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ON THE APPLICATION OF DEPENDENCE WITH COMPLETE CONNECTIONS FOR A CLASS OF SERIES REPRESENTATIONS FOR REAL NUMBERS

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Abstract

We investigate the analogous Gauss-Kuzmin type problem for a class of series representations for real numbers known as Lüroth series. We obtain significant asymptotic formulas for their digits and study some applications of the law of large numbers for the associated Markov chain which lead to important ergodic properties. Our approach is given in the context of the theory of stochastic dependence with complete connections.

1. Introduction

At first, we define a *general algorithm* introduced in Galambos [1] as follows. Let $\alpha_j(n) > 0$, $j = 1, 2, \ldots$ be a sequence of strictly decreasing functions of natural numbers n such that, for each j, $\alpha_j(1) = 1$ and $\alpha_j(n) \to 0$, as $n \to \infty$. Let also $\gamma_j(n)$ be another sequence of positive

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functions of n with the property that, for $n \geq 2$,

$$\alpha_j(n-1) - \alpha_j(n) \leq \gamma_j(n). \tag{1.1}$$

Then for an arbitrary real number $0 < x \leq 1$, we define the *integers* $d_j = d_j(x)$ and the *real numbers* x_j, $j = 1, 2, \ldots$ by the algorithm

$$x = x_1,\ \alpha_j(d_j) < x_j \leq \alpha_j(d_j - 1) \tag{1.2}$$

$$x_{j+1} = \{x_j - \alpha_j(d_j)\}/\gamma_j(d_j). \tag{1.3}$$

By the assumptions (1.1) and (1.2), it is implied that $0 < x_{j+1} \leq 1$. Putting

$$y_N = \alpha_1(d_1) + \gamma_1(d_1) \cdot \alpha_2(d_2) + \cdots + \gamma_1(d_1) \cdots \gamma_{N-1}(d_{N-1}) \cdot \alpha_N(d_N) \tag{1.4}$$

repeated application of (1.2) and (1.3) yields

$$x - y_N = \gamma_1(d_1) \cdot \gamma_2(d_2) \cdots \gamma_N(d_N) \cdot x_{N+1}. \tag{1.5}$$

Since by assumption, we have that $\gamma_j(n) > 0$, (1.4) and (1.5) imply

$$0 < y_N < x$$

and thus $y_N = y_N(x)$ has a finite limit $y(x)$ as $N \to +\infty$, that is, for each x the infinite series

$$y(x) = \alpha_1(d_1) + \gamma_1(d_1) \cdot \alpha_2(d_2) + \gamma_1(d_1) \cdot \gamma_2(d_2) \cdot \alpha_3(d_3) + \cdots \tag{1.6}$$

always converges and

$$0 < y(x) \leq x. \tag{1.7}$$

Definition 1.1. The infinite series (1.6) of $y(x)$ is called the (α, γ)-*expansion* of x.

Here the letters α and γ stand for the sequences $\alpha_j(n)$ and $\gamma_j(n)$, respectively. Also we have to notice that the (α, γ)-expansion $y(x)$ of x is not guaranteed to equal x and indeed, there are cases where the equality (1.7) fails. Such an example is given by Vervaat [11, pp. 101].

In order to give a necessary and sufficient condition for $y(x) = x$, we shall need the following definitions:

Definition 1.2. The vector $(k_1, k_2, \ldots, k_N)$ with positive integer components is called *realizable* with respect to the (α, γ)-expansion of real numbers if and only if there is at least one x in $(0, 1]$ such that $d_j(x) = k_j, 1 \le j \le N$.

Definition 1.3. An infinite sequence $k_1, k_2, \ldots$ of positive integers is called *realizable* with respect to the (α, γ)-expansion if and only if for each N, $(k_1, k_2, \ldots, k_N)$ is realizable.

The next theorem provides the suitable condition for all integers $d_j \ge 1, 1 \le j \le N$, under which $y(x) = x$, for all $x \in (0, 1]$.

Theorem 1.4. *The* (α, γ)*-expansion* $y(x)$ *of* x *satisfies the relation* $y(x) = x$, *for each* x *in* $(0, 1]$ *if and only if, for any realizable sequence* $k_1, k_2, \ldots$

$$\lim_{N \to +\infty} [\gamma_1(k_1) \cdot \gamma_2(k_2) \ldots \gamma_{N-1}(k_{N-1}) \cdot (\alpha_N(k_N - 1) - \alpha_N(k_N))] = 0.$$

We shall study a special case of the above-mentioned algorithm by specifying the functions $\alpha_j(n)$ and $\gamma_j(n)$.

