Far East J. Math. Sci. (FJMS) **27**(1) (2007), 57-69
This paper is available online at http://www.pphmj.com

NEW EXACT SOLUTIONS AND PEAKON LOCALIZED EXCITATIONS FOR THE $(2+1)$-DIMENSIONAL BROEK-KAUP SYSTEM WITH VARIABLE COEFFICIENTS

SONG-HUA MA, JIAN-PING FANG and CHUN-LONG ZHENG

Department of Physics
Zhejiang Lishui University
Lishui 323000, P. R. China
e-mail: msh6209@yahoo.com.cn

Abstract

By means of an extended mapping approach and a linear variable separation approach, a new family of exact solutions of the $(2+1)$-dimensional Broek-Kaup system with variable coefficients (VCBKK) is derived. Based on the derived solitary wave excitation, we obtain some special peakon localized excitations, then we discussed the interactions between two peakons in this short note.

1. Introduction

Soliton theory is one important aspect of nonlinear science [10, 12, 24]. Because of the wide applications of soliton in many natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, and condensed matter physics, etc., [9, 17, 25]

2000 Mathematics Subject Classification: 35Q51, 35Q53.

Keywords and phrases: extended mapping approach, variable separation approach, VCBKK system, peakon localized excitation, interactions between two peakons.

Received May 4, 2007

searching for exact and explicit solutions of a nonlinear physical model, especially for new exponentially localized structures like soliton solutions or for these excitations with novel properties is a very significant work. Since the concept of dromions was introduced by Boiti et al. [1] the study of soliton-like solutions in higher dimensions has attracted much more attention. Now several significant $(2+1)$- and $(3+1)$-dimensional models, such as $(2+1)$-dimensional Kadomtsev-Petviashvili equation [11], Davey Stewartson equation [4], generalized Korteweg-de Vries equation [22], asymmetric NNV equation [21], sine-Gordon equation [13], $(3+1)$-dimensional Korteweg-de Vries equation [14] and Jimbo-Miwa-Kadomtsev-Petviashvili equation [23] have been investigated and some special types of localized solutions for these models have also been obtained by means of different approaches, for instance, the bilinear method, the standard Painlevé truncated expansion, the method of "coalescence of eigenvalue" or "wavenumbers", the homogenous balance method, the variable separation method [15, 16, 26-28, 31-35], and the mapping method [5-7, 18-20], etc. From the above study of $(2+1)$- and $(3+1)$-dimensional models, one can see that there exist more abundant localized structures than in lower dimensions. This fact hints that there may exist new localized coherent structures that are unrevealed in some $(2+1)$-dimensional integrable models.

In this paper, by the extended mapping approach, we found the new exact solutions of $(2+1)$-dimensional VCBKK system

$$u_{ty} - a(t)[u_{xxy} - 2(uu_x)_y - 2v_{xx}] = 0,$$

$$v_t + a(t)[v_{xx} + 2(uv)_x] = 0, \tag{1}$$

where $a(t)$ is an arbitrary function of time t. It is evident that when $a(t) = 1$, the VCBKK system will be degenerated to the well-known $(2+1)$-dimensional Broer-Kaup-Kupershmidt (BKK) system, which may be derived from the inner parameter dependent symmetry constant of the Kadomtsev-Petviashvili model. When $y = x$, the $(2+1)$-dimensional

Broer-Kaup-Kupershmidt system is reduced to the usual $(1+1)$-dimensional BKK system, which is often used to describe the propagation of long waves in shallow water [30]. Using some suitable dependent and independent variable transformations, the $(2+1)$-dimensional BKK system can be further transformed to the $(2+1)$-dimensional dispersive long-water wave equation and $(2+1)$-dimensional Ablowitz-Naup-Newell-Segur system [2]. Actually the $(2+1)$-dimensional BKK system has been widely investigated in details by many researcher [29].

