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AN EXTENSION OF THE EULER-TYPE TRANSFORMATION FOR THE ${}_3F_2$ SERIES

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Abstract

The aim of this paper is to establish an extension of the Euler-type transformation for the ${}_3F_2$ hypergeometric function. This is achieved by application of a recently obtained summation formula for ${}_3F_2(1)$. An alternative simple proof is also given for the Kummer-type transformation for the series ${}_2F_2$ recently derived by Paris [6].

1. Introduction

In 1997, Exton [3] derived four interesting reduction formulas for the Kampé de Fériet function and, from one of these formulas, he deduced the following two results:

$$(1-x)^{-d}\,{}_3F_2\left(\begin{matrix} d,\ a,\ 1+\frac{1}{2}a; \\ b,\ \frac{1}{2}a; \end{matrix}\ -\frac{x}{1-x}\right) = {}_3F_2\left(\begin{matrix} d,\ b-a-1,\ 2+a-b; \\ b,\ 1+a-b; \end{matrix}\ x\right) \quad (1.1)$$

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and

$$e^{-x}\,{}_2F_2\left(\begin{matrix} a,\ 1+\frac{1}{2}a; \\ b,\ \frac{1}{2}a; \end{matrix}\ x\right) = {}_2F_2\left(\begin{matrix} b-a-1,\ 2+a-b; \\ b,\ 1+a-b; \end{matrix}\ -x\right). \tag{1.2}$$

These expressions are generalizations of the well-known Euler transformation [8, p. 31]

$$(1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\ \gamma-\beta;\ \gamma;\ -\frac{x}{1-x}\right) = {}_2F_1(\alpha,\ \beta;\ \gamma;\ x),$$

valid for complex x in the domain $|x| < 1,\ \ \mathrm{Re}(x) < \frac{1}{2}$, and Kummer's first theorem [1, Eq. (13.1.27)]

$$e^{-x}\,{}_1F_1(a;\ b;\ x) = {}_1F_1(b-a;\ b;\ -x)$$

valid for all finite values of x. The identity (1.1) is also given in [8, p. 66], where it is obtained by a different method, and (1.2) has been established in [4].

The result (1.2) has recently been extended to three independent parameters in [6] in the form

$$e^{-x}\,{}_2F_2\left(\begin{matrix} a,\ c+1; \\ b,\ c; \end{matrix}\ x\right) = {}_2F_2\left(\begin{matrix} b-a-1,\ f+1; \\ b,\ f; \end{matrix}\ -x\right), \tag{1.3}$$

where the parameter f depends on a nonlinear combination of the free parameters a, b and c given by

$$f = \frac{c(1+a-b)}{a-c}. \tag{1.4}$$

This analogue of the well-known Kummer transformation was established by means of an integral representation for ${}_2F_2(x)$ combined with an addition theorem for the confluent hypergeometric function ${}_1F_1(x+y)$. An alternative proof of (1.3) has been given by Miller [5] using a reduction formula for the Kampé de Fériet function.

The aim of this note is twofold. We employ a summation formula for a terminating ${}_3F_2(1)$ function to generalize the identity (1.1) to three independent parameters in the form

$$(1-x)^{-d}\,{}_3F_2\left(\begin{matrix} d,\ a,\ c+1; \\ b,\ c; \end{matrix}\ -\frac{x}{1-x}\right) = {}_3F_2\left(\begin{matrix} d,\ b-a-1,\ f+1; \\ b,\ f; \end{matrix}\ x\right), \tag{1.5}$$

where f is defined in (1.4) and x lies in the domain $|x| < 1$, $\operatorname{Re}(x) < \frac{1}{2}$. This generalization is seen to involve the same nonlinear combination f of the free parameters as that in (1.3). In addition, we supply another simple proof of the Kummer-type transformation (1.3).

2. Proof of the Euler-type Transformation (1.5)

To establish the result (1.5) we require the following

Lemma 1. *Let n be a nonnegative integer and a, b and c be complex parameters. Then*

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ c+1; \\ b,\ c; \end{matrix}\ 1\right) = \frac{(b-a-1)_n}{(b)_n}\frac{(f+1)_n}{(f)_n}, \tag{2.1}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ *is the Pochhammer symbol and* $f = c(1+a-b)/(a-c)$.