1.1. The Balkema-Oppenheim expansion

Let $\alpha_j(n) \equiv \alpha(n)$ be independent of j. Let further $h(n)$ be a positive integer valued function of n for $n \ge 2$. Then we define, for each $j \ge 1$,

$$\gamma_j(n) \equiv \gamma(n) = \{\alpha(n-1) - \alpha(n)\}/\alpha(h(n)), \; n \ge 2. \tag{1.1.1}$$

It is obvious that the resulting series

$$y(x) = \alpha(d_1) + \gamma(d_1) \cdot \alpha(d_2) + \gamma(d_1) \cdot \gamma(d_2) \cdot \alpha(d_3) + \cdots \tag{1.1.2}$$

is identical to (1.6) except that the subscripts for α and γ should be dropped. The (α, γ)-expansion (1.1.2) of real numbers is called *Balkema-Oppenheim expansion*.

The special case of $\alpha(n) = \frac{1}{n}$ leads to

$$\gamma(n) = h(n)/n(n-1).$$

Hence we have the following (α, γ)-expansion of $x \in (0, 1]$

$$y(x) = \frac{1}{d_1} + \frac{h(d_1)}{d_1(d_1 - 1)} \cdot \frac{1}{d_2} + \frac{h(d_1) \cdot h(d_2)}{d_1(d_1 - 1)d_2(d_2 - 1)} \cdot \frac{1}{d_3} + \cdots \qquad (1.1.3)$$

with $d_{j+1} > d_j - 1$, $d_j \geq 2$, for any $j \geq 1$.

The Balkema-Oppenheim (α, γ)-expansion (1.1.2) or its special case (1.1.3), $y(x)$ of x satisfies the equality $y(x) = x$ since Theorem 1.4 is directly applicable. In particular by putting $h(n) = 1$, for $n \geq 2$, we take the following (α, γ)-expansion of every real number in the left-open, right-closed unit interval $(0, 1]$

$$y(x) = \frac{1}{d_1} + \frac{1}{d_1(d_1 - 1)} \cdot \frac{1}{d_2} + \frac{1}{d_1(d_1 - 1)d_2(d_2 - 1)} \cdot \frac{1}{d_3} + \cdots \qquad (1.1.4)$$

$$\equiv (d_1, d_2 \ldots),$$

with d_j natural numbers such that $d_j \geq 2$, for any $j \geq 1$, which is called *Lüroth series* (see, e.g., [10], Satz 48).

Moreover, every real number $x \in (0, 1]$ has α unique representation in the form of a Lüroth series (1.1.4). This means that there exists a one-to-one correspondence between the elements $x \in (0, 1]$ and the sequences $(d_1, d_2, \ldots.)$, $d_n \geq 2$, $n = 1, 2, \ldots$. Specifically x is rational if and only if its sequence of digits $d_1, d_2, d_3, \ldots$ terminates or is periodic.

We should note that the above expansion can be generated by the operator $T : (0, 1] \to (0, 1]$ defined by the relation

$$T(x) = \left(\left[\frac{1}{x}\right] - 1\right) \cdot \left(-1 + x\left[\frac{1}{x}\right]\right). \qquad (1.1.5)$$

In fact if $d_1(x) = \left[\frac{1}{x}\right]$ and $d_{n+1} \equiv d_{n+1}(x) = \left[\frac{1}{T^n(x)}\right]$, for $T^n(x) \neq 0$, we obtain a unique finite or infinite representation for any real number $x \in (0, 1]$ as the one given by (1.1.4).

In addition we should mention that the digits d_n, $n = 1, 2, \ldots$ may be considered as random variables on $(0, 1]$ equipped with the σ-algebra $\mathcal{B}_{(0,1]}$ of all Borel subsets in $(0, 1]$. These are almost surely defined with respect to any probability measure on $\mathcal{B}_{(0,1]}$ assigning probability 0 to the set of rationals in $(0, 1]$ (in particular with respect to the ordinary Borel-Lebesgue measure λ).

