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THE WEYL GROUP OF THE 3-EXTENDED AFFINE ROOT SYSTEM $BC_l^{(2,2,1)}(2)\,(l \geq 1)$

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Abstract

We describe the Weyl group associated to the 3-extended affine root system $BC_l^{(2,2,1)}(2)$ [1], [5] in terms of the 3-extended affine diagram.

1. Introduction

In 1985, Saito [5] introduced the notion of an extended affine root system, and especially classified (marked) 2-extended affine root systems associated to the elliptic singularities, which are the root systems belong to a positive semi-definite quadratic form I whose radical has rank two. Therefore 2-extended affine root systems are also called *elliptic root systems*. In 1997, Allison et al. [1] also introduced the extended affine root systems associated to the extended affine Lie algebras and gave a complete description of them by using the concept of a semilattice. The generators and their relations of elliptic Weyl groups associated to the elliptic root systems were described from the viewpoint of a generalization

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of Coxeter groups by Saito and Takebayashi [6]. In the cases of the simply-laced extended affine root systems, Azam [3] has given a presentation of the corresponding Weyl groups. In [7], in the cases of simply-laced 3-extended affine root systems, and in [8-11], in the cases of the 3-extended affine root systems $C_l^{(1,1,1)}$, $C_l^{(1,1,2)}$, $C_l^{(1,2,2)}$ and $BC_l^{(2,1,1)}$, similarly to the cases of the elliptic root systems, we described the 3-extended affine Weyl groups in terms of the 3-extended affine diagrams. In this paper, we describe the Weyl group of the (marked) 3-extended affine root system $BC_l^{(2,2,1)}(2)$ in terms of the 3-extended affine diagram.

2. The 3-extended Affine Root System $BC_l^{(2,2,1)}(2)$

We recall the (marked) 3-extended affine root system $BC_l^{(2,2,1)}(2)$ [1], [5], which is given as follows:

$BC_l^{(2,2,1)}(2)$ $(l \geq 1)$

$$R: \quad \pm\varepsilon_i + nb + ma + kc \ (1 \leq i \leq l) \quad (n, m, k \in \mathbb{Z}),$$

$$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc \ (1 \leq i \leq l) \quad (n, m, k \in \mathbb{Z}),$$

$$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc \ (1 \leq i < j \leq l) \quad (n, m, k \in \mathbb{Z}).$$

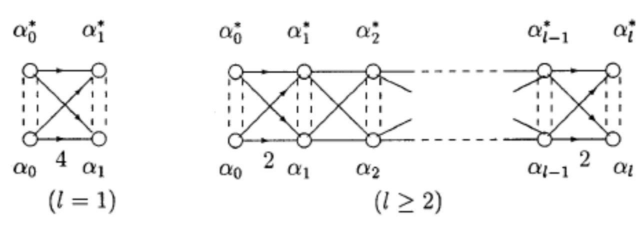
We set;

$$\alpha_0 = -2\varepsilon_1 + b, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \ (1 \leq i \leq l-1), \quad \alpha_l = \varepsilon_l,$$

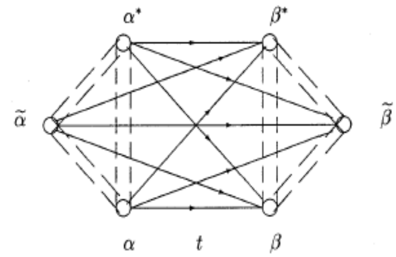
$$\alpha_i^* = \alpha_i + 2a \ (0 \leq i \leq l-1), \quad \alpha_l^* = \alpha_l + a, \ \tilde{\alpha}_i = \alpha_i + c \ (0 \leq i \leq l).$$

The elliptic diagram of $BC_l^{(2,2)}(2)(l \geq 1)$ is given as follows [5]:

$BC_l^{(2,2)}(2)(l \geq 1)$



We define the 3-extended affine diagram $\Gamma(R)$ by adding the vertices $\tilde{\alpha}_i$ $(0 \leq i \leq l)$ to the elliptic diagram in the above. So, we have the following subdiagram for all α, β s.t. $\langle \alpha, \beta^\vee \rangle = -t$, $\langle \alpha^\vee, \beta \rangle = -1$.



