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NICE POLYNOMIALS WITH FOUR ROOTS

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Abstract

Nice polynomials are polynomials whose coefficients, roots, and critical points are integers. We first define what it means for two nice polynomials to be equivalent, then we give the relations between the roots and critical points of all polynomials with four roots. This system of relations is the key to studying nice polynomials. We use these relations to derive the relations between the roots and critical points of all symmetric polynomials with four roots. We use these relations for symmetric polynomials to give a complete description of all nice symmetric polynomials with four roots by finding an explicit formula and counting the number of equivalence classes of such nice polynomials. We then give a necessary, but not sufficient, condition that nice symmetric polynomials with four roots with the first m derivatives having integer roots must satisfy. To conclude, we state several open problems about nice polynomials with four roots and about higher-order derivatives of nice symmetric polynomials with four roots.

1. Introduction

Nice polynomials are polynomials whose coefficients, roots, and critical points are integers. The first paper on nice polynomials [4],

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written in 1960, gives an explicit formula for all nice cubics. The derivation of this formula uses Pythagorean triples. Most mathematicians who had begun investigating this problem were interested in constructing polynomials with integer coefficients, roots, and critical points-polynomials that are “nice” for calculus students to sketch (see [1], for example). Most earlier papers had focused on nice cubics and quartics. We had found this problem of constructing nice polynomials of any degree worthy of further study.

In 1999 the problem of finding, constructing, and classifying nice polynomials was added to the list of unsolved problems [8] in *The American Mathematical Monthly*. Other papers soon followed, including the main paper [2] on nice polynomials, the paper [5] with a new approach to nice polynomials, the accepted paper [6] with several new results on nice symmetric and antisymmetric polynomials, and the accepted paper [7] with a complete description of all nice polynomials of any degree with three roots. The main result of [7] is a formula for finding all nice polynomials with three roots. The new approach to nice polynomials in [5] is the system of relations between the roots and critical points of polynomials. The new results in [5]-[7] and in this paper follow from these relations. Papers [5]-[7] and this paper prove that the key to studying nice polynomials is this system of relations.

The first paper on nice quartics [3], written in 1990, gives a formula for all nice symmetric quartics and also contains the first five known examples of nice nonsymmetric quartics with four distinct roots. The author of [5] has found over 300 examples of nice nonsymmetric quartics with a computer; and he has included two of those examples in his paper, these being the smallest *possible* example (the difference between the largest root and the smallest root is minimal) and the largest known example [5, Example B6]. Examples of higher degree with four roots have not been published. But such examples have been found, and we include some of these in this paper. But the problem of finding all nice polynomials with four roots, which we briefly consider, is by no means completely solved.

We do solve the problem of finding all nice symmetric polynomials of any degree with four roots, a natural extension of the problem in [3] of finding all nice symmetric quartics. To find a formula for all nice symmetric polynomials with four roots, we solve an equation relating the roots and critical points of all symmetric polynomials with four roots (see Lemma 4.1). For simplicity, we solve the equation in $\mathbb{Q}$ rather than in $\mathbb{Z}$; so our formula gives examples of polynomials with rational coefficients, roots, and critical points. Such polynomials we call $\mathbb{Q}$-*nice*. This does not cause a problem, however, since we can then find examples of nice polynomials by horizontally stretching $\mathbb{Q}$-nice polynomials so that we obtain a polynomial with integer coefficients, roots, and critical points.

In the following section, we discuss any necessary notation and terminology. We also define what it means for two $\mathbb{Q}$-nice polynomials and two nice polynomials to be equivalent. In Section 3, we give the relations between the roots and critical points of all polynomials with four roots. These relations are useful in computer searches for nice polynomials with four roots. Several new examples we have found using these relations are included. If it is possible to find a formula for nice polynomials with four roots, then these relations will help in deriving such a formula. In Section 4, we give a complete description of all nice symmetric polynomials with four roots; that is, we give a formula, conditions for the existence of such polynomials, and the number of equivalence classes. We illustrate these results with a few examples. In Section 5, we briefly discuss higher-order nicety properties of nice symmetric polynomials with four roots. In other words, we give a necessary (but not sufficient) condition that nice symmetric polynomials with four roots with the first m derivatives having integer roots must satisfy. Sufficient conditions have not been found. We conclude in Section 6 by stating several open problems about nice polynomials with four roots and about higher-order nicety properties of nice symmetric polynomials with four roots.