Define by

$$\frac{P_n(x)}{q_n(x)} = \frac{1}{d_1} + \sum_{k=2}^{n} \frac{1}{q_k}, \tag{1.1.6}$$

where $q_k = d_1(d_1 - 1) \ldots d_{k-1}(d_{k-1} - 1) \cdot d_k$ the nth convergent of x and put

$$\begin{aligned} S_n &= (d_n, d_{n-1}, \ldots, d_2, d_1) \\ &= \frac{1}{d_n} + \frac{1}{d_n(d_n - 1)} \cdot \frac{1}{d_{n-1}} \\ &\quad + \cdots + \frac{1}{d_n(d_n - 1)d_{n-1}(d_{n-1} - 1) \ldots d_2(d_2 - 1)} \cdot \frac{1}{d_1}. \end{aligned} \tag{1.1.7}$$

The aim of the present paper is to give the asymptotical properties of the sequences of $(d_n)_n, (S_n)_n$ by using the approach of stochastic dependence with complete connections.

The paper is organized as follows. In Section 2, we prove the existence of a random system with complete connections associated with the expansion (1.1.4) and identify its limit probability measure. In Section 3, we solve the version of the Gauss-Kuzmin type problem for the Lüroth series by using the theory of random systems with complete connections. Section 4 deals with the law of large numbers for the chain of infinite order of the above random system providing asymptotical properties for the digits d_n, $n = 1, 2, \ldots$ Finally in Section 5 by using the law of large numbers for the associated Markov chain of the suitable constructed random system with complete connections we arrive at results on approximation theory.

2. Construction of the Random System with Complete Connections

We are interested on the Lüroth expansion (1.1.4) for any irrational number $x \in (0, 1]$ and define by

$$\varphi_n(x) = \frac{1}{d_n(x)} + \frac{1}{d_n(x) \cdot (d_n(x) - 1)} \cdot \frac{1}{d_{n-1}(x)}$$

$$+ \frac{1}{d_n(x) \cdot (d_n(x) - 1) \cdot d_{n-1}(x) \cdot (d_{n-1}(x) - 1)} \cdot \frac{1}{d_{n-2}(x)}$$

$$+ \cdots + \frac{1}{d_n(x) \cdot (d_n(x) - 1) \ldots d_2(x) \cdot (d_2(x) - 1)} \cdot \frac{1}{d_1(x)}. \quad (2.1)$$

It is obvious that $(\varphi_n(\cdot))_n$ is a sequence of real random variables defined on $(0, 1]$, which satisfies the recursive relation

$$\varphi_{n+1} = \frac{1}{d_{n+1}} + \frac{1}{d_{n+1}(d_{n+1} - 1)} \cdot \phi_n, \quad n \geq 1. \quad (2.2)$$

In the next we shall need the following (for the proof, see, Galambos [1]):

Proposition 2.1. *The digits* $d_n(\cdot), n \geq 1$, *are stochastically independent and identically distributed random variables with respect to Lebesgue measure* λ *with*

$$\lambda(d_n = k) = \frac{1}{k(k-1)}, \; k \geq 2.$$

The relations (2.1), (2.2) and Proposition 2.1 lead us to the consideration of a random system with complete connections (RSCC)

$$\{((0, 1], \mathcal{B}_{(0, 1]}), (A, \mathcal{P}(A)), u, P\}, \quad (2.3)$$

where

$$u(x, n) = \frac{1}{n} + \frac{1}{n(n-1)} \cdot x, \; P(x, n) = \frac{1}{n(n-1)}, \; x \in (0, 1], \; n \geq 2,$$

$A = \{2, 3, \ldots\}$ and $\mathcal{P}(A)$ is the power set of A.

Let us now define for each real-valued function f on $(0, 1]$ the following positive numbers

$$|f| = \sup_{x \in (0, 1]} |f(x)|, \quad s(f) = \sup_{x_1 \neq x_2 \in (0, 1]} \left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} \right|.$$

Then the set $L((0, 1])$ of all real-valued functions defined on $(0, 1]$ such that $|f|$, $s(f)$ are finite positive numbers is a Banach space with respect to norm $\| f \| = |f| + s(f)$. We shall prove the following:

Theorem 2.2. *The random system* (2.3) *is an RSCC with contraction while its Markov operator is regular with respect to* $L((0, 1])$.

