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DEFINING RELATIONS OF THE SIMPLY-LACED 3-EXTENDED AFFINE LIE ALGEBRAS

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Abstract

We give the defining relations of the simply-laced 3-extended affine Lie algebras in terms of the generators associated to 3-extended affine diagrams.

1. Introduction

A toroidal Lie algebra $\mathfrak{u}$ is the universal central extension of a Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_m^{\pm 1}]$, where $\mathfrak{g}$ is one of the finite dimensional simple Lie algebras over $\mathbb{C}$ and $\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, ..., t_m^{\pm 1}]$ is the ring of Laurent polynomials in m variables $t_1, ..., t_m$ over $\mathbb{C}$. Let $A = (\langle \alpha_i, \alpha_j \rangle)_{1 \le i, j \le n}$ be any simply-laced finite Cartan matrix of rank $n \ge 2$, and $A^{[m]} = (\langle \alpha_i, \alpha_j \rangle)_{1 \le i, j \le n+m}$ be any m-fold affinization of A. Then Slodowy [7] introduced intersection matrix algebra $im(A^{[m]})$, and after that the following isomorphic has been established; $\mathfrak{u} \simeq im(A^{[m]})$ [1]. In vertex

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operator's method, Saito and Yoshii [6] constructed a Lie algebra $\mathfrak{g}(\Phi)$ attached to any m-extended homogeneous root system Φ as a certain subalgebra of $V_{Q(\Phi)}/DV_{Q(\Phi)}$, here V_Q is the lattice vertex algebra attached to a lattice Q and D is the derivation, (studied by Borcherds [2]). Especially, in the case of 2-extended root system (also called *elliptic root system*), $\mathfrak{g}(\Phi)$ is isomorphic to the 2-toroidal algebra. The m-toroidal algebra is presented in terms of infinite generators and infinite relations by Moody et al. [3, 4]. In particular, for the simply-laced elliptic root system Φ, Saito and Yoshii [6] also defined the elliptic algebra $\widetilde{e}(\Gamma(\Phi, G))$ which is isomorphic to $\widetilde{\mathfrak{g}}(\Phi)$, by Chevalley generators and generalized Serre relations. After that, Takebayashi [8] described the elliptic Lie algebra $\widetilde{e}(\Gamma(\Phi, G))$ by the extended elliptic Cartan matrix. Further, Yamane [11] described all elliptic Lie algebra with rank ≥ 2 by the Serre type relations. After that, Takebayashi [10] described them more simply by the completed elliptic diagrams. In the case of type $A_1^{(1,1)}$, Takebayashi [9] described the defining relations of its elliptic Lie algebra in terms of the elliptic diagram. In this paper, in the similar method as [9], we describe the 3-toroidal algebra $\mathfrak{g}_{tor}$ associated to the simply-laced Lie algebra $\mathfrak{g}$ in terms of the 3-extended affine diagram and call it the 3-*extended affine Lie algebra*.

2. Definition of the 3-toroidal Algebra $\mathfrak{g}_{tor}$

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{C}$ of rank n with a nondegenerate symmetric invariant bilinear form $(\cdot, \cdot)$. Let $R :=$ $\mathbb{C}[s^{\pm 1}, t^{\pm 1}, u^{\pm 1}]$ be the ring of Laurent polynomials in 3 variables s, t and u over $\mathbb{C}$. Let $\Omega_R^1 := Rds \oplus Rdt \oplus Rdu$ be an R-module with generators $df\ \forall f \in R$, and the relation $d(fg) = fd(g) + gd(f)$. Let $\bar{\cdot} : \Omega_R \to \Omega_R^1 / dR$ be the canonical projection, in which there holds the relation, $\overline{d(fg)} =$ $\overline{fd(g)} + \overline{gd(f)} = 0$. Then the Lie algebra $\mathfrak{g}_{tor} = \mathfrak{g} \otimes R \oplus (\Omega_R^1/dR)$, with Lie bracket $[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)\overline{(df)g}$, $[c, \mathfrak{g}_{tor}] = 0$,

$\forall\ c \in \Omega_R^1/dR,$ is the universal central extension of $\mathfrak{g} \otimes R,$ and called the 3-*toroidal algebra*. Let us recall the result by Moody et al. [4]. Let $A = (a_{ij})_{i,j=0}^{n}$ be a Cartan matrix of affine type $X_n^{(1)},$ Δ be the root system associated to A and Q be the free $\mathbb{Z}$-module on generators $\alpha_0, \ldots, \alpha_n$ of Δ.

Definition 2.1 (Moody et al. [4])**.** A *Lie algebra* $\tau(A)$ over $\mathbb{C}$ is defined by the following presentation.

Generators: $H_{i,k,m}, E_{i,k,m}, F_{i,k,m}$ $(0 \le i \le n,\ k,\ m \in \mathbb{Z})$ and $c_1,\ c_2$.

Relations: (0) c_1 and c_2 are central,

(I)

$$[H_{i,k,m},\ H_{j,l,p}] = (\alpha_i^\vee \mid \alpha_j^\vee)(kc_1 + mc_2)\delta_{k+l,0}\delta_{m+p,0}.$$

(II)

(i) $[H_{i,k,m},\ E_{j,l,p}] = (\alpha_i^\vee \mid \alpha_j)E_{j,k+l,m+p},$

(ii) $[H_{i,k,m},\ F_{j,l,p}] = -(\alpha_i^\vee \mid \alpha_j)F_{j,k+l,m+p},$

(III)

(i) $[E_{i,k,m},\ E_{i,l,p}] = [F_{i,k,m},\ F_{i,l,p}] = 0,$

(ii) $[E_{i,k,m}, F_{j,l,p}] = \delta_{ij}\left\{H_{i,k+l,m+p} + \dfrac{2}{(\alpha_i \mid \alpha_i)}(kc_1 + mc_2)\delta_{k+l,0}\delta_{m+p,0}\right\},$

(iii) $(adE_{i,0,0})^{-a_{ji}+1}E_{j,k,m} = 0\ (i \ne j),$

(iv) $(adF_{i,0,0})^{-a_{ji}+1}F_{j,k,m} = 0\ (i \ne j),$

where $0 \le i,\ j \le n,\ k,\ l,\ m,\ p \in \mathbb{Z},$ $(\cdot|\cdot)$ is a $\mathbb{Z}$-valued symmetric bilinear form on Q normalized by $(\alpha_0 \mid \alpha_0) = 2$ and $\alpha_i^\vee := \dfrac{2\alpha_i}{(\alpha_i \mid \alpha_i)},$ $a_{ij} = \dfrac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)}.$

Then the following has been established.

