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THE LAPLACIAN SPECTRAL RADIUS STUDY OF A SPECIAL TYPE OF GRAPHS

SHUHUA YIN

Institute of Mathematics
Zhejiang Wanli University
Ningbo 315100, P. R. China
e-mail: quitshy@hotmail.com

Abstract

Let G be a simple graph. Then the Laplacian matrix $L(G) = D(G) - A(G)$ is the difference of the diagonal of vertex degrees and the $0-1$ adjacency matrix. In this paper, we investigate the effect on the Laplacian spectral radius of a graph by grafting an edge, and give some results of bicyclic graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E = \{e_1, e_2, ..., e_m\}$ in which $n = |V|$ and $m = |E|$. Let $d(v_i)$ denote the degree of $v_i \in V$, $i = 1, 2, ..., n$, and $D = D(G)$ be the diagonal matrix of vertex degree. The matrix $A = A(G)$ denotes the adjacency matrix of the graph G, then the matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix of* G. One may also describe $L(G)$, by means of its quadratic form:

$$X^T L(G) X = \sum_{i<j,\, v_i v_j \in E(G)} (x_i - x_j)^2,$$

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where $X = (x_1, x_2, \ldots, x_n)^T$. So $L(G)$ is a symmetric, positive semidefinite matrix, denoted its eigenvalues by

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0.$$

The largest eigenvalue of $L(G)$ is called the *Laplacian spectral radius of the graph* G, denoted by $\mu(G)$.

The Laplacian spectral radius of graphs has many results [1, 2, 3, 4, 5].

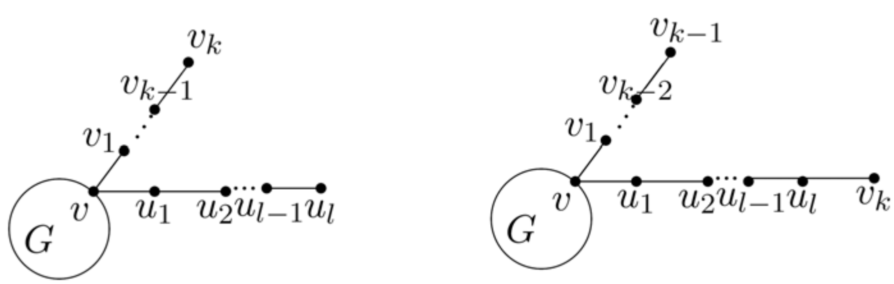
Lemma 1.1 [5]. *Let G be a graph on n vertices with at least one edge. Then* $\mu(G) \leq n$.

Lemma 1.2 [5]. *Let G be a connected graph on n vertices with at least one edge. Then* $\mu(G) \geq \Delta(G) + 1$, *where* $\Delta(G)$ *is the maximum degree of the graph G, with equality if and only if* $\Delta(G) = n - 1$.

Lemma 1.3 [5]. *Let G be a connected graph on n vertices with at least one edge. Then* $\mu(G) \leq d_1 + d_2$, *where* d_1, d_2 *are the maximum and the second maximum degree of the graph G.*

2. The Laplacian Spectrum of Graphs $G_{s,t}$ and Graph $G'_{s,t}$

In this section, we will consider the effect on the Laplacian spectral radius of a graph by grafting an edge. Guo [4] determined how the Laplacian spectral radius behaves when the graph is perturbed by adding or grafting edges (Figure 1).



$G_{k,l}$ $G_{k-1,l+1}$.

Figure 1. $G_{k,l}$ and $G_{k-1,l+1}$.

Lemma 2.1 [4]**.** *Let G be a connected graph on $n \geq 2$ vertices and v be a vertex of G. If $l \geq k \geq 1$, then $\mu(G_{k-1, l+1}) \geq \mu(G_{k, l})$, with equality if and only if there exists a unit eigenvector of $G_{k, l}$ corresponding to $\mu(G_{k, l})$ taking the value* 0 *on vertex v.*

Let $G_{s, t}$ denote the graph that is satisfied for conditions as follows: $v_s, v_t \in V(G_{s, t}), v_s v_t \notin E(G_{s, t}), d(v_s) > d(v_t)$ and v_s, v_t have the same neighboring vertex set N. Besides that, all the other vertices that are adjacent with v_s and v_t are dependent. v_0 is a dependent vertex of adjacent to v_t, and let $G_{s+1, t-1}$ be the graph obtained from $G_{s, t}$ by deleting the edge $v_t v_0$ and adding the edge $v_s v_0$ (Figure 2).