Proof. According to Norman's definition (see [10]) and since

$$\frac{dP(x, n)}{dx} = 0, \; \frac{du(x, n)}{dx} = \frac{1}{n(n-1)} < 1, \; n \geq 2,$$

we obtain that the random system (2.3) is an RSCC with contraction. Furthermore in order to prove that the associated Markov operator U is regular with respect to $L((0, 1])$, we have to prove the existence of a point $x_0 \in (0, 1)$ such that

$$\lim_{n \to \infty} \left| \sum\nolimits_n (x) - x_0 \right| = 0,$$

for any $x \in (0, 1]$, where $\sum_n (x)$ denotes the support of the measure $Q^n(x, .)$. (see Grigorescu and Iosifescu [3]). Here Q is the transition probability measure of the associated Markov chain, which is defined by (1.1.7) (or its equivalent recursive relation (2.2)).

Let x be an arbitrarily fixed number in $(0,1]$. Then by defining

$$x_1 = x, \quad x_{n+1} = \frac{1}{3} + \frac{1}{6} \cdot x_n, \; n \geq 1, \tag{2.4}$$

we obtain that $x_n \in (0, 1)$. Letting $n \to \infty$ in (2.4), we take that $x_0 = \frac{2}{5}$ and the proof is complete.

By using the above Theorem and Theorem 3.4.5 of Grigorescu and Iosifescu [3], we deduce that the RSCC (2.3) is uniformly ergodic and as a consequence there exists a unique limit probability measure on $\mathcal{B}_{(0,\,1]}$ which is identified by the following:

Theorem 2.3. *The limit probability measure of the RSCC* (2.3) *is identical to the Lebesgue measure* λ.

Proof. By virtue of uniqueness of the limit probability measure we have to show that it satisfies the equality

$$\int_0^1 Q(x,\, B) \cdot \lambda(dx) = \lambda(B), \tag{2.5}$$

for all $B \in \mathcal{B}_{(0,\,1]}$, where $Q(.\,,\,.)$ is defined by

$$Q(x,\, B) = \sum_{\substack{n \geq 2 \\ u(x,\, n) \in B}} P(x,\, n).$$

In fact it suffices to verify (2.5) only for $B = (0,\, t]$, where t ranges over the rationals of $(0,\, 1]$.

Let us fix $t_0 \in (0,\, 1]$. Then the inequality

$$\frac{n - 1 + x}{n \cdot (n - 1)} \leq t_0$$

is satisfied by the integers n such that

$$n \geq \left[\frac{t_0 + 1 + \sqrt{(t_0 + 1)^2 - 4t_0 \cdot (1 - x)}}{2t_0} \right] + 1, \quad x \in (0,\, 1].$$

If

$$n_0 \equiv n_0(x) = \min\left\{ n \geq 2 : \frac{n - 1 + x}{n \cdot (n - 1)} \leq t_0 \right\}, \quad x \in (0,\, 1],$$

then we get that

$$\int_0^1 Q(x,\, (0,\, t_0])dx = \int_0^1 \left(\sum_{n \geq n_0(x)} \frac{1}{n(n - 1)} \right) dx = \int_0^1 \frac{1}{n_0(x) - 1}\, dx. \tag{2.6}$$

Furthermore since

$$\frac{1}{t_0} \le \frac{1 + t_0 + \sqrt{(t_0 + 1)^2 - 4t_0 \cdot (1 - x)}}{2t_0} < \frac{1}{t_0} + 1, \tag{2.7}$$

we may find x_0 such that

$$\frac{1 + t_0 + \sqrt{(t_0 + 1)^2 - 4t_0 \cdot (1 - x_0)}}{2t_0} = \left[\frac{1}{t_0}\right].$$

Therefore, we obtain that

$$t_0 = \frac{n_0 - 1 + x_0}{n_0 \cdot (n_0 - 1)}.$$

In the next by further equalities like (2.6), we take

$$\int_0^{x_0} \frac{1}{n_0 - 1} dx + \int_{x_0}^1 \frac{1}{n_0} dx = \frac{1}{n_0 - 1} \cdot x_0 + \frac{1}{n_0} \cdot (1 - x_0)$$

$$= \frac{n_0 - 1 + x_0}{n_0 \cdot (n_0 - 1)} = t_0 = \lambda((0, t_0])$$

and the proof is complete.