Proposition 2.2 [3, 4]. $\tau(A) \simeq \mathfrak{g}_{tor}$.

Definition 2.3. In the case of simply-laced affine Lie algebra of type $X_n^{(1)}$, its 3-toroidal algebra $\mathfrak{g}_{tor}$ is presented as follows.

Generators: $H_{i,k,m}$, $E_{i,k,m}$, $F_{i,k,m}$, $(0 \leq i \leq n,\ k,\ m \in \mathbb{Z})$ and c_1, c_2.

Relations: (0) c_1 and c_2 are central,

(I)

$$[H_{i,k,m},\ H_{j,l,p}] = a_{ji}(kc_1 + mc_2)\delta_{k+l,0}\delta_{m+p,0},$$

(II)

(i) $[H_{i,k,m},\ E_{j,l,p}] = a_{ji}E_{j,k+l,p+m}$,

(ii) $[H_{i,k,m},\ F_{j,l,p}] = -a_{ji}F_{j,k+l,p+m}$,

(III)

(i) $[E_{i,k,m},\ E_{i,l,p}] = [F_{i,k,m},\ F_{i,l,p}] = 0$,

(ii) $[E_{i,k,m},\ F_{j,l,p}] = \delta_{ij}\{H_{i,k+l,m+p} + (kc_1 + mc_2)\delta_{k+l,0}\delta_{m+p,0}\}$,

(iii) $(adE_{i,0,0})^{1-a_{ji}} E_{j,k,m} = 0\ (i \neq j)$,

(iv) $(adF_{i,0,0})^{1-a_{ji}} F_{j,k,m} = 0\ (i \neq j)$.

Our main result is the following.

Theorem 2.4. *The simply-laced* 3-*extended affine Lie algebra is described as follows.*

Generators: e_i, e_i^*, $\widetilde{e}_i$, f_i, f_i^*, $\widetilde{f}_i$, h_i $(0 \leq i \leq n)$ *and* c_1, c_2.

Relations: (0) c_1 *and* c_2 *are central*,

(I)

(i) $[h_i,\ h_j] = 0$,

(ii) $[h_i,\ e_j] = a_{ji}e_j$, $[h_i,\ e_j^*] = a_{ji}e_j^*$, $[h_i,\ \widetilde{e}_j] = a_{ji}\widetilde{e}_j$,

$[h_i,\ f_j] = -a_{ji}f_j$, $[h_i,\ f_j^*] = -a_{ji}f_j^*$, $[h_i,\ \widetilde{f}_j] = -a_{ji}\widetilde{f}_j$,

(II)

(i) $[e_i, e_i^*] = [e_i, \widetilde{e}_i] = [e_i^*, \widetilde{e}_i] = 0, [f_i, f_i^*] = [f_i, \widetilde{f}_i] = [f_i^*, \widetilde{f}_i] = 0,$

(ii) $[e_i, f_j] = \delta_{ij} h_i, [e_i^*, f_j^*] = \delta_{ij}(h_i + c_1), [\widetilde{e}_i, \widetilde{f}_j] = \delta_{ij}(h_i + c_2),$

(iii) $[e_i^*, e_j] = [e_i, e_j^*], [\widetilde{e}_i, e_j] = [e_i, \widetilde{e}_j], [e_i^*, \widetilde{e}_j] = [\widetilde{e}_i, e_j^*],$

$[f_i^*, f_j] = [f_i, f_j^*], [\widetilde{f}_i, f_j] = [f_i, \widetilde{f}_j], [f_i^*, \widetilde{f}_j] = [\widetilde{f}_i, f_j^*],$

(III)

(i) $(adX_i)^{1-a_{ji}} X_j = 0,$ *if* $a_{ji} \leq 0,$

where $X_i, X_j \in \{e_i, e_i^*, \widetilde{e}_i\},$ *or* $\in \{f_i, f_i^*, \widetilde{f}_i\}.$

(ii) $[e_i, [e_i^*, \widetilde{e}_j]] = 0, [f_i, [f_i^*, \widetilde{f}_j]] = 0,$ *if* $a_{ji} = -1.$

In what follows, we prove Theorem 2.4. At first, we have the following.

Proposition 2.5. *The* 3-*toroidal algebra* $\mathfrak{g}_{tor}$ *can be described by finite generators as follows.*

Generators: $E_{i,0,0}, F_{i,0,0}, H_{i,p,q}$ $(0 \leq i \leq n,\ p, q = 0, \pm 1),$ *and* $c_1, c_2.$

Relations: (0) c_1 *and* c_2 *are central*,

(A)

$[H_{i,r,\mu}, H_{j,s,t}] = a_{ji}(rc_1 + \mu c_2)\delta_{r+s,0}\delta_{\mu+t,0},$

(B)

(i) $[H_{i,0,0}, E_{j,0,0}] = a_{ji} E_{j,0,0},$

(ii) $[H_{i,0,0}, F_{j,0,0}] = -a_{ji} F_{j,0,0},$

(iii) $[H_{i,p,q}, X_{i,0,0}] = -2[H_{j,p,q}, X_{i,0,0}]$ $(a_{ji} = -1),$

(iv) $adH_{i,p,q} adH_{j,-p,-q} X_{j,0,0} = 2a_{ji} X_{j,0,0}, ((p, q) \neq (0, 0)),$

(C)

(i) $adX_{i,0,0}(adH_{i,p,0})^k(adH_{i,0,q})^l X_{i,0,0} = 0$ $(k \geq 0,\ l \geq 0,\ k + l \geq 1)$,