2. New Exact Solutions to the $(2+1)$-dimensional VCBKK System

In this section, we give some exact solutions to the VCBKK system, including solitary wave solutions, trigonometric function solutions, rational solutions and Weierstrass function solutions.

Letting $f \equiv f(\xi(x))$, $g \equiv g(\xi(x))$, where $\xi \equiv \xi(x)$ is an undetermined function for the independent variables $x \equiv (x_0 = t, x_1, x_2, ..., x_m)$, the projective Riccati equation [3, 8] is defined by

$$f' = pfg, \quad g' = q + pg^2 - rf, \tag{2}$$

where $p^2 = 1$, q and r are two real constants. When $p = -1$ and $q = 1$, (2) reduces the coupled equations given in [3] and the following relation between f and g can be satisfied as $\delta = \pm 1$ and $q \neq 0$:

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2 + \delta}{q} f^2\right]. \tag{3}$$

Equation (2) had been discussed in [8]. In this paper, we discuss other cases.

Lemma. *If the condition of* (3) *holds with other choices of* δ, *then the projective Riccati equation* (2) *has following solutions*:

(a) *If* $\delta = -r^2$, *then the Weierstrass elliptic function solution is admitted*

$$f = \frac{q}{6r} + \frac{2}{pr}\wp(\xi), \quad g = \frac{12\wp'(\xi)}{q + 12p\,\wp(\xi)}. \tag{4}$$

Here $p = \pm 1$, *the Weierstrass elliptic function* $\wp(\xi) = \wp(\xi;\, g_2,\, g_3)$ *satisfies* $\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3$, *and* $g_2 = \frac{g^2}{12}$, $g_3 = \frac{pq^3}{216}$.

(b) *If* $\delta = -\frac{r^2}{25}$, *then the projective Riccati equation* (2) *has the Weierstrass elliptic function solution*

$$f = \frac{5q}{6r} + \frac{5pq^2}{72r\,\wp(\xi)}, \quad g = -\frac{q\,\wp'(\xi)}{\wp(\xi)\,(pq + 12\wp(\xi))}, \tag{5}$$

where $p = \pm 1$. *Both q and r in* (4) *and* (5) *are arbitrary constants.*

(c) *If* $\delta = h^2 - s^2$, *and* $pq < 0$, *then* (2) *has the solitary wave solution*

$$f = \frac{q}{r + s\cosh(\sqrt{-pq}\,\xi) + h\sinh(\sqrt{-pq}\,\xi)},$$

$$g = -\frac{\sqrt{-pq}}{p}\,\frac{s\sinh(\sqrt{-pq}\,\xi) + h\cosh(\sqrt{-pq}\,\xi)}{r + s\cosh(\sqrt{-pq}\,\xi) + h\sinh(\sqrt{-pq}\,\xi)}, \tag{6}$$

where $p = \pm 1$, *r, s and h are arbitrary constants.*

(d) *If* $\delta = -h^2 - s^2$, *and* $pq > 0$, *then* (2) *has the trigonometric function solution*

$$f = \frac{q}{r + s\cos(\sqrt{pq}\,\xi) + h\sin(\sqrt{pq}\,\xi)},$$

$$g = \frac{\sqrt{pq}}{p}\,\frac{s\sin(\sqrt{pq}\,\xi) - h\cos(\sqrt{pq}\,\xi)}{r + s\cos(\sqrt{pq}\,\xi) + h\sin(\sqrt{pq}\,\xi)}, \tag{7}$$

where $p = \pm 1$, *r, s and h are arbitrary constants.*

(e) *If* $q = 0$, *then* (2) *has the rational solution*

$$f = \frac{2}{pr\xi^2 + C_1\xi - C_2},$$

$$g = -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}, \tag{8}$$

where $p = \pm 1,\ r,\ C_1$ *and* C_2 *are arbitrary constants.*

We now introduce the mapping approach via the above projective Riccati equation. The basic ideal of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with the independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$, and the dependent variable u, in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{9}$$

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives. We assume that its solution is written as the standard truncated Painlevé expansion, namely

$$u = A_0(x) + \sum_{i=1}^{n} [A_i(x) f(\xi(x)) + B_i(x) g(\xi(x))] f^{i-1}(\xi(x)). \tag{10}$$

Here $A_0(x)$, $A_i(x)$, $B_i(x)\ (i = 1, \ldots, n)$ are arbitrary constants to be determined, and f, g satisfy the projective Riccati equation (2).