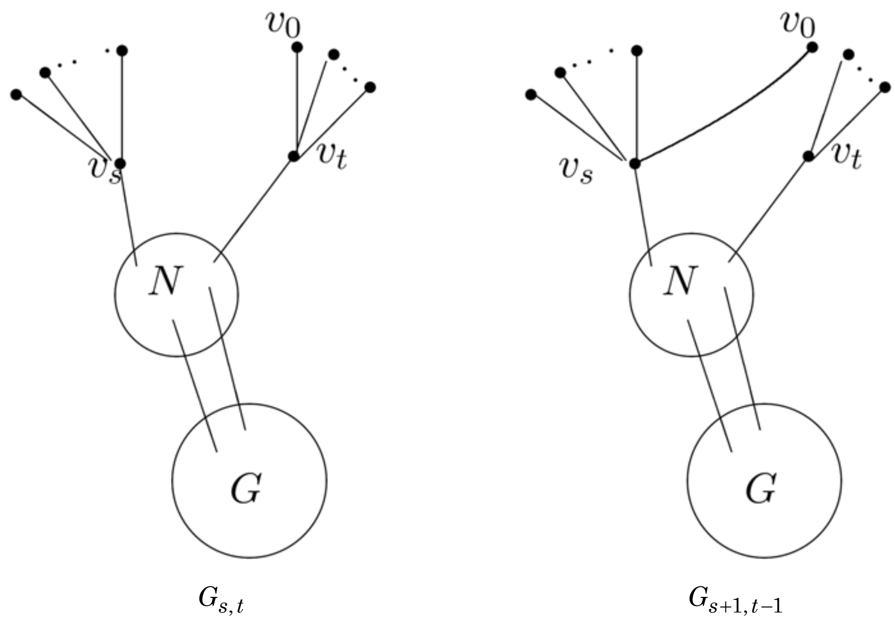


Figure 2. $G_{s, t}$ and $G_{s+1, t-1}$.

Now we consider the Laplacian spectrum of graph $G_{s, t}$.

Theorem 2.1. *Let X be a unit eigenvector of $G_{s, t}$ corresponding to $\mu(G_{s, t})$, where x_v corresponds to the vertex $v(v \in V(G_{s, t}))$. Then we have $|x_{v_s}| > |x_{v_t}|$ and $x_{v_s} x_{v_t} > 0$.*

Proof. By the definition of Laplacian matrix of graph G, we have $L(G_{s,t})X = \mu(G_{s,t})X$ and

$$[d(v_s) - \mu(G_{s,t})]x_{v_s} = \sum_{v_i \in N} x_{v_i} + \frac{[d(v_s) - |N|]x_{v_s}}{1 - \mu(G_{s,t})}, \tag{1}$$

$$[d(v_t) - \mu(G_{s,t})]x_{v_t} = \sum_{v_i \in N} x_{v_i} + \frac{[d(v_t) - |N|]x_{v_t}}{1 - \mu(G_{s,t})}. \tag{2}$$

By (1)-(2), we have

$$[\mu^2(G) - (1 + d(v_t))\mu(G_{s,t}) + |N|](x_{v_s} - x_{v_t}) = \mu(G_{s,t})[d(v_s) - d(v_t)]x_s. \tag{3}$$

According to Lemma 1.2 we have obtained $\mu^2(G_{s,t}) - (1 + d(v_t))\mu(G_{s,t}) + |N| > 0$, so $|x_{v_s}| > |x_{v_t}|$.

By (3) we have

$$(\mu^2(G_{s,t}) - (1 + d(v_s))\mu(G_{s,t}) + |N|)x_{v_s}$$
$$= (\mu^2(G_{s,t}) - (1 + d(v_t))\mu(G_{s,t}) + |N|)x_{v_t}. \tag{4}$$

Also note that $\mu^2(G_{s,t}) - (1 + d_{v_t})\mu(G_{s,t}) + |N| > 0$ and $\mu^2(G_{s,t}) - (1 + d_{v_s})\mu(G) + |N| > 0$, so $x_{v_s}x_{v_t} > 0$.