2. Preliminaries

The *type* of a polynomial is a list of the multiplicities of its distinct roots. For convenience, we often list the multiplicities in decreasing order. For example, all polynomials of the type $(4, 3, 2, 2)$ are of the form

$p(x) = a(x - r_1)^4(x - r_2)^3(x - r_3)^2(x - r_4)^2$, where r_1, r_2, r_3, and r_4 are all distinct.

Most of the earlier papers on nice or $\mathbb{Q}$-nice polynomials note that horizontal translations, horizontal or vertical stretches and compressions, and reflections over the coordinate axes transform a $\mathbb{Q}$-nice polynomial $p_1(x)$ into another $\mathbb{Q}$-nice polynomial $p_2(x)$. Each of these transformations has an inverse transformation which transforms $p_2(x)$ into $p_1(x)$. Two newly discovered transformations that behave similarly are the power transformation and its inverse, the root transformation [6, Theorem 2.1]: For any natural number n, a polynomial $p(x)$ is $\mathbb{Q}$-nice iff $[p(x)]^n$ is $\mathbb{Q}$-nice. It is clear that the root transformation transforms a $\mathbb{Q}$-nice polynomial $p(x)$ into another $\mathbb{Q}$-nice polynomial iff $p(x) = [q(x)]^n$ for some $\mathbb{Q}$-nice polynomial $q(x)$ and some natural number n. Otherwise, the n-th root of $p(x)$ is not a polynomial. Since any finite composition of these transformations and their inverses, which we call *equivalence transformations*, transform a $\mathbb{Q}$-nice polynomial $p_1(x)$ into another $\mathbb{Q}$-nice polynomial $p_2(x)$, we may define two $\mathbb{Q}$-nice polynomials $p_1(x)$ and $p_2(x)$ to be *equivalent* whenever $p_1(x)$ can be transformed into $p_2(x)$ and vice-versa by a finite composition of equivalence transformations. And two nice polynomials $p_1(x)$ and $p_2(x)$ are *equivalent* if, when considered $\mathbb{Q}$-nice polynomials, they are equivalent. Thus, whenever we count the number of $\mathbb{Q}$-nice (or nice) polynomials of a given type, we count the number of equivalence classes rather than the actual number of $\mathbb{Q}$-nice (or nice) polynomials of that given type.

Because horizontal stretches and compressions are equivalence transformations for $\mathbb{Q}$-nice polynomials and because all nice polynomials are $\mathbb{Q}$-nice polynomials, any result on the existence of or the number of equivalence classes of $\mathbb{Q}$-nice polynomials of a given type also holds for nice polynomials of the same type and vice-versa. Thus, we may use our formula for $\mathbb{Q}$-nice symmetric polynomials to determine the existence of and the number of equivalence classes of all types of nice symmetric polynomials with four roots.

3. Nice Polynomials with Four Roots

To begin, we consider the problem of finding all nice polynomials $p(x) \in \mathbb{Z}[x]$ with four distinct roots. Because of vertical stretches and compressions, we may assume $p(x)$ is monic. Thus, $p(x) = (x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3}(x - r_4)^{m_4}$ for some integers r_1, r_2, r_3, and r_4; and the derivative $p'(x) = d(x - r_1)^{m_1 - 1}(x - r_2)^{m_2 - 1}(x - r_3)^{m_3 - 1}(x - r_4)^{m_4 - 1}(x - c_1)(x - c_2)(x - c_3)$ for some integers c_1, c_2, and c_3.

The following lemma gives the relations between the roots and critical points of all polynomials with four roots. As [5]-[7] show, this system of relations is the key to studying nice polynomials.