3. The Gauss-Kuzmin Type Problem

Let us now define the *nth rank remainder* of the Lüroth expansion (1.1.4) given by the relation

$$r_n(x) = \frac{1}{d_n(x)} + \frac{1}{d_n(x) \cdot (d_n(x) - 1)} \cdot \frac{1}{d_{n+1}(x)} + \cdots. \tag{3.1}$$

If μ is an arbitrary monatomic probability measure on $\mathcal{B}_{(0,1]}$, then we may define the function

$$F_n(x) = F_n(x, \mu) = \mu(r_{n+1} < x), \tag{3.2}$$

for any $n = 0, 1, 2, \ldots, x \in (0, 1]$. It is obvious that $F_0(x) = \mu((0, x])$.

Since $0 < r_{n+2} < x$ if and only if

$$\frac{1}{d_{n+1}} < r_{n+1} < \frac{1}{d_{n+1}} + \frac{1}{d_{n+1} \cdot (d_{n+1} - 1)} \cdot x,$$

we may take the following associated *Gauss equation*

$$F_{n+1}(x) = \sum_{k \geq 2} \left[F_n\left(\frac{1}{k} + \frac{1}{k(k-1)} \cdot x \right) - F_n\left(\frac{1}{k} \right) \right]. \tag{3.3}$$

Assuming that F_0' exists and is bounded (that is, the measure μ has bounded density), we obtain by induction that F_n', $n \geq 1$, exist and are bounded too.

By taking the derivative of (3.3), we arrive at

$$F_{n+1}'(x) = \sum_{k \geq 2} \frac{1}{k(k-1)} \cdot F_n'\left(\frac{1}{k} + \frac{1}{k(k-1)} \cdot x \right), \tag{3.4}$$

for any $n = 0, 1, 2, \ldots$, $x \in (0, 1]$.

If $F_n'(x) \equiv f_n(x)$, $n = 0, 1, 2, \ldots$, $x \in (0, 1]$, then relation (3.4) becomes

$$f_{n+1}(x) = \sum_{k \geq 2} \frac{1}{k(k-1)} \cdot f_n\left(\frac{1}{k} + \frac{1}{k(k-1)} \cdot x \right). \tag{3.5}$$

So $f_{n+1} = Uf$, where U denotes the associated Markov operator of RSCC (2.3). Then

$$F_n(x) = \int_0^x U^n f_0(t) dt, \quad n = 0, 1, \ldots. \tag{3.6}$$

Here $f_0(x) = F_0'(x)$, $x \in (0, 1]$.

Now we can solve the Gauss-Kuzmin type problem for the Lüroth series representation given by the following:

Theorem 3.1. *If the density* $F_0' \in L((0, 1])$, *then there exist two positive constants c and* $q < 1$ *such that for each* $x \in (0, 1]$, $n = 0, 1, 2, \ldots$, *we have*

$$\mu(r_n < x) = (1 + \theta \cdot q^n) \cdot x, \tag{3.7}$$

where $\theta = \theta(n, x)$ *with* $|\theta| \leq c$.

Proof. Let F_0' be a Lipschitz function. Then $f_0 \in L((0, 1])$ and

$$U^\infty f_0 = \int_0^1 f_0(x) \lambda(dx) = \int_0^1 F_0'(x) dx = F_0(1) = 1.$$

Then by virtue of Theorem 2.2 and Lemma 3.1.22 of Grigorescu and Iosifescu [3] and by writing $T = U - U^{\infty}$, we take

$$U^n f_0(x) = U^{\infty} f_0(x) + T^n f_0(x) = 1 + \theta(n, x) \cdot q^n,$$

where $0 < q < 1$ and $|\theta| \leq c$, for $c > 0$. Then (3.7) follows from (3.2) and (3.6) and the proof is complete.

4. Asymptotical Behaviour of the Digits

On account of Theorems 2.2 and 2.3 it is possible to apply the law of large numbers to the chain of infinite order of RSCC (2.3), which is the sequence of digits $(d_n)_n$ defined by (1.1.5) (see [5]).