To determine u explicitly, one proceeds as follows: First similar to the usual mapping approach, we can determine n by balancing the highest-order partial differential terms with the highest nonlinear terms in (9). Second, substituting (10) together with (2) and (3) into the given NPDE, collecting the coefficients of polynomials of $f^i g^i$ and eliminate each of them, we can derive a set of partial differential equations for $A_0(x)$, $A_i(x)$, $B_i(x)\ (i = 1, \ldots, n)$ and $\xi(x)$. Third, to calculate $A_0(x)$, $A_i(x)$, $B_i(x)\ (i = 1, \ldots, n)$ and $\xi(x)$, we solve these partial differential equations. Finally, substituting $A_0(x)$, $A_i(x)$, $B_i(x)\ (i = 1, \ldots, n)$, $\xi(x)$ and the solutions (4)-(8) into (10), one obtains solutions of the given NPDE.

First, let us make a transformation of (1): $v = u_y$. Substituting this transformation into (1), yields

$$u_{ty} + a(t)[2u_y u_x + 2uu_{xy} + u_{xxy}] = 0. \tag{11}$$

Now we apply the mapping approach to (11). By the balancing procedure, ansatz (10) becomes

$$u = F + Gf(\xi(x,\ y,\ t)) + Hg(\xi(x,\ y,\ t)), \tag{12}$$

where F, G, H, and ξ are arbitrary functions of $(x,\ y,\ t)$ to be determined. Substituting (12) together with (2) and (3) into (11), collecting the coefficients of the polynomials of $f^i g^i$ $(i = 0, 1, 2, \ldots,\ j = 0, 1, 2, \ldots)$ and setting each of the coefficients equal to zero, we can derive a set of partial differential equations for F, G, H, and ξ. It is difficult to obtain the general solutions of these algebraic equations based on the solutions of (2). Fortunately, in the special case if setting $\xi = \chi(x,\ t) + \varphi(y)$, where $\chi \equiv \chi(x,\ t)$, $\varphi \equiv \varphi(y)$ are two arbitrary variable separated functions of $(x,\ t)$ and y, respectively, we can obtain solutions of (1).

Case 1. For $\delta = -r^2$, the Weierstrass elliptic function solutions are

$$u_1 = \frac{1}{2}\frac{\chi_t + a(t)\chi_{xx}}{a(t)\chi_x} + \frac{1}{2}p\chi_x g(\xi), \tag{13}$$

$$v_1 = -\frac{1}{2}p\varphi_y f(\xi), \tag{14}$$

where $p = \pm 1$, f, g are expressed by (4).

Case 2. For $\delta = -\frac{r^2}{25}$, another set of Weierstrass elliptic function solutions are found

$$u_2 = \frac{1}{2}\frac{\chi_t + a(t)\chi_{xx}}{a(t)\chi_x} + \frac{1}{5}\frac{\sqrt{-6pq}\,r\chi_x f(\xi)}{q} - \frac{1}{2}p\chi_x g(\xi), \tag{15}$$

$$v_2 = \frac{1}{5}\frac{\sqrt{-6pq}\,r\varphi_y f(\xi)}{q} - \frac{1}{2}p\varphi_y f(\xi), \tag{16}$$

where $p = \pm 1$, q and r are arbitrary constants, f, g are expressed by (5).