Lemma 3.1. *A polynomial* $p(x) = (x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3}(x - r_4)^{m_4}$ *of degree* $d = m_1 + m_2 + m_3 + m_4$ *with integer coefficients and with four integer roots is nice iff there exist integers* c_1, c_2, *and* c_3 *such that*

$$\sum_{i=1}^{4}(d - m_i)r_i = d\sum_{i=1}^{3} c_i, \tag{3.1}$$

$$\sum_{1 \le i < j \le 4}(d - m_i - m_j)r_i r_j = d\sum_{1 \le i < j \le 3} c_i c_j, \tag{3.2}$$

$$\sum_{1 \le i < j < k \le 4}(d - m_i - m_j - m_k)r_i r_j r_k = d\sum_{1 \le i < j < k \le 3} c_i c_j c_k. \tag{3.3}$$

Proof. Differentiating $p(x)$ by the product rule, we have $p'(x) = q(x)(x - r_1)^{m_1 - 1}(x - r_2)^{m_2 - 1}(x - r_3)^{m_3 - 1}(x - r_4)^{m_4 - 1}$, where $q(x) = m_1(x - r_2)(x - r_3)(x - r_4) + m_2(x - r_1)(x - r_3)(x - r_4) + m_3(x - r_1)(x - r_2)(x - r_4) + m_4(x - r_1)(x - r_2)(x - r_3)$. By definition, the derivative of $p(x)$ has the form $p'(x) = d(x - r_1)^{m_1 - 1}(x - r_2)^{m_2 - 1}(x - r_3)^{m_3 - 1}(x - r_4)^{m_4 - 1}(x - c_1)(x -$

$c_2)(x-c_3)$. Therefore, $q(x) = d(x-c_1)(x-c_2)(x-c_3)$. By expanding both forms of $q(x)$ and equating coefficients, we obtain the relations above.

Because of the horizontal translation, we may translate a nice polynomial $p(x)$ so that it has a root at 0. If we do assume 0 is a root, then the relations (3.1)-(3.3) above simplify considerably, so finding examples of nice nonsymmetric polynomials with four roots with a computer and proving existence or nonexistence of certain types become easier. For convenience, we state these relations with the assumption that 0 is a root of $p(x)$.

Lemma 3.2. *A polynomial* $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$ *of degree* d *with integer coefficients and with four integer roots is nice iff there exist integers* c_1, c_2, *and* c_3 *such that*

$$(d-m_1)r_1 + (d-m_2)r_2 + (d-m_3)r_3 = d(c_1+c_2+c_3), \tag{3.4}$$

$$(d-m_1-m_2)r_1r_2 + (d-m_1-m_3)r_1r_3 + (d-m_2-m_3)r_2r_3$$
$$= d(c_1c_2 + c_1c_3 + c_2c_3), \tag{3.5}$$

$$m_0r_1r_2r_3 = dc_1c_2c_3. \tag{3.6}$$

Proof. Let $r_4 = 0$ and relabel m_4 as m_0 and use Lemma 3.1 above.

Remark. We had derived the relations in Lemma 3.1 without the assumption that 0 is a root of $p(x)$ so that we can use these relations to derive immediately the relations for symmetric polynomials with four roots. In other words, relations (3.4)-(3.6) do not allow us to derive the relations for symmetric polynomials with four roots, as we will see in Section 4.

Application. Although a complete description of all nice polynomials with four roots is currently unknown (see Problems 6.1-6.3), the relations above, especially the relations (3.4)-(3.6), can help reduce the amount of work of a computer search for examples. Furthermore, if it is possible to find a formula for all nice polynomials with four roots, the relations (3.4)-(3.6) above will help in the derivation of this formula.