So if we consider a real measurable function g defined on $(0, 1]$ and put $f_n \equiv g \circ T^{n-1}$, $n = 1, 2, \ldots,$ where T is the transformation defined as in (1.1.5), we can take the following theorem which is a generalization of Kolmogorov's law of large numbers giving a connection between the measures λ and P_x (see Theorem 2.2.12 of Iosifescu and Theodorescu [4]). The interpretation of P_x for irrational numbers x is similar to that given in [4].

Theorem 4.1. *Assume that g is integrable* $\left(\text{that is, } \int_0^1 |g(x)| dx < \infty\right)$. *Then the series* $\left(\sum_{k=1}^{n} f_i\right)\Big/ n$ *converges* $P_x - \alpha.e.$ *to* $E_{\lambda}(g)$, *for every* $x \in (0, 1]$, *where*

$$E_{\lambda}(g) = \int_0^1 g(x) \cdot \lambda(dx).$$

Now we are ready to obtain some asymptotical results concerning the sequence $(d_n)_n$.

Proposition 4.2 (Relative frequency of digits)**.** *Let k be any positive integer with $k \geq 2$. Then*

$$\lim_{n \to \infty} \sum_{\substack{j \leq n \\ d_j = k}} 1 = \frac{1}{k(k-1)}, \; P_x - \alpha.e.$$

Proof. If we apply Theorem 4.1 to $g = I_{\{d_1 = k\}}$, where $\{d_1 = k\}$ $= \left[\frac{1}{k}, \frac{1}{k} + \frac{1}{k(k-1)}\right]$, we obtain

$$\lim_{n\to\infty} \sum_{\substack{j\le n \\ d_j = k}} 1 = \int_0^1 I_{\{d_1=k\}} \cdot \lambda(dx) = \int_{\{d_1=k\}} \lambda(dx) = \int_{1/k}^{1/k+1/k(k-1)} \lambda(dx) = \frac{1}{k(k-1)}, \, k \ge 2.$$

Proposition 4.3 (Geometric means of digits)**.** *For any* $x \in (0, 1]$, *we have*

$$\lim_{n\to\infty} \sqrt[n]{d_1 \cdot d_2 \ldots d_n} = \exp\left[\sum_{k\ge 2} \frac{\log k}{k \cdot (k-1)}\right]$$

$\left(\textit{Here } \sum_{k\ge 2} \frac{\log k}{k\cdot(k-1)} = c_1, \textit{ where } c_1 \approx 1.25\right).$

Proof. Since

$$\{d_1 = k\} = \left[\frac{1}{k}, \frac{1}{k} + \frac{1}{k(k-1)}\right]$$

by applying Theorem 4.1 to $g = \log d_1$, we obtain

$$E_\lambda(f_1) = \sum_{k\ge 2} \int_{\{d_1 = k\}} \log k \, \lambda(dx) = \sum_{k\ge 2} \int_{1/k}^{1/k+1/k(k-1)} \log k \lambda(dx)$$

$$= \sum_{k\ge 2} \frac{\log k}{k\cdot(k-1)}.$$

But

$$\frac{1}{n} \cdot \sum_{k=1}^{n} f_k = \frac{1}{n} \cdot \sum_{k=1}^{n} \log d_k = \frac{1}{n} \log\left(\prod_{k=1}^{n} d_k\right) = \log(d_1 \ldots d_n)^{1/n}$$

$$= \log \sqrt[n]{d_1 \ldots d_n}.$$

So

$$\lim_{n\to\infty} \log \sqrt[n]{d_1 \ldots d_n} = \sum_{k\ge 2} \frac{\log k}{k\cdot(k-1)} \text{ or } \lim_{n\to\infty} \sqrt[n]{d_1 \ldots d_n} = \exp\left(\sum_{k\ge 2} \frac{\log k}{k\cdot(k-1)}\right)$$

and the proof is complete.

5. Approximation Theory

In this section, we study the problem of approximation of an irrational number $x \in (0, 1]$ by the nth convergent of the corresponding Lüroth series representation. We shall focus on the sequence $(S_n)_n$, where S_n is defined as in (1.1.7).