(ii) $[E_{i,0,0},\ F_{j,0,0}] = \delta_{ij} H_{i,0,0}$,

(iii) $adF_{i,0,0} adH_{i,p,q} E_{i,0,0} = -2H_{i,p,q}$,

(iv) $adX_{i,0,0}(adH_{j,p,0})^k(adH_{j,0,q})^l Y_{j,0,0} = 0$

$$(i \neq j,\ k \geq 0,\ l \geq 0,\ k + l \geq 1),$$

(v) $(adX_{i,0,0})^2(adH_{j,p,0})^k(adH_{j,0,q})^l X_{j,0,0} = 0$

$$(a_{ji} = -1,\ k \geq 0,\ l \geq 0),$$

(D)

(i) $adH_{i,p,q} adE_{j,0,0}(adH_{j,s,0})^k(adH_{j,0,t})^l F_{j,0,0} = 0$

$$(k \geq 0,\ l \geq 0,\ k + l \geq 2),$$

(ii) $adE_{i,0,0}(adH_{i,p,0})^k(adH_{i,0,q})^l F_{i,0,0}$

$$= (-1)^{k+l+1} adF_{i,0,0}(adH_{i,p,0})^k(adH_{i,0,q})^l E_{i,0,0}$$

$$(k \geq 0,\ l \geq 0,\ k + l \geq 2),$$

(iii) $adX_{j,0,0} adX_{i,0,0}(adH_{i,p,0})^k(adH_{i,0,q})^l Y_{i,0,0}$

$$= 2^{k+l}(adH_{i,p,0})^k(adH_{i,0,q})^l X_{j,0,0}$$

$$(a_{ji} = -1,\ k \geq 0,\ l \geq 0,\ k + l \geq 1),$$

where

$$i,\ j = 0,\ \ldots,\ n, \quad r,\ s,\ \mu,\ t = 0,\ \pm 1, \quad p,\ q = \pm 1,$$

and

$$X_{i,0,0} \neq Y_{i,0,0} \in \{E_{i,0,0},\ F_{i,0,0}\}.$$

Proof. The proof that the relations in Proposition 2.5 can be obtained from the relations in Definition 2.3 is easy. Conversely, for a positive integer k, m and $p,\ q = \pm 1,$ we set

$$H_{i,pk,qm} = adE_{i,0}\left(-\frac{1}{2}adH_{i,p,0}\right)^{k}\left(-\frac{1}{2}H_{i,0,q}\right)^{m}F_{i,0,0}$$

$$\left(= -adF_{i,0,0}\left(\frac{1}{2}adH_{i,p,0}\right)^{k}\left(\frac{1}{2}adH_{i,0,q}\right)^{m}E_{i,0,0}\right)$$

(by (D) (ii)),

$$E_{i,pk,qm} = \left(\frac{1}{2}adH_{i,p,0}\right)^{k}\left(\frac{1}{2}adH_{i,0q}\right)^{m}E_{i,0,0}$$

and

$$F_{i,pk,qm} = \left(-\frac{1}{2}adH_{i,p,0}\right)^{k}\left(-\frac{1}{2}adH_{i,0,q}\right)^{m}F_{i,0,0},$$

then, similarly to [9], we can show that the relations in Definition 2.3 can be obtained from the relations in Proposition 2.5.

We can describe $\mathfrak{g}_{tor}$ by choosing another generators as follows.

Proposition 2.6. *The simply-laced 3-toroidal algebra* $\mathfrak{g}_{tor}$ *can be presented as follows.*

Generators: $H_{i,0,0},\ E_{i,0,0},\ E_{i,1,0},\ E_{i,0,1},\ F_{i,0,0},\ F_{i,-1,0},\ F_{i,0,-1}$ $(0 \le i \le n)$, *and* $c_1,\ c_2$.

Relations: (0) c_1 *and* c_2 *are central*,

(I)

$$[H_{i,0,0},\ H_{j,0,0}] = 0,$$

(II)

(i) $[H_{i,0,0},\ E_{j,p,q}] = a_{ji}E_{j,p,q},$

(ii) $[H_{i,0,0},\ F_{j,p,q}] = -a_{ji}F_{j,p,q},$

(iii) $adE_{i,0,0}adF_{i,p,q}F_{j,0,0} = F_{j,p,q}$ $(a_{ji} = -1,\ (p,\ q) = (-1,\ 0),\ (0,\ -1)),$

(iv) $adE_{i,p,q}adF_{i,0,0}F_{j,-p,-q} = F_{j,0,0}$ $(a_{ji} = -1,\ (p,\ q) = (1,\ 0),\ (0,\ 1)),$

(v) $adE_{i,0,0}adE_{j,p,q}F_{j,0,0} = E_{i,p,q}$ $(a_{ji} = -1, (p, q) = (1, 0), (0, 1))$,

(vi) $adE_{i,p,q}adE_{j,0,0}F_{j,-p,-q} = E_{i,0,0}$ $(a_{ji} = -1, (p, q) = (1, 0), (0, 1))$,

(vii) $adE_{i,0,0}adE_{i,0,0}F_{i,p,q} = -2adE_{i,0,0}adE_{j,0,0}F_{j,p,q}$

$$(a_{ji} = -1, (p, q) = (0, -1), (-1, 0)),$$

(viii) $adE_{i,p,q}adE_{i,p,q}F_{i,0,0} = -2adE_{i,p,q}adE_{j,p,q}F_{j,0,0}$

$$(a_{ji} = -1, (p, q) = (0, 1), (1, 0)),$$

(ix) $adF_{i,0,0}adE_{i,p,q}F_{i,0,0} = -2adF_{i,0,0}adE_{j,p,q}F_{j,0,0}$

$$(a_{ji} = -1, (p, q) = (0, 1), (1, 0)),$$

(x) $adF_{i,p,q}adE_{i,0,0}F_{i,p,q} = -2adF_{i,p,q}adE_{j,0,0}F_{j,p,q}$

$$(a_{ji} = -1, (p, q) = (-1, 0), (0, -1)),$$

(III)