We now include several new examples we have found using relations (3.4)-(3.6). The first example is of type (2, 1, 1, 1):

$$p(x) = x^2(x-60)(x-80)(x-105),$$
$$p'(x) = 5x(x-30)(x-70)(x-96). \tag{3.7}$$

This example is of type (3, 1, 1, 1):

$$p(x) = x^3(x-42)(x-48)(x-60),$$
$$p'(x) = 6x^2(x-24)(x-45)(x-56). \tag{3.8}$$

The final example we give is of type (4, 1, 1, 1):

$$p(x) = x^4(x-308)(x-420)(x-455),$$
$$p'(x) = 7x^3(x-210)(x-364)(x-440). \tag{3.9}$$

4. Nice Symmetric Polynomials with Four Roots

We now consider the problem of finding all nice symmetric polynomials with four roots. We call a polynomial $p(x)$ *symmetric* if there exists a unique number c, called the *center*, such that $p(c-x) = p(c+x)$ for all x. In [5] it is proven that the average of the roots of a nice polynomial is an integer. Since the average of the roots of a symmetric polynomial equals the center, the center of a nice symmetric polynomial is an integer, so we may center nice symmetric polynomials at the origin. The same can be said about $\mathbb{Q}$-nice symmetric polynomials.

If $p(x) \in \mathbb{Q}[x]$ is a $\mathbb{Q}$-nice symmetric polynomial with four roots and is centered at the origin, then $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ for some rational numbers r_1 and r_2; and the derivative $p'(x) = dx(x^2 - r_1^2)^{m_1-1}$ $(x^2 - r_2^2)^{m_2-1}(x^2 - c^2)$ for some rational number $c \neq 0$.

The following lemma gives the relations between the roots and critical points of all symmetric polynomials $p(x)$ with four roots. These relations follow directly from Lemma 3.1 but not from Lemma 3.2 since 0 is not a root of $p(x)$.

Lemma 4.1. *A symmetric polynomial* $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ *of degree* $d = 2m_1 + 2m_2$ *with rational coefficients and with four rational roots is* $\mathbb{Q}$*-nice iff there exists a nonzero rational number c such that*

$$m_2 r_1^2 + m_1 r_2^2 = (m_1 + m_2)c^2. \tag{4.1}$$

The following theorem gives a formula for all $\mathbb{Q}$-nice symmetric polynomials with four roots. To find a formula, we solve (4.1) for all rational numbers r_1, r_2, and c. By Lemma 4.1, all such solutions allow us to find representatives of all the equivalence classes of $\mathbb{Q}$-nice (and nice) symmetric polynomials with four roots. In this sense, the formula gives all examples of nice symmetric polynomials with four roots.

Theorem 4.2. *The symmetric polynomial* $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ *with* $p'(x) = dx(x^2 - r_1^2)^{m_1 - 1}(x^2 - r_2^2)^{m_2 - 1}(x^2 - c^2)$ *is* $\mathbb{Q}$*-nice iff*

$$c = \frac{m_2 a^2 + m_1 b^2}{2m_2 a - 2m_1 b}, \tag{4.2}$$

$$r_1 = c - a, \tag{4.3}$$

$$r_2 = c + b, \tag{4.4}$$

where a and b are positive rational numbers such that $2m_2 a - 2m_1 b \neq 0$ *or, equivalently,* $a \neq \left(\frac{m_1}{m_2}\right)b$.

Proof. To find all rational solutions of (4.1), first assume $r_1 < r_2$. By Rolle's theorem, there is a critical point c so that $r_1 < c < r_2$; thus, $c = r_1 + a$ and $c = r_2 - b$ for some positive rational numbers a and b. Substituting $c - a$ for r_1 and $c + b$ for r_2 in (4.1), we have $m_2(c^2 - 2ac + a^2) + m_1(c^2 + 2bc + b^2) = (m_1 + m_2)c^2$ which simplifies to $m_2 a^2 + m_1 b^2 = (2m_2 a - 2m_1 b)c$. Dividing both sides by $2m_2 a - 2m_1 b \neq 0$, we obtain (4.2) above. Equations (4.3) and (4.4) follow from the definition of a and b. It is easy to see that if $2m_2 a - 2m_1 b = 0$, then (4.1) has no solution. This theorem is an equivalence because Lemma 4.1 is.