Case 3. For $\delta = h^2 - s^2$ and $pq = -1$, the solitary wave solutions are

$$u_3 = -\frac{1}{2}\frac{\chi_t + a(t)\chi_{xx}}{a(t)\chi_x}$$

$$+\frac{1}{2}\frac{\chi_x[\sqrt{r^2 + h^2 - s^2} + h\cosh(\chi + \varphi) + s\sinh(\chi + \varphi)]}{r + s\cosh(\chi + \varphi) + h\sinh(\chi + \varphi)}, \tag{17}$$

$$v_3 = -\frac{1}{2}\frac{\chi_x\varphi_y[\cosh(\chi + \varphi)(h\sqrt{r^2 + h^2 - s^2} - sr)]}{[r + s\cosh(\chi + \varphi) + h\sinh(\chi + \varphi)]^2}$$

$$-\frac{1}{2}\frac{\chi_x\varphi_y[\sinh(\chi + \varphi)(s\sqrt{r^2 + h^2 - s^2} - hr) - s^2 + h^2]}{[r + s\cosh(\chi + \varphi) + h\sinh(\chi + \varphi)]^2} \tag{18}$$

with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$, r, s, h are arbitrary constants.

Case 4. For $\delta = -h^2 - s^2$, and $pq = 1$, the trigonometric function solutions are

$$u_4 = -\frac{1}{2}\frac{\chi_t + a(t)\chi_{xx}}{a(t)\chi_x}$$

$$+\frac{1}{2}\frac{\chi_x[\sqrt{s^2 + h^2 - r^2} + h\cos(\chi + \varphi) - s\sin(\chi + \varphi)]}{r + s\cos(\chi + \varphi) + h\sin(\chi + \varphi)}, \tag{19}$$

$$v_4 = -\frac{1}{2}\frac{\chi_x\varphi_y[\cos(\chi + \varphi)(h\sqrt{-r^2 + h^2 + s^2} + sr)]}{[r + s\cos(\chi + \varphi) + h\sin(\chi + \varphi)]^2}$$

$$-\frac{1}{2}\frac{\chi_x\varphi_y[\sin(\chi + \varphi)(-s\sqrt{-r^2 + h^2 + s^2} + hr) + s^2 + h^2]}{[r + s\cos(\chi + \varphi) + h\sin(\chi + \varphi)]^2} \tag{20}$$

with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$, r, s, h are arbitrary constants.

Case 5. For $q = 0,$ the rational solutions are

$$u_5 = -\frac{1}{2}\frac{\chi_t + a(t)\chi_{xx}}{a(t)\chi_x} + \frac{1}{2}\frac{\chi_x[p\sqrt{C_1^2 + 4C_2pr} + 2pr(\chi + \varphi) + C_1]}{pr(\chi + \varphi)^2 + C_1(\chi + \varphi) - C_2}, \tag{21}$$

$$v_5 = -\frac{1}{2}\frac{\chi_x\varphi_y[\sqrt{C_1^2 + 4C_2pr}(2p^2r\chi + 2p^2r\varphi + pC_1)]}{[pr(\chi + \varphi)^2 + C_1(\chi + \varphi) - C_2]^2} - \frac{1}{2}\frac{\chi_x\varphi_y[2p^2r^2(\chi + \varphi)^2 + 2prC_1(\chi + \varphi) + 2C_2pr + C_1^2]}{[pr(\chi + \varphi)^2 + C_1(\chi + \varphi) - C_2]^2} \tag{22}$$

with two arbitrary functions being $\chi(x, t)$ and $\varphi(y)$, $p = \pm 1$, C_1, C_2, and r are arbitrary constants.