3. The Weyl Group of the 3-extended Affine Root System $BC_l^{(2,2,1)}(2)$

The Weyl group of the 3-extended affine root system is defined as follows [1], [5]. Let V be an $(l+3)$-dimensional real vector space equipped with a positive semi-definite bilinear form. Let V^0 be the 3-dimensional radical of the form $\langle , \rangle$ and $(V^0)^*$ be the dual space of V^0. Set $V = \dot{V} \oplus V^0$, and $\tilde{V} = \dot{V} \oplus V^0 \oplus (V^0)^*$. Let $\{\varepsilon_1, ..., \varepsilon_l\}$ be the standard basis of $\dot{V}$ satisfying $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for all $i, j = 1, ..., l$. Define the bilinear form $\langle , \rangle$ on $\tilde{V}$ so that $\langle , \rangle$ extends the form on V and $\langle , \rangle$ is nondegenerate on $\tilde{V}$. For $\alpha \in R$, we define the reflection $w_\alpha \in GL(\tilde{V})$ by $w_\alpha(u) = u - \langle u, \alpha^\vee \rangle \alpha$ $(u \in \tilde{V})$ with $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. Set $\widetilde{W}_R = \langle w_\alpha \mid \alpha \in R \rangle \subseteq GL(\tilde{V})$. Then $\widetilde{W}_R$ is the Weyl group of the 3-extended affine root system R. We set $X_i = w_{\alpha_i} w_{\alpha_i + b}$ $(1 \leq i \leq l)$, $Y_i = w_{\alpha_i} w_{\alpha_i + 2a}$ $(1 \leq i \leq l-1)$, $Y_l = w_{\alpha_l + a}$, $Z_i = w_{\alpha_i} w_{\alpha_i + c}$ $(1 \leq i \leq l)$, then we have the following.

Proposition 3.1. *The Weyl group of the* 3*-extended affine root system* $BC_l^{(2,2,1)}(2)$ *is described as follows.*

Generators: $w_i := w_{\alpha_i}$, X_i, Y_i, Z_i $(1 \leq i \leq l)$, *and the central elements* η_1, η_2, η_3.

Relations:

(a)
$$\begin{cases} X_iX_j = X_jX_i \\ Y_iY_j = Y_jY_i \\ Z_iZ_j = Z_jZ_i, \end{cases}$$

(b)
$$\begin{cases} Y_iX_i = \eta_1^4 X_iY_i \ (1 \leq i \leq l) \\ Y_jX_i = \eta_1^{-2} X_iY_j \\ (i = j \pm 1, (j, i) \neq (l-1, l)) \\ Y_{l-1}X_l = \eta_1^{-4} X_lY_{l-1} \\ Y_jX_i = X_iY_j \ (i \neq j, j \pm 1), \end{cases}$$

(c)
$$\begin{cases} Z_iX_i = \eta_2^2 X_iZ_i \ (1 \leq i \leq l-1) \\ Z_lX_l = \eta_2^4 X_lZ_l \\ Z_jX_i = \eta_2^{-1} X_iZ_j \ (1 \leq i, j \leq l-1) \\ Z_lX_{l-1} = \eta_2^{-2} X_{l-1}Z_l \\ Z_{l-1}X_l = \eta_2^{-2} X_lZ_{l-1} \\ Z_jX_i = X_iZ_j \ (i \neq j, j \pm 1), \end{cases}$$

(d)
$$\begin{cases} Z_iY_i = \eta_3^2 Y_iZ_i \ (1 \leq i \leq l) \\ Z_jY_i = \eta_3^{-1} Y_iZ_j \\ (i = j \pm 1, (j, i) \neq (l, l-1)) \\ Z_lY_{l-1} = \eta_3^{-2} Y_{l-1}Z_l \\ Z_jY_i = Y_iZ_j \ (i \neq j, j \pm 1), \end{cases}$$