(i) $[E_{i,0,0}, E_{i,p,q}] = [F_{i,0,0}, F_{i,r,s}] = 0$,

$$((p, q) = (1, 0), (0, 1), (r, s) = (-1, 0), (0, -1)),$$

(ii) $[E_{i,1,0}, E_{i,0,1}] = [F_{i,-1,0}, F_{i,0,-1}] = 0$,

(iii) $adX_{i,0,0}(adA_{i,p,0})^k(adA_{i,0,q})^l X_{i,0,0} = 0$,

$$(k \geq 0, l \geq 0, k + l \geq 1),$$

(iv) $[E_{i,p,q}, F_{j,r,s}] = 0$ $(i \neq j)$,

(v) $[E_{i,p,q}, F_{i,-p,-q}] = H_{i,0,0} + \delta_{r,1}\delta_{q,1}(c_1 + c_2)$,

(vi) $adX_{i,0,0}(adA_{i,p,0})^k(adA_{i,0,q})^l Y_{j,0,0} = 0$,

$$(i \neq j, k \geq 0, l \geq 0, k + l \geq 1),$$

(vii) $(adX_{i,0,0})^{1-a_{ji}}(adA_{i,p,0})^k(adA_{i,0,q})^l X_{j,0,0} = 0$

$$(i \neq j, k \geq 0, l \geq 0),$$

(IV)

(i) $adA_{i,p,q}adE_{j,0,0}(adA_{j,r,0})^k(adA_{j,0,s})^l F_{j,0,0} = 0$

$$(k \geq 0,\ l \geq 0,\ k + l \geq 2),$$

(ii) $adE_{i,0,0}(adA_{i,p,0})^k(adA_{i,0,q})^l F_{i,0,0}$

$$= (-1)^{k+l+1} adF_{i,0,0}(adA_{i,p,0})^k(adA_{i,0,q})^l E_{i,0,0}$$

$$(k \geq 0,\ l \geq 0,\ k + l \geq 2),$$

(iii) $adX_{j,0,0}adX_{i,0,0}(adA_{i,p,0})^k(adA_{i,0,q})^l Y_{i,0,0}$

$$= 2^{k+l}(adA_{i,p,0})^k(adA_{i,0,q})^l X_{j,0,0}$$

$$(a_{ji} = -1,\ k \geq 0,\ l \geq 0,\ k + l \geq 1),$$

where

$$i,\ j,\ r = 0, 1, \quad s = 0, -1, \quad p,\ q = \pm 1,$$

$$adA_{i,1,0} := adE_{i,1,0}adF_{i,0,0} - adF_{i,0,0}adE_{i,1,0},$$

$$adA_{i,-1,0} := adE_{i,0,0}adF_{i,-1,0} - adF_{i,-1,0}adE_{i,0,0},$$

$$adA_{i,0,1} := adE_{i,0,1}adF_{i,0,0} - adF_{i,0,0}adE_{i,0,1},$$

$$adA_{i,0,-1} := adE_{i,0,0}adF_{i,0,-1} - adF_{i,0,-1}adE_{i,0,0}$$

and

$$X_{i,0,0} \neq Y_{i,0,0} \in \{E_{i,0,0},\ F_{i,0,0}\}.$$

Proof. We set

$$H_{i,1,0} = adE_{i,1,0}F_{i,0,0},$$

$$H_{i,0,1} = adE_{i,0,1}F_{i,0,0},$$

$$H_{i,-1,0} = adE_{i,0,0}F_{i,-1,0},$$

$$H_{i,0,-1} = adE_{i,0,0}F_{i,0,-1},$$

then we have

$$E_{i,1,0} = \frac{1}{2} adH_{i,1,0}E_{i,0,0},$$

$$E_{i,0,1} = \frac{1}{2} adH_{i,0,1}E_{i,0,0},$$

$$F_{i,-1,0} = -\frac{1}{2} adH_{i,-1,0}F_{i,0,0},$$

$$F_{i,0,-1} = -\frac{1}{2} adH_{i,0,-1}F_{i,0,0}$$

and

$$adH_{i,1,0} = adE_{i,1,0}adF_{i,0,0} - adF_{i,0,0}adE_{i,1,0},$$

$$adH_{i,0,1} = adE_{i,0,1}adF_{i,0,0} - adF_{i,0,0}adE_{i,0,1},$$

$$adH_{i,-1,0} = adE_{i,0,0}adF_{i,-1,0} - adF_{i,-1,0}adE_{i,0,0},$$

$$adH_{i,0,-1} = adE_{i,0,0}adF_{i,0,-1} - adF_{i,0,-1}adE_{i,0,0}.$$

From these, it is easy to see that the relations in Proposition 2.6 can be obtained from Proposition 2.5, since Proposition 2.5 gives a presentation of $\mathfrak{g}_{tor}$ in Definition 2.3. Conversely similarly to [9], we can show that the relations in Proposition 2.5 can be obtained from Proposition 2.6.

From the point of view of the 3-extended affine Lie algebra, we describe the 3-toroidal algebra $\mathfrak{g}_{tor}$. The 3-extended affine diagram is defined to be composed of all vertices α_i, α_i^*, $\widetilde{\alpha}_i$ $(0 \le i \le n)$. Associated to this diagram, we choose Chevalley generators of $\mathfrak{g}_{tor}$ such as $e_i = E_{i,0,0}$, $e_i^* = E_{i,1,0}$, $\widetilde{e}_i = E_{i,0,1}$, $f_i = F_{i,0,0}$, $f_i^* = F_{i,-1,0}$, $\widetilde{f}_i = F_{i,0,-1}$, $h_i = H_{i,0,0}$, $(0 \le i \le n)$ and c_1, c_2, then we call $\mathfrak{g}_{tor}$ the 3-*extended affine Lie algebra* and from Proposition 2.6, we have the following.

Proposition 2.7. *The simply-laced* 3-*extended affine Lie algebra is described as follows.*

Generators: $e_i,\ e_i^*,\ \tilde{e}_i,\ f_i,\ f_i^*,\ \tilde{f}_i,\ h_i\ (0 \leq i \leq n)$ *and* $c_1,\ c_2$.