3. Some Special Localized Excitations in the VCBKK System

Due to the arbitrariness of the functions $\chi(x, t)$ and $\varphi(y)$ included in the above cases, the physical quantities u and v may possess rich structures. For example, when $\chi = ax + ct$ and $\varphi = by,$ all the solutions of the above cases become simple travelling wave excitations. Moreover, based on the derived solutions, we may obtain rich localized structures such as peakons. In the following discussion, we merely analyze some special multi-peakons localized excitation and the interactions between two peakons of solution v_3 (18) in Case 3, namely

$$V = v_3 = -\frac{1}{2}\frac{\chi_x\varphi_y[\cosh(\chi + \varphi)(h\sqrt{r^2 + h^2 - s^2} - sr)]}{[r + s\cosh(\chi + \varphi) + h\sinh(\chi + \varphi)]^2} - \frac{1}{2}\frac{\chi_x\varphi_y[\sinh(\chi + \varphi)(s\sqrt{r^2 + h^2 - s^2} - hr) - s^2 + h^2]}{[r + s\cosh(\chi + \varphi) + h\sinh(\chi + \varphi)]^2}. \tag{23}$$

3.1. Multi-peakon excitations

According to the solution V (23), we first discuss its multi-peakon excitations. For instance, if we choose χ and φ as

$$\chi = 1 + \exp(-|x + ct + 2|) + 2\exp(-|x + ct - 1|), \quad \varphi = 1 + \exp(-|y - 1|), \tag{24}$$

and

$$\chi = 1 + 0.5\exp(-|x + ct + 2|) + 0.8\exp(-|x + ct - 1|) + \exp(-|x + ct - 3|)$$
$$+ 2\exp(-|x + ct - 4|),$$
$$\varphi = 1 + \exp(-|y - 1|), \tag{25}$$

we can obtain the two-peakon and four-peakon excitations for the physical quantity V of equation (23) presented in Figures 1(a) and 1(b) with fixed parameters $h = 2$, $s = 1$, $r = 0$, and $t = 0$.

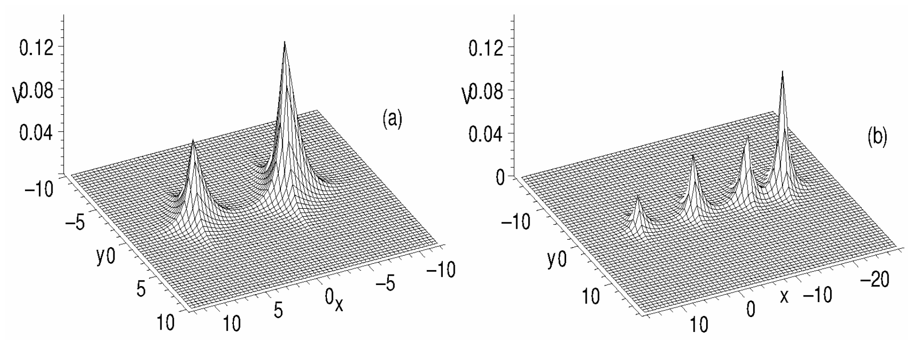

Figure 1. A plot of multi-peakon structures for the physical quantity V given by the solution (23) with the choice (24), (25) and $h = 2$, $s = 1$, $r = 0$, $t = 0$.

3.2. Interactions between two peakons

According to the solution V (23), if we choose χ and φ as

$$\chi = 1 + \exp(-|x + 3t - 1|) + 2\exp(-|x - t - 1|), \quad \varphi = 1 + \tanh(-|y - 1|), \tag{26}$$

then we can obtain a solitary wave solution of equation (1) with elastic behaviour. Figure 2 shows an evolutional profile corresponding to the physical quantity V of equation (23). From Figure 2 and through detailed analysis, we find that the shapes, amplitudes and velocities of the two peakons are completely after their interaction.