(e)
$$\begin{cases} w_iX_i = X_i^{-1}w_i \ (1 \leq i \leq l) \\ w_iX_j = X_jX_iw_i \\ (j = i \pm 1, (i, j) \neq (l-1, l)) \\ w_{l-1}X_l = X_{l-1}^2X_lw_{l-1} \\ w_iX_j = X_jw_i \ (j \neq i, i \pm 1), \end{cases}$$

(f)
$$\begin{cases} w_iY_i = Y_i^{-1}w_i \ (1 \leq i \leq l) \\ w_iY_j = Y_jY_iw_i \\ (j = i \pm 1, (i, j) \neq (l, l-1)) \\ w_lY_{l-1} = Y_{l-1}Y_l^2w_l \\ w_iY_j = Y_jw_i \ (j \neq i, i \pm 1), \end{cases}$$

Z_j and w_i satisfy the same relations as the relations of X_j and w_i.

Proof. We define η_1, η_2 and η_3 as follows:

$$\eta_1(u) = u + \langle u, a\rangle b - \langle u, b\rangle a,$$

$$\eta_2(u) = u + \langle u, c\rangle b - \langle u, b\rangle c,$$

$$\eta_3(u) = u + \langle u, 2c\rangle a - \langle u, 2a\rangle c.$$

Then, using the following formula;

for

$$X_i = w_{\alpha_i} w_{\alpha_i + b}, \quad Y_j = w_{\alpha_i} w_{\alpha_j + a}, \quad \alpha_i^{\vee} = p\alpha_i,$$

$$Y_j X_i Y_j^{-1} X_i^{-1}(u) = u + \langle \alpha_i, \alpha_j^{\vee}\rangle(-\langle u, pb\rangle a + \langle u, pa\rangle b),$$

they are similarly checked as the cases of [8-11].

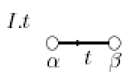
From Proposition 3.1, we obtain the following.

Theorem 3.2. *The Weyl group of the* 3-*extended affine root system* $BC_l^{(2,2,1)}(2)$ *is described as follows.*

Generators: *for each* $\alpha \in \Gamma(R)$, *we attach a generator* $a_\alpha := w_\alpha$. *For simplicity, we shall write* $a, a^*, \tilde{a}, b, b^*, \tilde{b}, \ldots$ *instead of* a_α, a_{α^*}, $a_{\tilde{\alpha}}$, a_β, a_{β^*}, $a_{\tilde{\beta}}, \ldots$.

Relations:

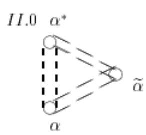
$$\Rightarrow \quad a^2 = 1$$



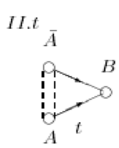
$$t = 0 \quad \Rightarrow \quad (ab)^2 = 1$$

$$t = 1 \quad \Rightarrow \quad (ab)^3 = 1$$

$$t = 2 \quad \Rightarrow \quad (ab)^4 = 1$$



$$\Rightarrow \quad (a\tilde{a}a^*)^2 = (\tilde{a}a^*a)^2 = (a^*a\tilde{a})^2$$

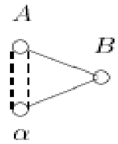


$$t = 1 \quad \Rightarrow \quad (AB\overline{A}B)^3 = 1$$

$$t = 2^{\pm 1} \quad \Rightarrow \quad (AB\overline{A}B)^2 = 1$$

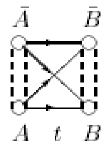
$$t = 4^{\pm 1} \quad \Rightarrow \quad (A\overline{A}B)^2 = (\overline{A}BA)^2 = (BA\overline{A})^2,$$

$$\textit{where } A \neq \overline{A} \in \{\alpha, \alpha^*, \tilde{\alpha}\}$$



$$\Rightarrow \quad (AaAB)^3 = 1, \text{ where } A = \tilde{\alpha},\ B = \beta^*,$$
$$\text{or } A = \alpha^*,\ B = \tilde{\beta}$$

III.t



$t = 1 \quad \Rightarrow \quad A\overline{B}A = B\overline{A}B$

$t = 2,\ \alpha^* = \alpha + 2a, \beta^* = \beta + 2a \ \Rightarrow\ \overline{A}B\overline{A} = A\overline{B}A$