Remarks. (1) Because Theorem 4.2 is an equivalence, we can find representatives of all the equivalence classes of the type specified by m_1 and m_2 just by picking positive rational numbers a and b where $a \neq \left(\frac{m_1}{m_2}\right)b$ and using formulas (4.2)-(4.4) above.

(2) The values a and b given in formulas (4.2)-(4.4) can be interpreted geometrically as follows: The number a gives the distance between the critical point c and the root r_1, and the number b gives the distance between c and the root r_2. If we were to multiply (4.2), (4.3), and (4.4) by $2m_2a - 2m_1b$ to clear fractions (which is a horizontal stretch or compression) so that our formula gives examples of nice polynomials, then a and b would lose this geometric interpretation. For this reason, we prefer the rational forms of (4.2)-(4.4) given above.

Before we give a few examples we have found using formulas (4.2)-(4.4), we need to make a few comments. Since stretching or compressing $p(x)$ horizontally by a factor of $k \in \mathbb{Q}$ results in an equivalent $\mathbb{Q}$-nice polynomial, we may stretch or compress so that $p(x)$ has integer roots and so that the greatest common divisor of all the nonzero roots and nonzero critical points equals 1. Since $p(x)$ and $[p(x)]^n$ are equivalent, we may take the n-th root so that the greatest common divisor of the multiplicities of the roots of $p(x)$ equals 1. Thus, we may use as a representative of any equivalence class a monic nice polynomial whose greatest common divisor of all the nonzero roots and critical points equals 1 and whose greatest common divisor of the multiplicities of the roots equals 1. Such a nice polynomial is in *reduced form*.

We now give several examples we have found using formulas (4.2)-(4.4). All these examples are in reduced form.

Example 4.3. If we specify the type (3, 3, 1, 1) and choose $a = 1$ and $b = 5$, then our formulas give $r_1 = -26/7$, $r_2 = 16/7$, and $c = -19/7$ if $m_1 = 3$ and $m_2 = 1$. Note that this example is equivalent to $p(x) = (x^2 - 26^2)^3(x^2 - 16^2)$ with derivative $p'(x) = 8x(x^2 - 26^2)^2(x^2 - 19^2)$. If

$m_1 = 1$ and $m_2 = 3$, then our formulas give $r_1 = -8$, $r_2 = -2$, and $c = -7$. Note that this example is equivalent to $p(x) = (x^2 - 2^2)^3 (x^2 - 8^2)$ with derivative $p'(x) = 8x(x^2 - 2^2)^2(x^2 - 7^2)$.

Example 4.4. If we specify the type $(7, 7, 5, 5)$ and choose $a = 2$ and $b = 7$, then our formulas give $r_1 = -173/26$, $r_2 = 61/26$, and $c = -121/26$ if $m_1 = 7$ and $m_2 = 5$. Note that this example is equivalent to $p(x) = (x^2 - 173^2)^7(x^2 - 61^2)^5$ with derivative $p'(x) = 24x(x^2 - 173^2)^6 (x^2 - 61^2)^4(x^2 - 121^2)$. If $m_1 = 5$ and $m_2 = 7$, then our formulas give $r_1 = -17/2$, $r_2 = 1/2$, and $c = -13/2$. Note that this example is equivalent to $p(x) = (x^2 - 17^2)^5(x^2 - 1)^7$ with derivative $p'(x) = 24x(x^2 - 17^2)^4(x^2 - 1)^6(x^2 - 13^2)$.

We now use Theorem 4.2 to complete our description of all nice symmetric polynomials with four roots. That is, we now count the number of equivalence classes of all types of nice symmetric polynomials with four roots.