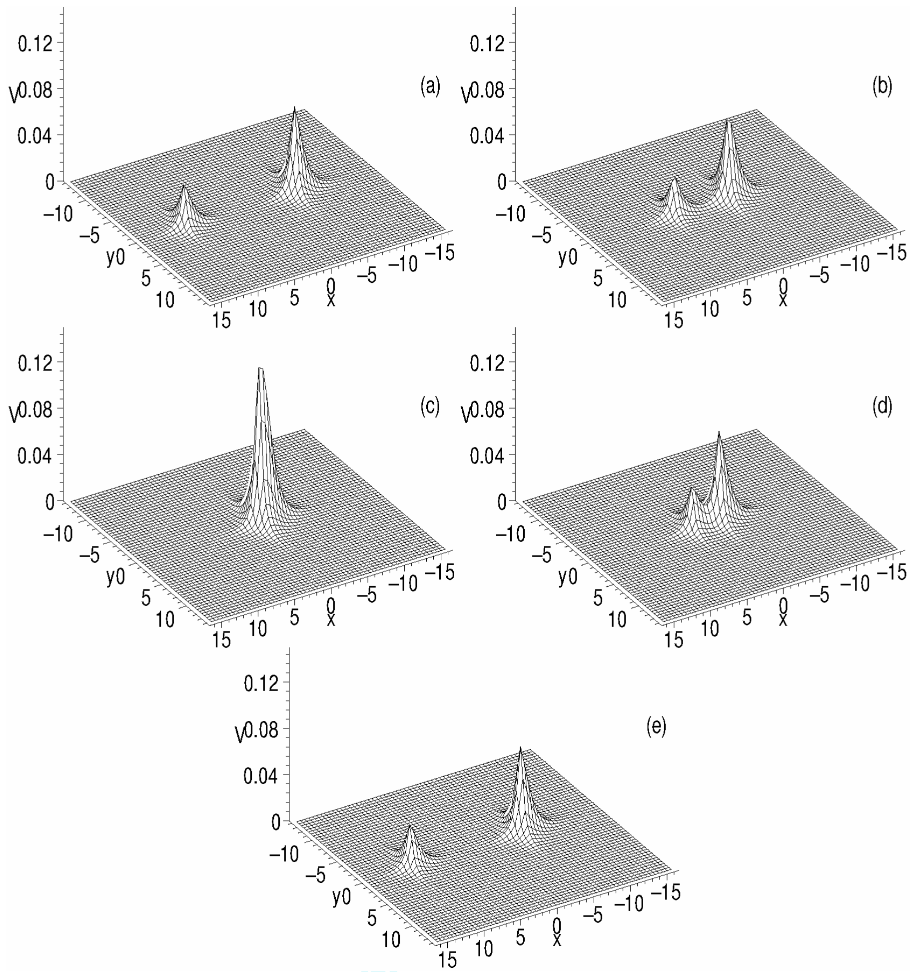

Figure 2. The evolutional plot of two peakons for the solution V (23) under the condition (26) with fixed parameters $h = 2$, $s = 1$, $r = 0$, at different times (a) $t = -5$, (b) $t = -2.5$, (c) $t = 0$, (d) $t = 1$, (e) $t = 5$, respectively.

Generally, the interactions between solitons are completely elastic as Figure 2. While for some specific cases, the interactions between solitons are nonelastic. For example, if χ and φ are chosen to be

$$\chi = 1 + 3.5\operatorname{cosech}(-|x + t - 1|) + 1.2\operatorname{cosech}(-|x + 0.3t - 1|),$$

$$\varphi = 1 + \tanh(-|y - 1|), \tag{27}$$

and $h = 2,\ s = 1,\ r = 0$ in equation (23), we can obtain another type of solitary wave solution of equation (1) with nonelastic behaviour. The two peakons move with the same direction, but their velocities are different. One peakon catch up with the other and they are in collision with each other. From Figure 3 we can see that the shapes and amplitudes of two peakons are changed after their collisions. What is more, after their departure, the distance of the two peakons becomes farther and farther.