$\tilde{\alpha} = \alpha + c, \tilde{\beta} = \beta + c,$

where $\{A = \alpha, \overline{A} = \alpha^*, B = \beta, \overline{B} = \beta^*\},$

$\{A = \alpha^*, \overline{A} = \tilde{\alpha}, B = \beta^*, \overline{B} = \tilde{\beta}\}$ or

$\{A = \alpha, \overline{A} = \tilde{\alpha}, B = \beta, \overline{B} = \tilde{\beta}\}$

$t = 2, \quad \alpha^* = \alpha + 2a, \beta^* = \beta + a \ \Rightarrow\ b^*ab^* = ba^*b$

$\tilde{\alpha} = \alpha + c, \tilde{\beta} = \beta + c \qquad \tilde{a}b\tilde{a} = a\tilde{b}a$

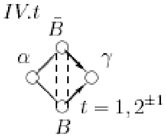
$t = 4, \quad \alpha^* = \alpha + 2a, \beta^* = \beta + a$

$\tilde{\alpha} = \alpha + c, \tilde{\beta} = \beta + c$

$\Rightarrow aa^*bb^* = a^*bb^*a = bb^*aa^* = b^*aa^*b$

$\tilde{a}b\tilde{a} = a\tilde{b}a$

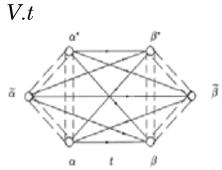
IV.t



$\Rightarrow \quad (aBa\overline{B}c\overline{B})^2 = 1, \quad (a\overline{B}aBcB)^2 = 1,$

where $B \neq \overline{B} \in \{\beta, \beta^*, \tilde{\beta}\}$

V.t



$t = 2, \quad \alpha^* = \alpha + 2a, \quad \beta^* = \beta + a$

$\tilde{\alpha} = \alpha + c, \quad \tilde{\beta} = \beta + c$

$\Rightarrow \ \tilde{b}ab^*\tilde{a} = b^*\tilde{a}ba^*, \quad \tilde{b}a^*\tilde{a}b^* = a^*\tilde{a}b^*b$

$\tilde{b}b^*\tilde{a}a = a^*\tilde{b}b^*\tilde{a} = b^*\tilde{a}a^*b$

$t = 4, \qquad aa^*bb^*\tilde{a} = \tilde{a}aa^*bb^*,$

$aa^*bb^*\tilde{b} = \tilde{b}aa^*bb^*$

$\Rightarrow \ (a\tilde{a})^2b\tilde{b}a^* = a^*(a\tilde{a})^2b\tilde{b},$

$(a\tilde{a})^2b\tilde{b}b^* = b^*(a\tilde{a})^2b\tilde{b}$

$a^*\tilde{a}b^*b = \tilde{b}a^*\tilde{a}b^*.$

Proof. In the case of V.4, we check the above relations, since the 4 others are similarly checked as the cases of [8-11]

$$\text{(V.4)}\quad a^*(a\tilde{a})^2 b\tilde{b}(u) = u - \left\langle u, \frac{1}{2}\alpha + a - c\right\rangle\alpha - \langle u, -2c\rangle\beta - \langle u, \alpha + 2\beta\rangle c$$

$$- \langle u, \alpha + 2a\rangle a = (a\tilde{a})^2 b\tilde{b}a^*(u).$$

$$aa^*bb^*\tilde{a}(u) = u - \left\langle u, \frac{1}{2}\alpha + \frac{1}{2}c - a\right\rangle\alpha - \langle u, -2a\rangle\beta - \langle u, \alpha + 2\beta\rangle a$$

$$- \left\langle u, \frac{1}{2}\alpha + \frac{1}{2}c\right\rangle c = \tilde{a}aa^*bb^*(u).$$

$$a^*\tilde{a}b^*b(u) = u - \left\langle u, a - \frac{1}{2}c\right\rangle\alpha - \langle u, 2a\rangle\beta - \langle u, -\alpha - 2c - 2\beta\rangle a$$

$$- \left\langle u, \frac{1}{2}\alpha + \frac{1}{2}c + 2a\right\rangle c = \tilde{b}a^*\tilde{a}b^*(u).$$

The others are similarly checked.