This completes the proof.

Applying Theorem 2.1, we can prove

Theorem 2.2. $\mu(G_{s,t}) < \mu(G_{s+1,t-1})$.

Proof. Let X be a unit eigenvector of $G_{s,t}$ corresponding to $\mu(G)$. Then

$$X^T L(G_{s+1,t-1})X = X^T L(G_{s,t})X + (x_{v_s} - x_{v_0})^2 - (x_{v_t} - x_{v_0})^2$$
$$= X^T L(G_{s,t})X + (x_{v_s} - x_{v_t})(x_{v_s} + x_{v_t} - 2x_{v_0}),$$

since $[1 - \mu(G_{s,t})]x_{v_0} = x_{v_t}$, so $x_{v_0}x_{v_t} < 0$. And by Theorem 2.1, we have $(x_{v_s} - x_{v_t})(x_{v_s} + x_{v_t} - 2x_{v_0}) > 0$, so $X^T L(G_{s+1,t-1})X > X^T L(G_{s,t})X$, then we obtain

$$\mu(G_{s+1,t-1}) = \max_{\| Y \|=1,\ Y^T e=0} Y^T L(G_{s+1,t-1})Y > X^T L(G_{s,t})X = \mu(G_{s,t}).$$

This completes the proof.

Let $G'_{s,t}$ denote the graph that is satisfied for conditions as follows: $v_s, v_t \in V$, $v_s v_t \in E(G'_{s,t})$, $d(v_s) > d(v_t)$ and v_s, v_t have the same neighboring vertex set N. Besides that, all the other vertices that are adjacent with v_s and v_t are dependent, but $d(v_s) \neq \Delta(G'_{s,t})$. v_0 is a dependent vertex of adjacent to v_t, and let $G'_{s+1,t-1}$ be the graph obtained from $G'_{s,t}$ by deleting the edges $v_t v_0$ and adding the edges $v_s v_0$ (Figure 3).

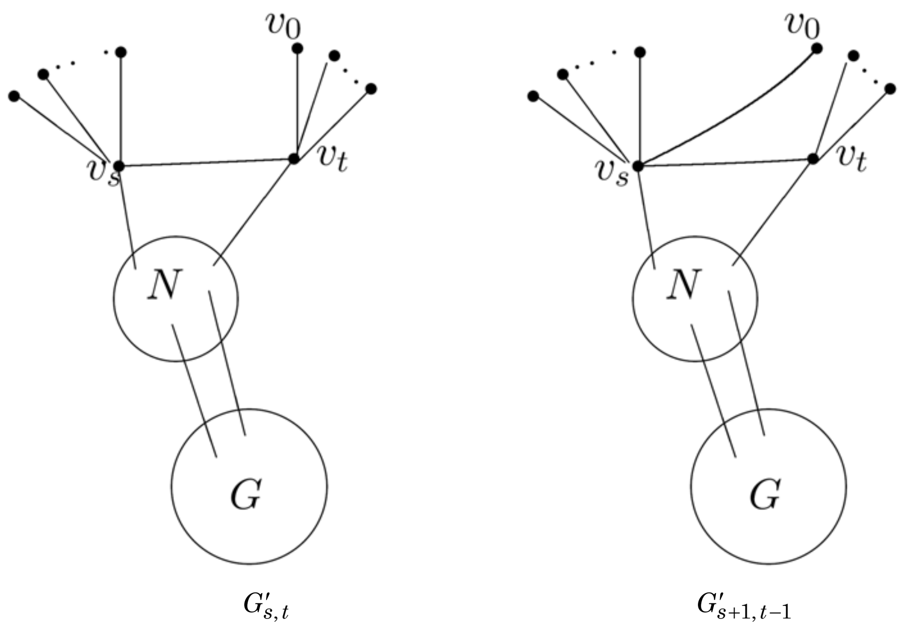


Figure 3. $G'_{s,t}$ and $G'_{s+1,t-1}$.

