JP Jour. Algebra, Number Theory & Appl. **8**(1) (2007), 107-113

SOME PROPERTIES OF A TRIANGULAR ARRAY OF BINOMIAL COEFFICIENTS

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Abstract

We obtain some results concerning a triangular array of positive integers that is obtained by presenting Pascal's triangle in left-justified fashion, and then looking at diagonals that ascend from left to right.

1. Introduction

Consider the infinite triangular array of which the first 15 rows are given below:

1							
1	1						
1	2						
1	3	1					
1	4	3					
1	5	6	1				
1	6	10	4				
1	7	15	10	1			
1	8	21	20	5			
1	9	28	35	15	1		
1	10	36	56	35	6		
1	11	45	84	70	21	1	
1	12	55	120	126	56	7	
1	13	66	165	210	126	28	1
1	14	78	220	330	252	84	8

2000 Mathematics Subject Classification: 11A99.

Keywords and phrases: Pascal's triangle, binomial coefficients.

Received February 20, 2007

The entry in the nth row, in the kth position from the left, is $\binom{n-k}{k}$, where $0 \le k \le \left[\frac{n}{2}\right]$. We might mention that this array can be obtained from Pascal's triangle by looking at diagonals that ascend from left to right. In addition, Pascal's triangle can be obtained from this array by looking at diagonals that descend from left to right. The entries in this triangular array have a number of interesting properties. First of all, it is known that

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} = F_{n+1}, \tag{1}$$

where F_n denotes the nth Fibonacci number (See [3, p. 179, identity (54)]). Secondly, let $\tau(n)$ denote Ramanujan's tau function, which may be defined for $x \in C, |x| < 1$ by

$$\sum_{n=0}^{\infty} \tau(n+1)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{24}. \tag{2}$$

If p is prime and $n \ge 2$, then

$$\tau(p^n) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} p^{11k} \tau(p)^{n-2k} \tag{3}$$

(See [1, identity (4.1)]). In this note, we derive some additional properties of this array.

2. Preliminaries

If p is prime and $n \in N$, then we define $o_p(n) = k$ if $p^k | n$ but $p^{k+1} \nmid n$. We also define $t_p(n)$ to be the sum of the digits of n to the base p, that is,

$$n = \sum_{i=0}^{r} a_i p^i \to t_p(n) = \sum_{i=0}^{r} a_i.$$

If $1 \le k \le n$, then

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \tag{4}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{5}$$

$$\cos\frac{\pi(2n+1)}{6} = \sin\frac{\pi(n+2)}{3} \text{ for all integers } n \tag{6}$$

$$o_p\left(\binom{n}{k}\right) = \frac{t_p(k) + t_p(n-k) - t_p(n)}{p-1}. \tag{7}$$

Remarks. (4) is known as *Pascal's Identity*; (5) is a trigonometric identity that may be used in the proof that the derivative of sin x is cos x. (6) is easily verified by letting $n \equiv 0, 1, 2 \pmod 3$. Regarding (7), see [2, p. 69].

3. The Main Results

Our first theorem concerns the effect of putting alternating signs on the adjacent entries in each row of our triangular array, namely:

Theorem 1.

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} = \varepsilon = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } n \equiv 2 \pmod 3 \\ -1 & \text{if } n \equiv 0 \pmod 3). \end{cases}$$

Proof. (Induction on n.) We will prove an equivalent, more compact statement, namely

$$\sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} = \frac{2}{\sqrt{3}} \sin\frac{\pi(n+1)}{3}.$$

The statement is easily verified for $n = 1$. Invoking (4), we have

$$(-1)^j \binom{n-j}{j-1} + (-1)^j \binom{n-j}{j} = (-1)^j \binom{n+1-j}{j}.$$

Therefore

$$\sum_{j\geq 1}(-1)^j\binom{n-j}{j-1}+\sum_{j\geq 0}(-1)^j\binom{n-j}{j}=\sum_{j\geq 0}(-1)^j\binom{n+1-j}{j}.$$

Making a change of variable in the leftmost sum, we have

$$\sum_{j\geq 0}(-1)^{j+1}\binom{n-1-j}{j}+\sum_{j\geq 0}(-1)^j\binom{n-j}{j}=\sum_{j\geq 0}(-1)^j\binom{n+1-j}{j},$$

that is,

$$\sum_{j\geq 0}(-1)^j\binom{n-j}{j}-\sum_{j\geq 0}(-1)^j\binom{n-1-j}{j}=\sum_{j\geq 0}(-1)^j\binom{n+1-j}{j}.$$

By the induction hypothesis, we have

$$\frac{2}{\sqrt{3}}\left(\sin\frac{\pi(n+1)}{3}-\sin\frac{\pi n}{3}\right)=\sum_{j\geq 0}(-1)^j\binom{n+1-j}{j}.$$

Invoking (5), we have

$$\sum_{j\geq 0}(-1)^j\binom{n+1-j}{j}=\frac{2}{\sqrt{3}}\left(2\cos\frac{\pi(2n+1)}{6}\sin\frac{\pi}{6}\right)=\frac{2}{\sqrt{3}}\cos\frac{\pi(2n+1)}{6}.$$

The conclusion now follows from (6).