Next, we show that the relations in Theorem 3.2 are the defining relations of $\widetilde{W}_R$. We denote by $\widetilde{W}(\Gamma(R))$ the group defined by the generators and relations in Theorem 3.2. Let $N(R)$ be the smallest normal subgroup of $\widetilde{W}(\Gamma(R))$ containing $a_\alpha\tilde{a}_\alpha \ \forall\alpha \in \Gamma(R)$. Then one has a natural isomorphism

$$\widetilde{W}(\Gamma(R))/N(R) \cong \widetilde{W}(R_{el}).$$

The left hand side is a group obtained from $\widetilde{W}(\Gamma(R))$ by substituting $\tilde{a}$, $\tilde{b}$, etc. by a, b, etc. Therefore, it is isomorphic to the central extension $\widetilde{W}(R_{el})$ of the elliptic Weyl group associated to the elliptic root system R_{el} [6]. For the proof of Theorem 3.2, we prepare the following.

Lemma 3.3. *We set* $\gamma_3 := w_0\tilde{w}_0w_0w_0^*\tilde{w}_0w_0w_0^*w_0$. *Then*

(i) γ_3 *is a central element in* $\widetilde{W}(\Gamma(R))$.

(ii) *When* $l = 1$,

$$\gamma_3 = w_1 w_1^* w_0 \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0,$$

and

$$\gamma_3^2 = w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1$$

$$= w_1 \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1 w_1^* w_1 (= w_1 w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1).$$

(iii) $N(R)$ *is an abelian group generated by* $T_\alpha := a_\alpha \widetilde{a}_\alpha \ \ \forall \alpha \in \Gamma(R)$ *and* γ_3.

Proof.

(i) When $l = 1$, noting that

$$\gamma_3 = w_0 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 w_0^* w_0$$

$$= w_0 w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0$$

$$(\text{by } (\widetilde{w}_0 w_0 w_0^*)^2 = (w_0 w_0^* \widetilde{w}_0)^2 = (w_0^* \widetilde{w}_0 w_0)^2),$$

$$w_1 \gamma_3 = w_1 w_0 w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0$$

$$= w_1 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 w_0^*$$

$$= \widetilde{w}_0 w_0 \widetilde{w}_1 w_0^* \widetilde{w}_0 w_0 w_0^*$$

$$(\text{by } w_1 \widetilde{w}_0 w_0 = \widetilde{w}_0 w_0 \widetilde{w}_1)$$

$$= \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_1^* w_1 w_1^* w_0 w_0^*$$

$$(\text{by } \widetilde{w}_1 w_0^* \widetilde{w}_0 = w_0^* \widetilde{w}_0 w_1^* w_1 w_1^*)$$

$$= \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 w_0^* w_1 = \gamma_3 w_1$$

$$(\text{by } w_1 w_1^* w_0 w_0^* = w_1^* w_0 w_0^* w_1).$$

$$w_1^*\gamma_3^{-1} = w_1^*\widetilde{w}_0 w_0^* w_0 \widetilde{w}_0 w_0^* w_0$$

$$= w_1 w_1^* \widetilde{w}_0 w_0^* \widetilde{w}_1 w_0 \widetilde{w}_0 w_0^* w_0$$

(by $w_1^*\widetilde{w}_0 w_0^* = w_1 w_1^* \widetilde{w}_0 w_0^* \widetilde{w}_1$)

$$= w_1 w_1^* \widetilde{w}_0 w_0^* w_0 \widetilde{w}_0 w_1 w_0^* w_0$$

(by $\widetilde{w}_1 w_0 \widetilde{w}_0 = w_0 \widetilde{w}_0 w_1$)

$$= w_1 w_1^* \widetilde{w}_0 w_0^* w_0 \widetilde{w}_0 w_1 w_0^* w_0 w_1^* w_1^*$$

$$= w_0^* w_0 \widetilde{w}_0 w_0^* w_0 \widetilde{w}_0 w_1^* = \gamma_3^{-1} w_1^*$$

(by $w_0 w_0^* w_1 w_1^* \widetilde{w}_0 = \widetilde{w}_0 w_0 w_0^* w_1 w_1^*$).