Relations: (0) c_1 *and* c_2 *are central*,

(A)

$$[h_i, h_j] = 0,$$

(B)

(i) $[h_i, e_j] = a_{ji}e_j, [h_i, e_j^*] = a_{ji}e_j^*, [h_i, \tilde{e}_j] = a_{ji}\tilde{e}_j,$

(ii) $[h_i, f_j] = -a_{ji}f_j, [h_i, f_j^*] = -a_{ji}f_j^*, [h_i, \tilde{f}_j] = -a_{ji}\tilde{f}_j,$

(iii) $[e_i^*, e_j] = [e_i, e_j^*], [\tilde{e}_i, e_j] = [e_i, \tilde{e}_j], [e_i^*, \tilde{e}_j] = [\tilde{e}_i, e_j^*],$

(iv) $[f_i^*, f_j] = [f_i, f_j^*], [\tilde{f}_i, f_j] = [f_i, \tilde{f}_j], [f_i^*, \tilde{f}_j] = [\tilde{f}_i, f_j^*],$

(v) $[X_i, [X_i, Y_i^*]] = -2[X_i, [X_j, Y_j^*]],$

$[X_i^*, [X_i^*, Y_i]] = -2[X_i^*, [X_j^*, Y_j]]\ (a_{ji} = -1),$

(vi) $[X_i, [X_i, \tilde{Y}_i]] = -2[X_i, [X_j, \tilde{Y}_j]],$

$[\tilde{X}_i, [\tilde{X}_i, Y_i]] = -2[\tilde{X}_i, [\tilde{X}_j, Y_j]]\ (a_{ji} = -1),$

(C)

(i) $[X_i, X_i^*] = [X_i, \tilde{X}_i] = [X_i^*, \tilde{X}_i] = 0,$

(ii) $adX_i(adA_{i,p,0})^k(adA_{i,0,q})^l X_i = 0\ (k \geq 0,\ l \geq 0,\ k + l \geq 1),$

(iii) $[e_i, f_j] = [e_i, f_j^*] = [e_i, \tilde{f}_j] = [e_i^*, f_j] = [e_i^*, f_j^*] = [e_i^*, \tilde{f}_j]$

$= [\tilde{e}_i, f_j] = [\tilde{e}_i, f_j^*] = [\tilde{e}_i, \tilde{f}_j] = 0 \quad (i \neq j),$

(iv) $[e_i, f_i] = h_i, [e_i^*, f_i^*] = h_i + c_1, [\tilde{e}_i, \tilde{f}_i] = h_i + c_2,$

(v) $adX_i(adA_{i,p,0})^k(adA_{i,0,q})^l Y_j = 0\ (i \neq j,\ k \geq 0,\ l \geq 0,\ k + l \geq 1),$

(vi) $(adX_i)^{1-a_{ji}}(adA_{i,p,0})^k(adA_{i,0,q})^l X_j = 0\ (i \neq j,\ k \geq 0,\ l \geq 0),$

(D)

(i) $adA_{i,p,q}ade_j(adA_{j,r,0})^k(adA_{j,0,s})^l f_j = 0$ $(k \geq 0,\ l \geq 0,\ k + l \geq 2)$,

(ii) $ade_i(adA_{i,p,0})^k(adA_{i,0,q})^l f_i$

$$= (-1)^{k+l+1} adf_i(adA_{i,p,0})^k(adA_{i,0,q})^l e_i$$

$$(k \geq 0,\ l \geq 0,\ k + l \geq 2),$$

(iii) $adX_j adX_i(adA_{i,p,0})^k(adA_{i,0,q})^l Y_i = 2^{k+l}(adA_{i,p,0})^k(adA_{i,0,q})^l X_j$ $(i \neq j,\ k \geq 0,\ l \geq 0,\ k + l \geq 1)$,

where

$$i,\ j = 0, 1, \quad p,\ q,\ r,\ s = \pm 1,$$

$$adA_{i,1,0} = ade_i^* adf_i - adf_i ade_i^*,$$

$$adA_{i,0,1} = ad\widetilde{e}_i adf_i - adf_i ad\widetilde{e}_i,$$

$$adA_{i,-1,0} = ade_i adf_i^* - adf_i^* ade_i,$$

$$adA_{i,0,-1} = ade_i ad\widetilde{f}_i - ad\widetilde{f}_i ade_i$$

and

$$X_i \neq Y_i \in \{e_i,\ f_i\}.$$

Proof. We set $e_\alpha := e_i$, $e_\beta := e_j$ and show that (B) (iii) $[e_\alpha, e_\beta^*] = [e_\alpha^*, e_\beta]$ and $[\widetilde{e}_\alpha, e_\beta^*] = [e_\alpha^*, \widetilde{e}_\beta]$, can be obtained from Proposition 2.6.

$$\begin{aligned} ade_\alpha e_\beta^* &= ade_\alpha ade_\beta ade_\alpha^* f_\alpha \quad \text{(by (II) (v))} \\ &= -ade_\alpha adf_\alpha ade_\beta e_\alpha^* \\ &= -(adf_\alpha ade_\alpha + adh_\alpha) ade_\beta e_\alpha^* \\ &= -adh_\alpha ade_\beta e_\alpha^* \\ &= -(ade_\beta adh_\alpha - ade_\beta) e_\alpha^* \\ &= -2ade_\beta e_\alpha^* + ade_\beta e_\alpha^* \\ &= ade_\alpha^* e_\beta = [e_\alpha^*, e_\beta]. \end{aligned}$$

$$
\begin{aligned}
ad\tilde{e}_\alpha e^*_\beta &= ad\tilde{e}_\alpha ade_\beta ade^*_\alpha f_\alpha \\
&= ade_\alpha ad\tilde{e}_\beta ade^*_\alpha f_\alpha \\
&= -ade_\alpha adf_\alpha ad\tilde{e}_\beta e^*_\alpha \\
&= -(adf_\alpha ade_\alpha + adh_\alpha) ad\tilde{e}_\beta e^*_\alpha \\
&= -adh_\alpha ad\tilde{e}_\beta e^*_\alpha \\
&= -(ad\tilde{e}_\beta adh_\alpha - ad\tilde{e}_\beta) e^*_\alpha \\
&= -2ad\tilde{e}_\beta e^*_\alpha + ad\tilde{e}_\beta e^*_\alpha \\
&= ade^*_\alpha \tilde{e}_\beta = [e^*_\alpha, \tilde{e}_\beta].
\end{aligned}
$$

Conversely, all relations in Proposition 2.6 can be obtained from Proposition 2.7.