Corollary 4.5. *For every degree* $d \geq 4$ *and every type* (m_1, m_1, m_2, m_2) *such that* $2m_1 + 2m_2 = d$, *there exist infinitely many equivalence classes of nice symmetric polynomials of this type.*

Proof. If a and b are positive rational numbers, then, by (4.3) and (4.4), $r_1 \neq c$ and $r_2 \neq c$. Hence, to find all values for a and b where r_1^2, r_2^2, and c^2 are not all distinct, all we need are the values for a and b where $r_1 = \pm r_2$, where $r_1 = 0$, and where $r_2 = 0$. Solving the equation $r_1 = \pm r_2$ for a and b, we have that $a = \pm b$. But $a \neq -b$ since a and b are positive. Thus, the ratio $a/b = 1 > 0$ fails to give four distinct roots. Now we solve the equation $r_1 = 0$ for a/b, which is equivalent to solving $m_2a^2 + m_1b^2 = 2m_2a^2 - 2m_1ab$. Rewrite this as $m_1b^2 = m_2a^2 - 2m_1ab$. Dividing both sides by $b^2 \neq 0$, we now have $m_1 = m_2\left(\frac{a}{b}\right)^2 - 2m_1\frac{a}{b}$.

Thus, there exist at most two ratios $a/b > 0$ so that $r_1 = 0$. Similarly, by solving the equation $r_2 = 0$ for a/b, we obtain $m_2\left(\frac{a}{b}\right)^2 + 2m_2\frac{a}{b} = m_1$, so there exist at most two ratios $a/b > 0$ where $r_2 = 0$. Thus, at most five ratios $a/b > 0$ fail to give four distinct roots.

Since all examples $p(x)$ found by our formula are monic and since $p(x)$ is even, examples found by our formula are equivalent only by horizontal stretches or compressions. Since horizontal stretches and compressions preserve the ratio a/b and since at most five ratios $a/b > 0$ fail to give four distinct roots, we can choose infinitely many different ratios $a/b > 0$ that give four distinct roots, regardless of the type, allowing us to find infinitely many equivalence classes of nice symmetric polynomials of any given type with four roots.

The following example illustrates Corollary 4.5.

Example 4.6. If we specify the type (3, 3, 2, 2) and choose $a = 1$ and $b = 4$, then our formulas give $r_1 = -7/2$, $r_2 = 3/2$, and $c = -5/2$ if $m_1 = 3$ and $m_2 = 2$. Note that this example is equivalent to $p_1(x) = (x^2 - 7^2)^3(x^2 - 3^2)^2$, $p_1'(x) = 10x(x^2 - 7^2)^2(x^2 - 3^2)(x^2 - 5^2)$. If we choose $a = 1$ and $b = 2$, then our formulas give $r_1 = -11/4$, $r_2 = 1/4$, and $c = -7/4$. Note that this example is equivalent to $p_2(x) = (x^2 - 11^2)^3(x^2 - 1)^2$, $p_2'(x) = 10x(x^2 - 11^2)^2(x^2 - 1)(x^2 - 7^2)$. Both these examples are inequivalent examples of the type (3, 3, 2, 2).

The following formula for all nice symmetric quartics, found in [3], was originally derived by the use of Pythagorean triples. We can give an alternate derivation of this formula by using formulas (4.2)-(4.4).

Corollary 4.7. *A symmetric quartic $p(x)$ is nice iff*

$$p(x) = [x^2 - (n^2 - m^2 - 2mn)^2][x^2 - (n^2 - m^2 + 2mn)^2],$$

$$p'(x) = 4x[x^2 - (m^2 + n^2)^2]$$

for some positive integers m and n.

Proof. Using (4.2)-(4.4) with $m_1 = m_2 = 1$ and positive integers a and b we have $c = \dfrac{a^2 + b^2}{2a - 2b}$, $r_1 = \dfrac{b^2 - a^2 + 2ab}{2a - 2b}$, and $r_2 = \dfrac{a^2 - b^2 + 2ab}{2a - 2b}$. By the second part of the proof of Corollary 4.5, each equivalence class is determined by the ratio a/b so we may take a and b to be positive integers. If we stretch $p(x)$ horizontally by $2a - 2b$, we have $c = a^2 + b^2$, $r_1 = b^2 - a^2 + 2ab$, and $r_2 = a^2 - b^2 + 2ab$. Since $r_2^2 = (-r_2)^2$, $p(x) = [x^2 - (b^2 - a^2 + 2ab)^2][x^2 - (b^2 - a^2 - 2ab)^2]$, and $p'(x) = 4x[x^2 - (a^2 + b^2)^2]$. This polynomial is nice because a and b are integers. This is the formula mentioned above with a and b in place of m and n.