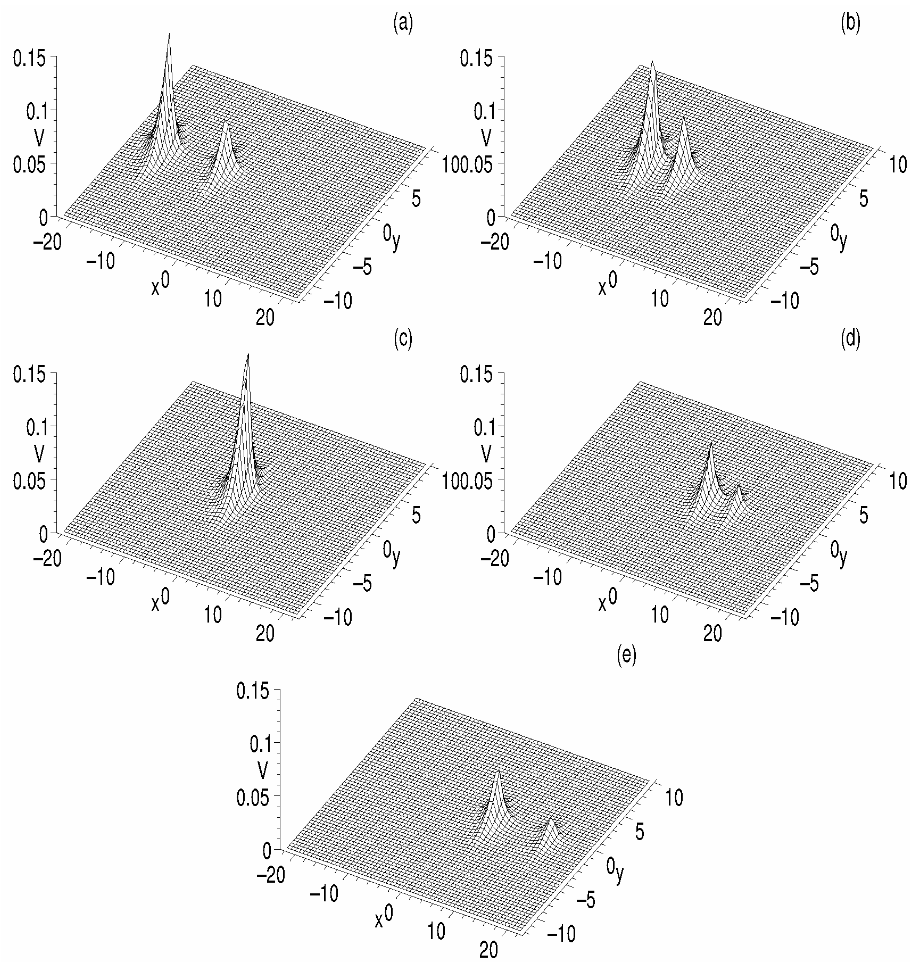

Figure 3. The evolutional plot of two peakons for the solution V (23) under the condition (27) with fixed parameters $h = 2,\ s = 1,\ r = 0,$ at different times (a) $t = -15$, (b) $t = -8$, (c) $t = 0$, (d) $t = 8$, (e) $t = 15$, respectively.

4. Summary and Discussion

In summary, via an extended mapping approach and a special variable separation approach, we find some new exact solutions of the $(2+1)$-dimensional Broek-Kaup system with variable coefficients. Based on the derived solitary wave solution v_3 (18), we obtain some special peakon excitations. Then we discussed the interactions between two peakons. Additionally, using the piecewise function, Zheng and Zhu recently obtained some peakon excitations in the new $(2+1)$-dimensional long dispersive wave system [35]. Along with the above line, we use the piecewise function to get the new peakon excitations of VCBKK system. Especially, the phenomenon showed in Figure 3 of two peakons running after each other and in collision with each other has never been reported in other literature. Since the wide applications of the soliton theory, to learn more about the localized excitations and their applications in reality is worthy of study further.