Lastly, we show that all relations in Proposition 2.7 can be obtained from Theorem 2.4.

(B)

(v) $[e_\alpha, [e_\alpha, f^*_\alpha]] = -2[e_\alpha, [e_\beta, f^*_\beta]]$

$$
\begin{aligned}
(ade_\alpha)^2 f^*_\alpha &= (ade_\alpha)^2 adf_\alpha adf^*_\beta e_\beta \quad (\text{by } f^*_\alpha = [f_\alpha, [f^*_\beta, e_\beta]]) \\
&= ade_\alpha (adf_\alpha ade_\alpha + adh_\alpha) adf^*_\beta e_\beta \\
&= ade_\alpha adf_\alpha ade_\alpha adf^*_\beta e_\beta \quad (\text{by } adh_\alpha adf^*_\beta e_\beta = 0) \\
&= (adf_\alpha ade_\alpha + adh_\alpha) ade_\alpha adf^*_\beta e_\beta \\
&= (ade_\alpha adh_\alpha + 2ade_\alpha) adf^*_\beta e_\beta \quad (\text{by } (ade_\alpha)^2 adf^*_\beta e_\beta = 0) \\
&= 2ade_\alpha adf^*_\beta e_\beta = -2[e_\alpha, [e_\beta, f^*_\beta]].
\end{aligned}
$$

(C) (v), (vi)

(a) (1) $ade_\alpha(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0,$

(2) $(adf_\alpha)^2(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0.$

(b) (1) $ade_\alpha(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\beta = 0,$

(2) $(adf_\alpha)^2(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\beta = 0.$

Proof. (a) We prove (1), (2) and the following simultaneously by the induction on k.

(3) $ade_\alpha^*(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0,$

(4) $(adf_\alpha^*)^2(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0,$

(5) $adf_\alpha adf_\alpha^*(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0.$

At first, we show that $(ade_\alpha)^3 f_\alpha^* = 0$.

$$\begin{aligned}
\text{l.h.s.} &= (ade_\alpha)^3 adf_\alpha adf_\beta^* e_\beta \quad (\text{by } f_\alpha^* = adf_\alpha adf_\beta^* e_\beta) \\
&= (ade_\alpha)^2(adf_\alpha ade_\alpha + adh_\alpha) adf_\beta^* e_\beta \\
&= (ade_\alpha)^2 adf_\alpha ade_\alpha adf_\beta^* e_\beta \quad (\text{by } adh_\alpha adf_\beta^* e_\beta = 0) \\
&= ade_\alpha(adf_\alpha ade_\alpha + adh_\alpha) ade_\alpha adf_\beta^* e_\beta \\
&= ade_\alpha adh_\alpha ade_\alpha adf_\beta^* e_\beta \quad (\text{by } (ade_\alpha)^2 e_\beta = 0) \\
&= (adh_\alpha ade_\alpha - 2ade_\alpha) ade_\alpha adf_\beta^* e_\beta = 0.
\end{aligned}$$

From this, we obtain the following.

(∗)

$$\begin{aligned}
&(ade_\alpha)^3 adf_\alpha^* - 3(ade_\alpha)^2 adf_\alpha^* ade_\alpha \\
&+ 3ade_\alpha adf_\alpha^*(ade_\alpha)^2 - adf_\alpha^*(ade_\alpha)^3 = 0.
\end{aligned}$$

(1)

$$ade_\alpha(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} f_\beta$$

$$= (ade_\alpha)^2 adf_\alpha^* ade_\alpha adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k-1} f_\beta$$

$$= \frac{1}{3}((ade_\alpha)^3 adf_\alpha^* + 3ade_\alpha adf_\alpha^* (ade_\alpha)^2$$

$$- adf_\alpha^*(ade_\alpha)^3) adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k-1} f_\beta \quad \text{(by } (*))$$

$$= \frac{1}{3}(ade_\alpha)^3 (adf_\alpha^*)^2 (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k-1} f_\beta = 0$$

(by the induction hypothesis).

(2)

$$(adf_\alpha)^2 (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} f_\beta$$

$$= adf_\alpha adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k adh_\alpha f_\beta$$

$$= adf_\alpha adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0.$$

(3)

$$ade_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} f_\beta$$

$$= ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k adh_\alpha^* f_\beta \quad (h_\alpha^* := h_\alpha + c_1)$$

$$= ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\beta = 0.$$

(4) and (5) are similarly proved.

(b) We prove (1), (2) and the following simultaneously.

(3) $ade_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\beta = 0,$

(4) $(adf_\alpha^*)^2 (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\beta = 0,$

(5) $adf_\alpha adf_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\beta = 0.$

(1)

$$ade_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^{k+1} f_\beta$$

$$= ade^*_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^k adh_\alpha f_\beta$$

$$= ade^*_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^k f_\beta = 0.$$

(2)

$$(adf_\alpha)^2(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^{k+1} f_\beta$$

$$= (adf_\alpha)^2 ade^*_\alpha adf_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^k f_\beta$$

$$= \frac{1}{3}((adf_\alpha)^3 ade^*_\alpha + 3adf_\alpha ade^*_\alpha (adf_\alpha)^2$$

$$- ade^*_\alpha (adf_\alpha)^3)(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^k f_\beta = 0.$$

(3)

$$ade^*_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^{k+1} f_\beta$$

$$= (ade^*_\alpha)^2 adf_\alpha ade^*_\alpha adf_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^{k-1} f_\beta$$

$$= \frac{1}{3}((ade^*_\alpha)^3 adf_\alpha + 3ade^*_\alpha adf_\alpha (ade^*_\alpha)^2$$

$$- adf_\alpha (ade^*_\alpha)^3)adf_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^{k-1} f_\beta = 0$$

(by the induction hypothesis and (2)).