The others and the case of $l \geq 2$ are similarly checked.

(ii) $\quad w_1 w_1^* w_0 \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0$

$$= w_0^* w_0^* w_1 w_1^* w_0 \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0$$

$$= w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0$$

(by $w_0^* w_1 w_1^* w_0 = w_0 w_0^* w_1 w_1^*$, $w_0 w_0^* w_1 w_1^* \widetilde{w}_0 = \widetilde{w}_0 w_0 w_0^* w_1 w_1^*$)

$$= w_0 w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 = \gamma_3.$$

$$\gamma_3^2 = (w_0 w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0)^2$$

$$= w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0,$$

here $\widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0$

$$= w_1 \widetilde{w}_1 \widetilde{w}_1 w_1 \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0$$

$$= w_1 \widetilde{w}_1 w_0^* \widetilde{w}_0 \widetilde{w}_1 w_1$$

(by $w_0^* \widetilde{w}_1 w_1 (\widetilde{w}_0 w_0)^2 = \widetilde{w}_1 w_1 (\widetilde{w}_0 w_0)^2 w_0^*$)

so, $\gamma_3^2 = w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1$.

$$\gamma_3^2 = (w_0 w_0^* \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0)^2$$

$$= w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0$$

$$= w_1^* w_1 w_1 w_1^* w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_0^* \widetilde{w}_0 w_0 \widetilde{w}_0$$

$$= w_1^* w_1 \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_1 w_1^* w_0 \widetilde{w}_0 w_0 \widetilde{w}_0$$

(by $w_1 w_1^* w_0 w_0^* \widetilde{w}_0 = \widetilde{w}_0 w_1 w_1^* w_0 w_0^*$)

$$= w_1^* \widetilde{w}_1 \widetilde{w}_1 w_1 \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 w_1 w_1^* w_0 \widetilde{w}_0 w_0 \widetilde{w}_0$$

(by $\widetilde{w}_1 w_1 (\widetilde{w}_0 w_0)^2 w_1^* = w_1^* \widetilde{w}_1 w_1 (\widetilde{w}_0 w_0)^2$)

$$= w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1 w_1 = w_1 w_1^* \widetilde{w}_1 w_1 w_1^* \widetilde{w}_1$$

(by $(w_1^* \widetilde{w}_1 w_1)^2 = (w_1 w_1^* \widetilde{w}_1)^2$).

(iii) We show that the subgroup generated by T_α ($\forall \alpha \in \Gamma(R)$) and γ_3 is closed under the adjoint action Ad_{a_α} $\forall \alpha \in \Gamma(R)$. When $l = 1$,

$$\begin{aligned} Ad_{w_0}(T_{w_1}) &= w_0 w_1 \widetilde{w}_1 w_0 \\ &= w_0 \widetilde{w}_0 w_0 w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 w_0 \widetilde{w}_0 \widetilde{w}_0 \\ &= w_0 \widetilde{w}_0 w_0 \widetilde{w}_0 w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 w_0 \widetilde{w}_0 \\ &= T_{w_0}^4 T_{w_1}. \end{aligned}$$

$$\begin{aligned} Ad_{w_0^*}(T_{w_1}) &= w_0^* w_1 \widetilde{w}_1 w_0^* \\ &= w_0 w_0 w_0^* w_1 \widetilde{w}_1 w_0^* w_0 \widetilde{w}_1 w_1 w_1 \widetilde{w}_1 w_0 \\ &= \gamma_3^2 w_0 w_1 \widetilde{w}_1 w_0 = \gamma_3^2 T_{w_0}^4 T_{w_1}. \end{aligned}$$

$$Ad_{\widetilde{w}_0}(T_{w_1}) = \widetilde{w}_0 w_1 \widetilde{w}_1 \widetilde{w}_0$$

