existence of approximate and acceptable values. Moreover, this activity allows to make conjectures and to work on criteria concerning the quality of the approximation.

In every class we proposed the activity following the outline below:

Everybody has an A4-sized sheet of paper.

The teacher marks a point on a paper and shows it to pupils.

S/he asks pupils a faithful reproduction on their own papers of her/his drawing and set the following rules:

(a) the pupil who marks a point exactly in the same place as the teacher wins;

(b) to find the winner, each sheet will be placed on the teacher's one.

The pupil play.

The teacher marks the set of points on a transparent paper (with a number to identify each pupil).

Probably nobody reaches the teacher's point.

In this case, before the pupils can see their points, the teacher asks them to discuss, how it is possible to decide who won and to make the final results[7]. After they chose the new rules, the teacher shows to each student her/his point.

Then the pupils play again, trying to improve their results.

At the end, each pupil has to answer the following questions:

«Having seen what happened today, why can we accept a result which is not exact? When?»

Figure 9. The target game outline.

By this activity we would like to undermine the certainty of always finding exactly the result of a problem and to criticize the absolute power

[7] The expected answer is to draw on a transparent sheet a sequence of concentric circles around the point originally marked by the teacher. This strategy can be suggested from pupils' experience of target practice or playing darts and it can give a graphical evidence of the results. In doubtful cases other measure are permitted. At the end it is possible the case of many winners.

of formulas. Our aim is the acceptance of methods of approximate resolution necessary to solve effectively some kinds of problems. We would also like to shift the attention from the search for a precise result to the search for an acceptable one, on the basis of the quality of the approximation.

Besides these cognitive aims, from a meta-cognitive point of view, it is very important the discussion and the legitimization of a new system of rules that allows to choose the winners. The possible preview strategies of the pupils, when they play for the second time, could be:

- to play randomly (as they probably did the first time),
- to play by estimating the distance between the new point and the teacher's point,
- to play, taking into account:
  - the position of one's own point in relation with the other ones,
  - the rules established, i.e., the sequence of concentric circles settled.

We must remark that going from first to last strategy means having an evolution from a static vision (approximation as a rudimentary operation for measuring) to a dynamic vision of approximation (process which could be always improved).

Using last strategy implies the consciousness of the existence of, on the one hand, a convergent sequence of points and, on the other hand, an iterative process. Approximation takes on the dynamic role of a measuring process which can always be improved.

The target game and its a posteriori *analysis*

This activity was proposed in 7 classes to pupils aged 14 to 17.

In all the classes the development was similar: after the first step where the first system of rules did not work, the pupils suggested a new system of rules in order to choose the winner as the person (the point) who nears most the teacher (i.e., the teacher's marked point).

The main didactical result we obtained by this activity was the acceptance of the need for approximation and therefore the aim of the activity can be considered as reached. It might seem to be a trivial result, on the contrary, this is the first 'indispensable' step if we wish, by a long learning process, to give a meaning to the construction of the concept of limit. It is noteworthy that this activity is mathematically rich: pupils are faced by an intuition of circular neighbourhoods of a point (different from the standard notion of neighbourhood as an interval). Another choice of detecting distance between teacher's and pupil's marks could use the 'natural' Cartesian coordinates given by the edges of the sheet and a scale, but in this way there is an application of Pythagorean theorem (Fischbein [22]).

Some answers to the questions «Having seen what happened today, why can we accept a result which is not exact? When?» show that the pupils are conscious of the need of a different way of reasoning, when a problem could not have a precise result and that this different way of reasoning can be acceptable: *We can accept it because we can have a* "*tolerance*", *i.e.*, *a* "*margin*" *which allows us to consider the activity as a success* (grade 9).

We can accept an imprecise result because sometimes we cannot have a precise one: *when*, *after we tried to arrive to the better result* (*approximation*) *we cannot have a precise one* (grade 11).

In a 14 years old pupils class (grade 9), before the beginning of the game, the pupils said that no winner was possible: the activity became an occasion for thinking on the reasons of the impossibility of having a winner.

Some Final Remarks

Thanks to the activity of 'The Little Lake' we provoked our students by leading them to question certain beliefs.

Thanks to the 'Target Game' we were able to raise doubts on those beliefs, by using a rich and unusual problematic context.

Pupils accepted the need for approximation as a process, but we are not sure if they accepted approximation as a mathematical result perhaps as a consequence of their own previous school education.

It is possible to plan other activities to justify the use of approximation even in cases in which the presence of a formula seems to suggest the exactitude of the result. In particular, we experienced some activities about rectangles divided into squares and about the area of a triangle (Andriani et al. [3]).

Until now our empirical research had mainly a diagnostic character: on the one hand, we found some favourable intuitions (in the younger pupils), on the other hand some wrong beliefs (in the older students). It means that something is not working effectively in frequent school practice; it is necessary to start again from good intuitions and develop them to begin a lengthy and constructive process for the establishment of the concept of limit. For this process it is important, in our opinion, to revalue the reasoning in terms of approximation as an essential and integral part of mathematics, and not as an expedient for lacking exact methods.

The ultimate aim of our research is to construct a 'conceptual map' of activities that teachers can perform in order to develop the students' pre-conceptions connected with limit, at any school level. In this way our research, starting from the diagnosis about the difficulties of the concept of limit become a construction research about the same concept.

We believe, for example, that the measure problem of an irregular shape (length, area and volume) (Marchini [34]) can offer a real opportunity to gradually familiarise with the idea of limit, because it naturally leads to the approximation methods, taking account also of the historical development of the infinitesimal calculus, where many remarkable applications preceded the correct arrangement of the general theory (Grugnetti and Jaquet [27]).

However, we think that the teaching practice (at least in Italy) dedicates today too much time to determining measure for special classes of polygons and polyhedra and delays unnecessarily, until the last year of the secondary school, the study of the general measure problem. Most of the exercises found in Italian textbooks are just related to using a formula or a special technique. This fact increases the above mentioned wrong beliefs in students. It is important to present significant problems which cannot be immediately translated into mathematical terms and where more conjectures and argumentation than calculations are required.

In "*Didactique*", *the value of a lesson is in what it permits future lessons to achieve*, *the things that would not work if this lesson had not taken place*. *This value shows up in terms of possibilities for the students* (*opportunities for learning*) *and possibilities given to the teacher*. (Brousseau [8]).

Moreover, today's secondary school does not exploit manual abilities, which the students learned to use in the primary school and neglects the approximation techniques.

But, in our opinion it is important, on the one hand, to go on using empirical measure and improving them with more advanced tools, such as computers, and on the other hand, to give some example of historically relevant techniques before developing the general theory of calculus.

All the activities we mentioned fulfil a primary need: reinforcing the students' pre-conceptions and leading their intuitions in view of a useful and justified approach to the analysis. We know that many researchers agree with this position, e.g., CREM [13] says, about the definition of limit: *It is interesting to construct Analysis in terms of* "*mental objects*" *until it is clear that these are no more sufficient and it is possible to understand the reasons why the* ε-δ-*formalism is required*.

These remarks require teachers to have deep knowledge of contents, even in their historical development, in connection with the learning process of pupils. Moreover, the teacher must have the 'strategic' competence of interpreting unexpected answers of the pupils and continually reintroducing stimuli. This awareness can be achieved only through a complete and constant teacher education, ultimately renewing teaching practice in Italy.

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