(4) and (5) are similarly proved.

(C)

(ii) (a) (1) $ade_\alpha(ade_\alpha adf^*_\alpha - adf^*_\alpha ade_\alpha)^k e_\alpha = 0.$

(b) (1) $ade_\alpha(ade^*_\alpha adf_\alpha - adf_\alpha ade^*_\alpha)^k e_\alpha = 0.$

Proof. (a) We prove (1) and the following simultaneously by the induction on k.

(2) $ade_\alpha ade_\beta (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha = 0.$

At first, from the relation

$$[e_\alpha, [e_\alpha, f_\alpha^*]] = -2[e_\alpha, [e_\beta, [e_\beta, f_\beta]]],$$

we obtain the following.

(**)

$$\begin{aligned}(ade_\alpha)^2 adf_\alpha^* &= 2ade_\alpha adf_\alpha^* ade_\alpha - adf_\alpha^* (ade_\alpha)^2 \\ &\quad - 2(ade_\alpha ade_\beta adf_\beta^* - ade_\alpha adf_\beta^* ade_\beta \\ &\quad + adf_\beta^* ade_\beta ade_\alpha - ade_\beta adf_\beta^* ade_\alpha).\end{aligned}$$

(1)

$$\begin{aligned}&ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} e_\alpha \\ &= (ade_\alpha)^2 adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha \\ &= 2ade_\alpha adf_\beta^* ade_\beta (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha = 0,\end{aligned}$$

(by (**) and (C) (v)).

(2)

$$\begin{aligned}&ade_\alpha ade_\beta (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} e_\alpha \\ &= ade_\alpha ade_\beta ade_\alpha adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha \\ &= \frac{1}{2} (ade_\alpha)^2 ade_\beta adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha\end{aligned}$$

(by $(ade_\alpha)^2 e_\beta - 2ade_\alpha ade_\beta ade_\alpha + ade_\beta (ade_\alpha)^2 = 0$)

$$= -\frac{1}{2}(ade_\alpha)^2 ade_\beta adf_\alpha^*(ade_\alpha adf_\alpha^*)^{k-1}(ade_\alpha)^2 f_\alpha^*$$

$$= -\frac{1}{2}(ade_\alpha)^2 ade_\beta adf_\alpha^*(ade_\alpha)^2(adf_\alpha^* ade_\alpha)^{k-1} f_\alpha^*$$

(by iterating, $ade_\alpha adf_\alpha^*(ade_\alpha)^2$

$$= (ade_\alpha)^2 adf_\alpha^* ade_\alpha + \frac{1}{3}(adf_\alpha^*(ade_\alpha)^3 - (ade_\alpha)^3 adf_\alpha^*))$$

$$= -(ade_\alpha)^2 adf_\alpha^* ade_\alpha ade_\beta ade_\alpha (adf_\alpha^* ade_\alpha)^{k-1} f_\alpha^*$$

(by $ade_\beta(ade_\alpha)^2 = 2ade_\alpha ade_\beta ade_\alpha - (ade_\alpha)^2 ade_\beta$)

$$= -\frac{1}{3}(ade_\alpha)^3 adf_\alpha^* ade_\beta ade_\alpha (adf_\alpha^* ade_\alpha)^{k-1} f_\alpha^*$$

$$= -\frac{1}{3}(ade_\alpha)^3 ade_\beta (adf_\alpha^* ade_\alpha)^k f_\alpha^*$$

$$= -\frac{1}{3}(ade_\alpha)^3 ade_\beta (adf_\alpha^* ade_\alpha - ade_\alpha adf_\alpha^*)^k f_\alpha^* = 0 \quad \text{(by (C) (v))}.$$

(b) We prove (1) and the following simultaneously.

(2) $ade_\alpha^*(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha = 0,$

(3) $ade_\alpha^* ade_\beta^*(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha = 0,$

(4) $ade_\alpha^* ade_\beta^*(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha^* = 0.$

(1)

$$ade_\alpha(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k+1}$$

$$= ade_\alpha^*(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k adh_\alpha e_\alpha$$

$$= 2ade_\alpha^*(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha = 0.$$

(2)

$$ade_{\alpha}^{*}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k+1}e_{\alpha}$$

$$= (ade_{\alpha}^{*})^{2}adf_{\alpha}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k}e_{\alpha}$$

$$= 2adf_{\beta}ade_{\alpha}^{*}ade_{\beta}^{*}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k}e_{\alpha} = 0$$

(by the relation replacing e_{α}, e_{β}, f_{α}^{*}, f_{β}^{*} with e_{α}^{*}, e_{β}^{*}, f_{α}, f_{β} in (**)).

(3)

$$ade_{\alpha}^{*}ade_{\beta}^{*}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k+1}e_{\alpha}$$

$$= ade_{\alpha}^{*}ade_{\beta}^{*}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k}adh_{\alpha}e_{\alpha}^{*}$$

$$= 2ade_{\alpha}^{*}ade_{\beta}^{*}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k}e_{\alpha}^{*} = 0.$$

(4) is similarly proved as (a) (2) by replacing e_{α}, e_{β}, f_{α}^{*}, with e_{α}^{*}, e_{β}^{*}, f_{α}, respectively.