This special summation theorem has been derived recently by Miller [5] by two different methods. The first proof relies on use of the result $(c+1)_m/(c)_m = 1 + (m/c)$ to reduce the above ${}_3F_2(1)$ function to the sum of two Gauss functions, which may then be summed using Gauss' theorem [1, Eq. (15.1.20)]. Thus we find

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ c+1; \\ b,\ c; \end{matrix}\ 1\right) = \sum_{m=0}^{\infty}\frac{(-n)_m(a)_m}{(b)_m m!}\left(1+\frac{m}{c}\right)$$

$$= {}_2F_1(-n,\ a;\ b;\ 1) - \frac{na}{bc}\,{}_2F_1(-n+1,\ a+1;\ b+1;\ 1)$$

$$= \frac{\Gamma(b)\Gamma(b-a+n-1)}{\Gamma(b-a)\Gamma(b+n)}\left\{b-a+n-1-\frac{na}{c}\right\}$$

which, upon a little algebraic simplification, yields the result stated. Miller's second proof makes use of Kummer's two-term transformation for the ${}_3F_2(1)$ function given in [2, p. 142, Cor. 3.3.5].

Let the domain D of the complex x-plane be specified by $|x| < 1$, $\mathrm{Re}(x) < \frac{1}{2}$. Then, for $\mathrm{Re}(x) < \frac{1}{2}$, we have upon expansion of the ${}_3F_2$ series in (1.5)

$$\mathcal{F}(x) \equiv (1-x)^{-d}\,{}_3F_2\left(\begin{matrix} d,\ a,\ c+1; \\ b,\ c; \end{matrix}\ \frac{-x}{1-x}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n (a)_n (c+1)_n}{(b)_n (c)_n n!} (-x)^n (1-x)^{-n-d}.$$

Application of the binomial theorem for $(1-x)^{-n-d}$ valid in $|x| < 1$, followed by interchange of the order of summation, then shows that $\mathcal{F}(x)$ can be written as the absolutely convergent double sum when $x \in D$

$$\mathcal{F}(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-)^n (a)_n (c+1)_n}{(b)_n (c)_n} \frac{(d)_{m+n}}{m!\,n!} x^{m+n} \qquad (x \in D), \tag{2.2}$$

where we have used $(d)_n (n+d)_m = (d)_{m+n}$.

If we now employ the result [7, p. 56, Lemma 10]

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A(m, n) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} A(m-n, m) \tag{2.3}$$

which enables an absolutely convergent double sum to be summed diagonally, we find from (2.2) that

$$\mathcal{F}(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(-)^n (a)_n (c+1)_n}{(b)_n (c)_n} \frac{(d)_m x^m}{(m-n)!\,n!}.$$

Using

$$(m-n)! = (-)^n\, m!/(-m)_n, \tag{2.4}$$

we finally obtain

$$\mathcal{F}(x) = \sum_{m=0}^{\infty} \frac{(d)_m x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (a)_n (c+1)_n}{(b)_n (c)_n n!} = \sum_{m=0}^{\infty} \frac{(d)_m x^m}{m!} {}_3F_2\left(\begin{matrix} -m, \ a, \ c+1; \\ b, \ c; \end{matrix} \ 1 \right).$$

Now if we employ the result (2.1) to sum the ${}_3F_2(1)$ function we obtain

$$\mathcal{F}(x) = \sum_{m=0}^{\infty} \frac{(d)_m (b-a-1)_m (f+1)_m}{(b)_m (f)_m} \frac{x^m}{m!} = {}_3F_2\left(\begin{matrix} d, \ b-a-1, \ f+1; \\ b, \ f; \end{matrix} \ x \right)$$

for $x \in D$, where f is defined in (1.4) This completes the proof of (1.5).

3. Alternative Proof of the Kummer-type Transformation (1.3)

For all finite complex values of x, we can express the left-hand side of (1.3) as an absolutely convergent double sum by expanding the functions e^{-x} and ${}_2F_2(x)$ as

$$\mathcal{G}(x) \equiv e^{-x} {}_2F_2\left(\begin{matrix} a, \ c+1; \\ b, \ c; \end{matrix} \ x \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_n (c+1)_n}{(b)_n (c)_n} \frac{(-)^m x^{m+n}}{m! \, n!}.$$

Application of (2.3) and (2.4) then leads to

$$\mathcal{G}(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-m)_n (a)_n (c+1)_n}{(b)_n (c)_n} \frac{(-x)^m}{m! \, n!}$$

$$= \sum_{m=0}^{\infty} \frac{(-x)^m}{m!} {}_3F_2\left(\begin{matrix} -m, \ a, \ c+1; \\ b, \ c; \end{matrix} \ 1 \right).$$

Upon summing the ${}_3F_2(1)$ function by (2.1), we find

$$\mathcal{G}(x) = \sum_{m=0}^{\infty} \frac{(b-a-1)_m (f+1)_m}{(b)_m (f)_m} \frac{(-x)^m}{m!} = {}_2F_2\left(\begin{matrix} b-a-1, \ f+1; \\ b, \ f; \end{matrix} \ -x \right),$$

where f is defined in (1.4). This provides a simple, direct proof of the Kummer-type transformation (1.3).

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