5. Higher-order Nicety Properties

When we study nice polynomials, we study both $p(x)$ and its derivative $p'(x)$. It is, therefore, natural to extend this problem by considering higher-order derivatives as well. If $p(x)$ and its first two derivatives have integer roots, then we say $p(x)$ is *doubly nice*. If $p(x)$ and its first three derivatives have integer roots, then $p(x)$ is *triply nice*. In general, $p(x)$ is *m-th order nice* if $p(x)$ and its first m derivatives have integer roots. If all nonconstant derivatives have integer roots, then we say that $p(x)$ is *totally nice*. The paper [5] uses these same definitions.

The idea of considering higher-order derivatives leads us to the following question: Which types of nice symmetric polynomials with four roots are doubly nice? triply nice? m-th order nice (for some choice of $m > 1$)? totally nice? This question is not completely answered, but we do have one result that tells us that certain types of nice symmetric polynomials with four roots are not m-th order nice. We then conclude that no nice symmetric polynomials with four roots are totally nice. The proof of this uses the following result from an earlier paper [6, Theorem 3.3]: If $p(x)$ is a nice symmetric or antisymmetric polynomial of degree d

with an odd number of roots with the center root having multiplicity m_0, then d/m_0 is the square of a rational number.

We now state and prove these results.

Proposition 5.1. *Let* $m \geq 2$ *be a positive even integer. If the degree* $d > m+1$ *of a nice symmetric polynomial* $p(x)$ *is not of the form* $n^2 + (m-1)$ *for some odd integer n, then* $p(x)$ *is not m-th order nice.*

Proof. Suppose the degree of $p(x)$ is not of the form $n^2 + (m-1)$, so $d - (m-1)$ is not a perfect square. The $(m-1)$st derivative of $p(x)$ is, therefore, an antisymmetric polynomial of degree $d - (m-1)$ with the center root having multiplicity $m_0 = 1$. Since $\frac{d-(m-1)}{m_0} = d - (m-1)$ is not the square of a rational number, by [6, Theorem 3.3], the $(m-1)$st derivative is not a nice polynomial, so either $p^{(m-1)}(x)$ or $p^{(m)}(x)$ does not have integer roots. Therefore, $p(x)$ is not m-th order nice.

The following corollary is a direct result of the above proposition. The proof is simple, so we omit it.

Corollary 5.2. *There are no totally nice symmetric polynomials with four roots.*

Nice symmetric quartics and sextics are of special interest, so we state the following result about these nice symmetric polynomials.

Corollary 5.3. *Nice symmetric quartics and sextics are not doubly nice.*

Remark. The condition stated in Proposition 5.1 is necessary but not sufficient. To see this, note that both of the nice polynomials stated in Example 4.6 are not doubly nice, yet both are nice symmetric polynomials of degree $10 = 3^2 + (2-1)$. Finding sufficient conditions in Proposition 5.1 is difficult in general because $p'(x)$, which is antisymmetric, can have up to seven roots and almost nothing is known about nice antisymmetric polynomials with seven roots.

6. Open Problems

We now conclude by stating several open problems about nice polynomials with four roots.

Problem 6.1. Which types of nice nonsymmetric polynomials with four roots exist? The only types whose existence is currently known are the types $(1, 1, 1, 1)$, $(2, 1, 1, 1)$, $(3, 1, 1, 1)$ and $(4, 1, 1, 1)$.

Problem 6.2. Find a formula for all nice polynomials with four roots.

Problem 6.3. Suppose nice polynomials of the type (m_0, m_1, m_2, m_3) exist. How many equivalence classes exist for the type (m_0, m_1, m_2, m_3)?

The following open problem comes from Section 5.

Problem 6.4. Are there nice symmetric polynomials with four roots that are doubly nice? If so, which ones? What about triply nice? Or m-th order nice (where m is any positive integer greater than 1)?

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