As a result of Theorem 1, we are able to deduce several corollaries. The first corollary concerns the sums of the even-indexed, and of the odd-indexed terms in each row of the array.

Corollary 1.

(a) $$\sum_{k=0}^{\left[\frac{n}{4}\right]}\binom{n-2k}{2k}=\frac{1}{2}(F_{n+1}+\varepsilon)$$

(b) $$\sum_{k=0}^{\left[\frac{n-2}{4}\right]}\binom{n-1-2k}{2k+1}=\frac{1}{2}(F_{n+1}-\varepsilon),$$

where ε *is defined as in Theorem* 1.

Proof. This follows from (1) and from Theorem 1.

Corollary 2. *Let* $r(n)$ *denote the number of odd terms in the nth row of our triangular array. Then*

$$r(n) \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 2 \pmod 3 \\ 1 \pmod 2 & \text{otherwise.} \end{cases}$$

Proof. This follows immediately from Theorem 1.

Our next result also concerns the parity of the entries in our triangular array.

Theorem 2. $2 \mid \binom{2^n - 1 - k}{k}$ *for all* k *such that* $1 \le k \le 2^{n-1} - 1$.

Proof. We will actually prove a stronger result, namely

$$o_2\left(\binom{2^n - 1 - k}{k}\right) = t_2(k)$$

for $1 \le k \le 2^{n-1} - 1$. Now

$$t_2(k) + t_2(2^n - 1 - k) = t_2(2^n - 1) = n$$

for all k such that $1 \le k \le 2^n - 1$ because when k is added to $2^n - 1 - k$, there is no carry. Therefore we have

$$t_2(2^n - 1 - k) = n - t_2(k);$$

$$t_2(2^n - 1 - 2k) = n - t_2(2k).$$

But $t_2(2k) = t_2(k)$, which implies

$$t_2(2^n - 1 - 2k) = t_2(2^n - 1 - k) = n - t_2(k). \tag{8}$$

Now according to (7), we have

$$o_2\left(\binom{2^n - 1 - k}{k}\right) = t_2(k) + t_2(2^n - 1 - 2k) - t_2(2^n - 1 - k).$$

The conclusion now follows from (7).

Our final result is as follows:

Theorem 3. *If* $n \geq 1$, *then* $\binom{3^n - k}{k} \equiv 0,\ 2 \pmod 3$ *for all k such that* $1 \leq k \leq \frac{3^n - 1}{2}$.

Proof: First we note that

$$\binom{3^n - k}{k} = \prod_{j=1}^{k} \frac{3^n + 1 - k - j}{k + 1 - j} \equiv (-1)^k \binom{2k-1}{k-1} \pmod 3.$$

Therefore it suffices to show that $\binom{2k-1}{k-1} \not\equiv (-1)^k \pmod 3$ for all $k \geq 1$.

We have

$$\binom{2k+1}{k} = \frac{4k+2}{k+1}\binom{2k-1}{k-1} \text{ hence}$$

$$\binom{2k+1}{k} \equiv \frac{k+2}{k+1}\binom{2k-1}{k-1} \pmod 3. \tag{9}$$

We will use induction on k. First observe that $\binom{1}{0} = 1 \not\equiv (-1)^1 \pmod 3$.

Now assume $\binom{2k-1}{k-1} \not\equiv (-1)^k \pmod 3$.

If $k \equiv 0 \pmod 3$, then (9) implies

$$\binom{2k+1}{k} \equiv 2\binom{2k-1}{k-1} \equiv -1\binom{2k-1}{k-1} \pmod 3.$$

Since $\binom{2k-1}{k-1} \not\equiv (-1)^k \pmod 3$ by induction hypothesis, it follows that $\binom{2k+1}{k} \not\equiv (-1)^{k+1} \pmod 3$.

If $k \equiv 1 \pmod 3$, then (9) implies $\binom{2k+1}{k} \equiv 0 \not\equiv (-1)^{k+1} \pmod 3$.

If $k \equiv 2 \pmod 3$, then $k = 3m + 2$, so

$$\binom{2k-1}{k-1} = \binom{6m+3}{3m+1}.$$

Now

$$o_3\left(\binom{6m+3}{3m+1}\right) = \frac{1}{2}(t_3(3m+1) + t_3(3m+2) - t_3(6m+3)).$$

Since the ternary representation of $t_3(3n+i)$ ends in i, where $i \in \{1, 2\}$, it follows that a carry occurs in the addition of $t_3(3m+1)$ and $t_3(3m+2)$. This implies that $o_3\left(\binom{6m+3}{3m+1}\right) > 0$, hence $\binom{2k-1}{k-1} \equiv 0 \not\equiv (-1)^k \pmod 3$.

References

[1] J. M. Gandhi, The nonvanishing of Ramanujan's τ-function, Amer. Math. Monthly 68 (1961), 757-760.

[2] N. Robbins, Beginning Number Theory, 2nd ed., Jones & Bartlett, 2006.

[3] S. Vajda, Fibonacci and Lucas Numbers and the Golden Section, Ellis Horwood Ltd., 1989.