From the relation (1.1.7) it follows that

$$S_n = \frac{1}{d_n} + \frac{1}{d_n \cdot (d_n - 1)} \cdot S_{n-1}, \; n \geq 1, \text{ with } S_0 \equiv 0. \tag{5.1}$$

Since $S_n = u(S_{n-1}, d_n)$, where u is defined by (2.3), we have that the sequence $(S_n)_n$ is the associated Markov chain of the RSCC (2.3). So according to Proposition 2.1.4 of Iosifescu and Theodorescu [4], for any $x \in (0, 1]$ there exists a probability P_x (with the same interpretation as in the previous paragraph) such that the associated Markov chain of the RSCC (2.3) is exactly the sequence $(S_n)_n$. This gives us the possibility to state for an arbitrary real continuous function x defined on $(0, 1]$ a strong law of large numbers for the sequence $(h(S_n))_n$ with respect to $\widetilde{P}_x$. (See Theorem 2.2.15 of Iosifescu and Theodorescu [4].)

Theorem 5.1. *For any* $x \in (0, 1]$ *and real continuous function* h *defined on* $(0, 1]$, *we have*

$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} h(S_k) = \int_0^1 h(x) \lambda(dx), \; \widetilde{P}_x - \alpha.e.$$

So in order to prove asymptotical results regarding the chain $(S_n)_n$ it suffices to specify the form of function h.

Proposition 5.2. *For any* $x \in (0, 1]$,

$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} S_k = \frac{1}{2}, \; \widetilde{P}_x - \alpha.e.$$

Proof. By applying Theorem 5.1 to $h(x) = x$, we have

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} S_k = \int_0^1 x\lambda(dx) = \int_0^1 xdx = \frac{1}{2}.$$

Proposition 5.3. *For any* $x \in (0, 1]$,

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} \log S_k = -1,\ \widetilde{P}_x - \alpha.e.$$

Proof. By using Theorem 5.1 for $h(x) = x^{\varepsilon} \cdot \log|x|$, $x \in (0, 1]$, $\varepsilon > 0$, we have that

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} S_k^{\varepsilon} \log S_k = \int_0^1 x^{\varepsilon} \log|x|\lambda(dx),\ \widetilde{P}_x - \alpha.e.$$

Letting $\varepsilon \to 0$, we get

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} \log S_k = \int_0^1 \log|x|\lambda(dx),\ \widetilde{P}_x - \alpha.e. \tag{5.2}$$

By partial integration the right member of (5.2) becomes equal to -1 and the proof is complete.

Corollary 5.4. *For any* $x \in (0, 1]$, *we have*

$$\lim_{n\to\infty} n^{-1} \log\left| x - \frac{P_n(x)}{q_n(x)} \right| \le -1,\ \widetilde{P}_x - \alpha.e.$$

Proof. According to the definition (1.1.6) of the nth convergent of x, we have that

$$\left| x - \frac{P_n(x)}{q_n(x)} \right| = \prod_{k=1}^{n} \frac{1}{d_k \cdot (d_k - 1)} \cdot |T^n(x)|,\ n \ge 1.$$

Since $|T^n(x)| \le 1$, for any $x \in (0, 1]$, we obtain

$$\left| x - \frac{P_n(x)}{q_n(x)} \right| \le \prod_{k=1}^{n} \frac{1}{d_k \cdot (d_k - 1)}.$$

But S_k defined by the recursive relation (5.1) satisfies the inequality

$$S_k \geq \frac{1}{d_k \cdot (d_k - 1)}.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log S_k \geq - \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log[d_k \cdot (d_k - 1)]$$

or

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{P_n(x)}{q_n(x)} \right| \leq - \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \log[d_k \cdot (d_k - 1)] \leq -1$$

and the proof is complete.

Remark 1. Theorem 5.1 does not enable us to find the order of approximation of a real number by its Lüroth expansion. This result is obtained by applying the associated ergodic theorem for the measure-preserving and ergodic transformation T defined by (1.1.5) (see Jager and Vroedt [6]).

By virtue of this theorem it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log\left(x - \frac{P_n(x)}{q_n(x)} \right) = -d_1.$$

Therefore,

$$x - \frac{P_n(x)}{q_n(x)} \sim e^{-d_1 \cdot n}, \text{ a.e., where } d_1 \approx 2.03.$$

Remark 2. A random system with complete connections is a special case of infinite order chain [2, 8]. An alternative way for investigating higher-order Markov chains is the construction of multiple Markov chains through collections of directed circuits and positive weights named as *higher-order circuit chains* [7, 9].

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