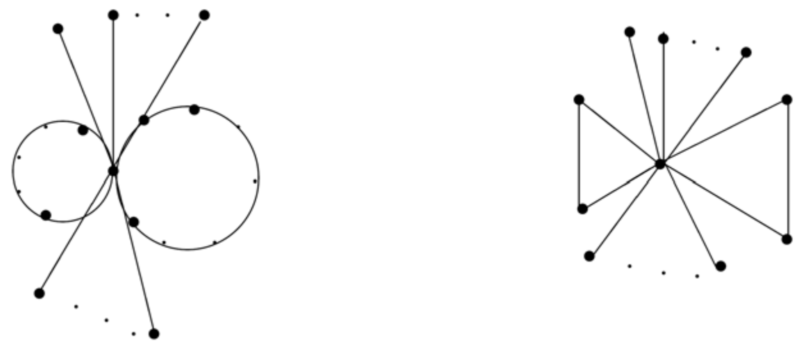
By the same way, we can get a corollary as follows:

Corollary 2.1. $\mu(G'_{s,t}) < \mu(G'_{s+1,t-1})$.

3. The Laplacian Spectral Radius of a Type of Bicyclic Graphs

Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one.

Denote by C_p the cycle with n vertices. Let $A_{p+q-1}(p, q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by identifying vertices u_0 of C_p and v_0 of C_q. Bicyclic graph $A_n(p, q)$ can be obtained from $A_{p+q-1}(p, q)$ by attaching trees to some vertices of $A_{p+q-1}(p, q)$ and $|V[A_n(p, q)]| = n$. Specially, the bicyclic graph $S_n(p, q)$ obtained from $A_{p+q-1}(p, q)$ by attaching a star graph $K_{1, n-p-q-1}$ to the vertex u_0 (Figure 4).



$S_n(p, q)$ $S_n(3, 3)$

Figure 4. $S_n(p, q)$ and $S_n(3, 3)$.

Applying Lemmas 1.2, 1.3, we easily obtain Corollary 3.1, 3.2.

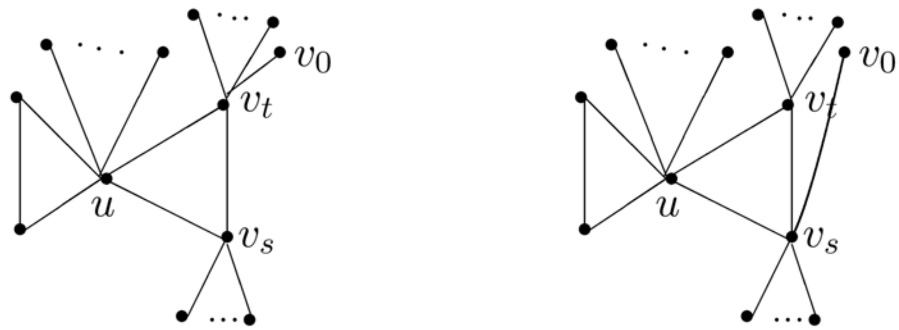
Corollary 3.1. $\mu(S_n(3, 3)) = n \geq \mu(A_n(p, q))$, *the equality holds if and only if* $A_n(p, q)$ *is isomorphic to* $S_n(3, 3)$.

Corollary 3.2. $\mu(S_n(p-1, q)) > \mu(S_n(p, q))$, *where* $p \geq 4$.

Proof. The maximum degree of $S_n(p, q)$ is $n - p - q + 5$ and the second largest is 2. From Lemmas 1.2 and 1.3, we have $n - p - q + 6 \leq \mu(S_n(p, q)) \leq n - p - q + 7$. Clearly, $n - p - q + 7 \leq \mu(S_n(p-1, q)) \leq n - p - q + 8$.

This completes the proof.

Let graph $A_n^0(3, 3)$ (Figure 5) be a bicyclic graph, $d(u) = \Delta(A_n^0(3, 3))$ and $d(v_s) > d(v_t)$, $v_t v_0 \in E(A_n^0(3, 3))$. $A_n^1(3, 3)$ obtained from $A_n^0(3, 3)$ by deleting edge $v_t v_0$ and adding edge $v_s v_0$.



$A_n^0(3, 3)$ $A_n^1(3, 3)$

Figure 5. $A_n^0(3, 3)$ and $A_n^1(3, 3)$.

By Corollary 2.1, we obtained

Corollary 3.3. $\mu(A_n^1(3, 3)) > \mu(A_n^0(3, 3))$.

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