(D) (i)

(1) $(ade_{\beta}^{*}adf_{\beta} - adf_{\beta}ade_{\beta}^{*})ade_{\alpha}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k}f_{\alpha} = 0,$

(2) $(ade_{\beta}^{*}adf_{\beta} - adf_{\beta}ade_{\beta}^{*})ade_{\alpha}(ade_{\alpha}adf_{\alpha}^{*} - adf_{\alpha}^{*}ade_{\alpha})^{k}f_{\alpha} = 0,$

(3) $(ade_{\beta}adf_{\beta}^{*} - adf_{\beta}^{*}ade_{\beta})ade_{\alpha}(ade_{\alpha}^{*}adf_{\alpha} - adf_{\alpha}ade_{\alpha}^{*})^{k}f_{\alpha} = 0,$

(4) $(ade_{\beta}adf_{\beta}^{*} - adf_{\beta}^{*}ade_{\beta})ade_{\alpha}(ade_{\alpha}adf_{\alpha}^{*} - adf_{\alpha}^{*}ade_{\alpha})^{k}f_{\alpha} = 0.$

Proof. Noting that

$$(ade_{\beta}^{*}adf_{\beta} - adf_{\beta}ade_{\beta}^{*})ade_{\alpha} = adh_{\beta}ade_{\alpha}^{*} = ade_{\alpha}^{*}adh_{\beta} - ade_{\alpha}^{*},$$

(1) and (2) are trivial.

(3)

$$(ade_\beta adf_\beta^* - adf_\beta^* ade_\beta) ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= (ade_\beta^* adf_\beta^* - adf_\beta^* ade_\beta^*) ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k-1} adh_\alpha f_\alpha$$

$$= -2adh_\beta^* ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k-1} f_\alpha \quad (h_\beta^* := h_\beta + c_1)$$

$$= -2(ade_\alpha adh_\beta^* - ade_\alpha)(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k-1} f_\alpha = 0.$$

(4)

$$(ade_\beta adf_\beta^* - adf_\beta^* ade_\beta) ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha$$

$$= (ade_\beta adf_\beta - adf_\beta ade_\beta) ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha^*$$

$$= adh_\beta ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha^*$$

$$= (ade_\alpha adh_\beta - ade_\alpha)(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha^* = 0.$$

(D) (ii)

(a) $ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha$

$$= (-1)^{k+1} adf_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha,$$

(b) $ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$

$$= (-1)^{k+1} adf_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha.$$

Proof. We prove by the induction on k.

(a)

$$ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} f_\alpha$$

$$= -ade_\alpha adf_\alpha^* ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha$$

$$\text{(by } adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha = 0)$$

$$= (-1)^{k+2} ade_\alpha adf_\alpha^* adf_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha$$

(by the induction hypothesis)

$$= (-1)^{k+2} (adf_\alpha ade_\alpha + adh_\alpha) adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha$$

$$= (-1)^{k+2} (adf_\alpha ade_\alpha adf_\alpha^* + adf_\alpha^* adh_\alpha - 2adf_\alpha^*)(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha$$

$$= (-1)^{k+2} adf_\alpha ade_\alpha adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha$$

$$= (-1)^{k+2} adf_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} e_\alpha.$$

(b)

$$ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k+1} f_\alpha$$

$$= -ade_\alpha adf_\alpha ade_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= -(adf_\alpha ade_\alpha + adh_\alpha) ade_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= -adf_\alpha ade_\alpha^* ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= (-1)^{k+2} adf_\alpha ade_\alpha^* adf_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha$$

$$= (-1)^{k+2} adf_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k+1} e_\alpha.$$

(D) (iii)

(a) $ade_\beta ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha$

$$= 2^k (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\beta,$$

(b) $ade_\beta ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$

$$= 2^k (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\beta.$$

Proof. (a)

$$ade_\beta ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} f_\alpha$$

$$= -ade_\beta ade_\alpha adf_\alpha^* ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha \quad \text{(by (C) (ii))}$$

$$= (-1)^{k+2} ade_\beta ade_\alpha adf_\alpha^* adf_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha \quad \text{(by (D) (ii))}$$

$$= (-1)^{k+2} ade_\beta (adf_\alpha ade_\alpha + adh_\alpha) adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha$$

$$= (-1)^{k+2} ade_\beta adf_\alpha ade_\alpha adf_\alpha^* (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\alpha$$

$$= (-1)^{k+2} adf_\alpha^* ade_\beta ade_\alpha (-1)^{k+1} ade_\alpha (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha$$

$$= -adf_\alpha^* (2ade_\alpha ade_\beta ade_\alpha - (ade_\alpha)^2 ade_\beta)(ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k f_\alpha$$

$$= -2adf_\alpha^* ade_\alpha 2^k (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^k e_\beta$$

$$= 2^{k+1} (ade_\alpha adf_\alpha^* - adf_\alpha^* ade_\alpha)^{k+1} e_\beta.$$

(b)

$$ade_\beta ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^{k+1} f_\alpha$$

$$= -ade_\beta ade_\alpha adf_\alpha ade_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= -ade_\beta ade_\alpha adf_\alpha ade_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= -ade_\beta^* ade_\alpha adf_\alpha ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= (-1)^{k+2} ade_\beta^* ade_\alpha (adf_\alpha)^2 (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha$$

$$= (-1)^{k+2} ade_\beta^* (adf_\alpha ade_\alpha + adh_\alpha) adf_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha$$

$$= (-1)^{k+2} adf_\alpha ade_\beta^* ade_\alpha adf_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\alpha$$

$$= -adf_\alpha ade_\beta^* (ade_\alpha)^2 (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= -adf_\alpha (2ade_\alpha ade_\beta^* ade_\alpha - (ade_\alpha)^2 ade_\beta^*)(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha$$

$$= -2adf_\alpha ade_\alpha^* ade_\beta ade_\alpha (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k f_\alpha \quad \text{(by (C) (v))}$$

$$= -2^{k+1} adf_\alpha ade_\alpha^* (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k e_\beta$$

(by the induction hypothesis)

$$= 2^{k+1} (ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha)^{k+1} e_\beta.$$

In the above, the cases of exchanging e_α with f_α are similar, and the more general cases such as replacing $(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k$ with $(ade_\alpha^* adf_\alpha - adf_\alpha ade_\alpha^*)^k (ad\widetilde{e}_\alpha adf_\alpha - adf_\alpha ad\widetilde{e}_\alpha)^l$ are similarly shown, so the proof is completed.

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