$$= w_0 w_1 \widetilde{w}_1 w_0 = T_{w_0}^4 T_{w_1}.$$

$$Ad_{w_1^*}(T_{w_1}) = w_1^* w_1 \widetilde{w}_1 w_1^*$$

$$= w_1 w_1 w_1^* w_1 \widetilde{w}_1 w_1^* w_1 \widetilde{w}_1 w_1 w_1 \widetilde{w}_1 w_1$$

$$= \gamma_3^{-2} \widetilde{w}_1 w_1 = \gamma_3^{-2} T_{w_1}^{-1}.$$

$$Ad_{w_1}(T_{w_0}) = w_1 w_0 \widetilde{w}_0 w_1$$

$$= w_1 \widetilde{w}_1 w_0 \widetilde{w}_0 = T_{w_0} T_{w_1}.$$

$$Ad_{w_1^*}(T_{w_0}) = w_1^* w_0 \widetilde{w}_0 w_1^*$$

$$= w_1 w_1 w_1^* w_0 \widetilde{w}_0 w_1^* w_1 \widetilde{w}_0 w_0 w_0 \widetilde{w}_0 w_1$$

$$= \gamma_3 w_1 w_0 \widetilde{w}_0 w_1 = \gamma_3 T_{w_0} T_{w_1}.$$

$$Ad_{\widetilde{w}_1}(T_{w_0}) = \widetilde{w}_1 w_0 \widetilde{w}_0 \widetilde{w}_1$$

$$= w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 = T_{w_0} T_{w_1}.$$

$$Ad_{w_0^*}(T_{w_0}) = w_0^* w_0 \widetilde{w}_0 w_0^*$$

$$= w_0 w_0 w_0^* w_0 \widetilde{w}_0 w_0^* w_0 \widetilde{w}_0 w_0 w_0 \widetilde{w}_0 w_0$$

$$= \gamma_3^{-1} \widetilde{w}_0 w_0 = \gamma_3^{-1} T_{w_0}^{-1}.$$

The case of $l \geq 2$ is similarly checked.

The proof of Theorem 3.2

Let N be a subgroup generated by Z_i $(1 \leq i \leq l)$, η_2 and η_3. Then N is a normal subgroup of $\widetilde{W}_R$ and there is an isomorphism $\widetilde{W}_R / N \cong \widetilde{W}(R_{el})$, where $\widetilde{W}(R_{el}) \cong \langle w_i, X_i, Y_i\ (1 \leq i \leq l), \eta_1 \rangle$. So we have the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & N(R) & \to & \widetilde{W}(\Gamma(R)) & \to & \widetilde{W}(R_{el}) & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N & \to & \widetilde{W}_R & \to & \widetilde{W}(R_{el}) & \to & 1 \end{array}$$

By the same argument in the case of the elliptic Weyl group [6], noting that η_2 is generated by Y_i $(1 \leq i \leq l)$, we see that the first arrow is an isomorphism. Therefore the middle arrow is also an isomorphism.

Let us denote $\dot{w}_\alpha$ be the reflection in $GL(V)$ such that $w_\alpha|_V = \dot{w}_\alpha$, and set $W_R = \langle \dot{w}_\alpha \,|\, \alpha \in R \rangle$. Then, from the same argument in the elliptic case [6] and Proposition 3.1, we see the following.

Proposition 3.4. (i) *The central elements* γ_1 *and* $\gamma_2 \in \widetilde{W}(\Gamma(R))$ *corresponding to* η_1 *and* η_2 *are given as follows*:

$$\gamma_1 = w_0 w_0^* w_1 w_1^* \cdots w_l w_l^*,$$

$$\gamma_2 = (w_0 \widetilde{w}_0 w_1 \widetilde{w}_1 \cdots w_{l-1} \widetilde{w}_{l-1})^2 w_l \widetilde{w}_l.$$

(ii) *We have an isomorphism* $\widetilde{W}(\Gamma(R))/\langle \gamma_1, \gamma_2, \gamma_3 \rangle \cong W_R$.

Proof. (i) is directly checked and (ii) is trivial.

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