Acknowledgement

The project supported by National Natural Science Foundation of China under Grant No. 10172056, the Natural Science Foundation of Zhejiang Province Grant No. Y604106, the Natural Science Foundation of Zhejiang Lishui University under Grant No. KZ05010.

References

[1] M. Boiti, L. M. Martiona and F. Pempinelli, Phys. Lett. A 132 (1988), 116.

[2] C. L. Chen and Y. S. Li, Theor. Phys. 38 (2002), 129.

[3] R. Conte and M. Musette, J. Phys. A 25 (1992), 5609.

[4] A. Davey and K. Stewartson, Proc. R. Soc. London A 338 (1974), 17.

[5] J. P. Fang and C. L. Zheng, Chin. Phys. 14 (2005), 0669.

[6] J. P. Fang and C. L. Zheng, Z. Naturforsch. 60a (2005), 245.

[7] J. P. Fang, C. L. Zheng and Q. Liu, Commun. Theor. Phys. 43 (2005), 246.

[8] Z. T. Fu, S. D. Liu and S. K. Liu, Chaos Solitons Fractals 20 (2004), 301.

[9] M. Gedalin, T. C. Scott and Y. B. Band, Phys. Rev. Lett. 78 (1997), 448.

[10] J. P. Gollb and M. C. Cross, Nature 404 (2000), 710.

[11] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. 15 (1970), 539.

[12] Y. S. Kivshar and B. A. Melomend, Rev. Modern Phys. 61 (1989), 765.

[13] B. G. Konopelchenko and C. Rogers, Phys. Lett. A 158 (1991), 391.

[14] S. Y. Lou, J. Phys. A 29 (1996), 5989.

[15] S. Y. Lou and X. B. Hu, J. Math. Phys. 38 (1997), 6401.

[16] S. Y. Lou and X. Y. Tang, Chin. Phys. Lett. 19 (2002), 770.

[17] I. Loutsenko and D. Roubtsov, Phys. Rev. Lett. 78 (1997), 3011.

[18] S. H. Ma and J. P. Fang, Acta Phys. Sinica 55 (2006), 5611 (in Chinese).

[19] S. H. Ma and J. P. Fang, Acta Phys. Sinica 56 (2007), 0620 (in Chinese).

[20] S. H. Ma, X. H. Wu, J. P. Fang and C. L. Zheng, Z. Naturforsch. 61a (2006), 249.

[21] L. P. Nizhnik, Sov. Phys. Dokl. 25 (1980), 706.

[22] R. Radha and M. Lakshmanan, J. Math. Phys. 35 (1994), 4746.

[23] H. Y. Ruan and S. Y. Lou, J. Math. Phys. 38 (1997), 3123.

[24] G. I. Stegemant and M. Segev, Science 286 (1999), 1518.

[25] M. Tajiri and H. Maesono, Phys. Rev. E 55 (1997), 3351.

[26] X. Y. Tang and S. Y. Lou, Phys. Rev. E 66 (2002), 046601.

[27] S. Wang, X. Y. Tang and S. Y. Lou, Chaos Solitons Fractals 19 (2004), 769.

[28] S. Wang, Y. S. Xu and S. Y. Lou, Chin. Phys. 14 (2003), 1049.

[29] J. P. Ying and S. Y. Lou, Z. Naturforsh. 56a (2001), 619.

[30] V. E. Zakharov and J. Li, Appl. Mech. Tech. Phys. 9 (1998), 190.

[31] J. F. Zhang and C. L. Zheng, Commun. Theor. Phys. 38 (2002), 37.

[32] J. F. Zhang, C. L. Zheng and J. P. Fang, Chin. Phys. Lett. 4 (2003), 448.

[33] C. L. Zheng, Commun. Theor. Phys. 40 (2003), 25.

[34] C. L. Zheng and J. F. Zhang, Commun. Theor. Phys. 40 (2003), 385.

[35] C. L. Zheng and J. M. Zhu, Commun. Theor. Phys. 39 (2003), 261.