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MULTIVARIATE EXTENDED GENERALIZED DISTRIBUTIONS OF ORDER k

GREGORY A. TRIPSIANNIS[1], **AFRODITI A. PAPATHANASIOU**[1] **and ANDREAS N. PHILIPPOU**[2]

[1]Department of Medical Statistics
Faculty of Medicine
Democritus University of Thrace
68100 Alexandroupolis, Greece

[2]Department of Mathematics
University of Patras
Patras, Greece

Abstract

In a generalized sequence of order k, which is an extension of independent trials with multiple outcomes, we introduce the multivariate extended generalized distributions of order k, which generalize the multivariate generalized negative binomial distribution of Jain [3], the multivariate extended negative binomial distribution of order k of Philippou and Antzoulakos [11] and the multivariate generalized negative binomial distribution of order k, type I, of Tripsiannis et al. [19]. This new distribution includes as special cases six new multivariate extended distributions of order k and gives rise in the limit to multivariate extended generalized logarithmic series and Poisson distributions of order k. Moments of these distributions are obtained and graphs are presented.

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1. Introduction

In five pioneering papers, Philippou and Muwafi [12], Philippou et al. [15], Philippou [9] and Philippou et al. [13, 14] introduced the study of univariate and multivariate distributions of order k, based on outcomes of independent and identical binary trials or trials with multiple outcomes. Since then, the subject matter received a lot of attention from many researchers. For comprehensive reviews at the time of publication we refer to Johnson et al. [6, 7]. Based on a generalized sequence of order k, which is an extension of independent trials with multiple outcomes, Philippou and Antzoulakos [11] derived the multivariate extended negative binomial and logarithmic series distributions of order k, generalizing the respective work of Aki [1] on univariate extended distributions of order k.

Recently, Tripsiannis et al. [19] introduced the multivariate generalized negative binomial and Polya distributions of order k, type I, as the distribution of a first-passage event in a sequence of independent trials with multiple outcomes, generalizing to the multivariate case the work of Tripsiannis et al. [17]. These two distributions include as special or limiting cases several known and new distributions of the same order and type. Jain and Consul [4], Jain and Gupta [5] and Consul and Jain [2] introduced and studied the generalized negative binomial, logarithmic series and Poisson distributions, respectively, while Jain [3] extended these distributions to the multivariate case.

In the present paper, we extend several results of Tripsiannis et al. [17, 19], Philippou and Antzoulakos [11], Aki [1], Jain [3], Jain and Consul [4], Jain and Gupta [5] and Consul and Jain [2]. In Section 2, we derive a multivariate generalized negative binomial distribution of order k, say $MEGN\,B_k(\cdot)$, which generalizes the multivariate generalized negative binomial distribution of Jain [3] to distributions of order k, the multivariate extended negative binomial distribution of order k of Philippou and Antzoulakos [11] to generalized distributions, and the multivariate generalized negative binomial distribution of order k, type I, of Tripsiannis et al. [19] to the case of dependent trials. We do it by counting multidimensional lattice paths in a generalized sequence of order k and employing a first passage approach (see Theorem 2.1 and Definition 2.2). We next obtain two limiting cases of $MEGN\ B_k(\cdot)$ (see Propositions 2.1 and 2.2), which provide, respectively, multivariate generalized logarithmic series and Poisson distributions of the same order (see Definitions 2.3 and 2.4). Means and variances-covariances of these distributions are obtained in Section 3. In Section 4, we introduce, as special cases of $MEGN\ B_k(\cdot)$, six new multivariate distributions of order k, and we relate asymptotically $MEGN\ B_k(\cdot)$ to the multivariate Poisson $(MP_{k,I}(\lambda_1, \ldots, \lambda_m))$ distribution of order k, type I, of Philippou et al. [13]. Finally, graphs of $MEGN\ B_k(\cdot)$ are presented (Figure 4.1). We mention that all the corresponding univariate generalized distributions of order k are also new.

In order to avoid unnecessary repetitions, we mention here that in this paper $x_{11}, \ldots, x_{mk}$ are non-negative integers as specified. In addition, whenever sums and products are taken over i and j, ranging from 1 to m and from 1 to k, respectively, we shall omit these limits for notational simplicity.

2. Multivariate Extended Generalized Distributions of Order k

In the present section we obtain two multivariate extended distributions of order k, by employing the generalized sequence of the same order of Philippou and Antzoulakos [11], which we introduce next.

Definition 2.1. An infinite sequence $\{Y_n\}_{n=0}^{\infty}$ of $\{0, 1, \ldots, m\}$-valued random variables is said to be the *generalized sequence* of order k with parameters $q_{11}, \ldots, q_{mk}$ $(0 < q_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$, $q_{1j} + \cdots + q_{mj} < 1)$, if

(1) $Y_0 \neq 0$ almost surely, and

(2) $P(Y_n = i \mid Y_0 = y_0, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}) = q_{ij}$ $(1 \le i \le m)$,

for any positive integer n, where $j = r - k[(r-1)/k]$, r is the smallest positive integer which satisfies $y_{n-r} \neq 0$, and $[x]$ denotes the greatest integer in x.

It follows from the definition that

$$P(Y_n = 0 \mid Y_0 = y_0, Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}) = p_j \quad (1 \le j \le k),$$

where

$$p_j = 1 - \sum_i q_{ij} \ (1 \le j \le k),$$

and j is as above.

The generalized sequence of order k reduces to the case of independent trials with $m+1$ possible outcomes, if $p_1 = \cdots = p_k = p = 1 - \sum_i q_i$, and to the binary sequence of the same order of Aki [1], for $k = 1$. Furthermore, according to the case of independent trials, Y_n is sometimes called *nth trial* and the outcomes "0" and "i" $(1 \le i \le m)$ are called *success* (S) and *failure of type-i* (F_i), respectively.

In the following theorem, we employ a first passage approach to derive the multivariate extended generalized negative binomial distribution of order k.

Theorem 2.1. *Let* $\{Y_n\}_{n=0}^{\infty}$ *be a generalized sequence of order k with parameters* $q_{11}, \ldots, q_{mk}$, *and consider the random variables* X_{ij} $(1 \le i \le m$ *and* $1 \le j \le k)$ *and* L_k $(k \ge 1)$ *denoting, respectively, the number of events*

$$e_{ij} = \underbrace{S \cdots S}_{j-1} F_i \quad \text{and} \quad \tilde{e}_k = \underbrace{S \cdots S}_{k}.$$

Let X_i $(1 \le i \le m)$ *be a random variable denoting the number of failures of type-i and the total number of successes which precedes directly the occurrences of failures of type-i, but do not belong to any success run of length k, that is,* $X_i = \sum_j jX_{ij}$. *Trials are continued until* $n + \sum_i \sum_j \mu_i X_{ij}$ $(n > 0$ *and* $\mu_i \ge -1)$ *non-overlapping success runs of length k appear for the first time, that is, at any trial t* $(1 \le t \le \sum_i X_i + k(n + \sum_i \sum_j \mu_i X_{ij}) - 1)$, *the condition* $A = \{L_k^{[t]} < n + \sum_i \sum_j \mu_i X_{ij}^{[t]}$, *where* $X_{ij}^{[t]}$ *and* $L_k^{[t]}$ *are the numbers of events* e_{ij} *and* $\tilde{e}_k$, *respectively, in the first t trials*$\}$, *is satisfied.*

Then, for $x_i = 0, 1, \ldots, (1 \le i \le m)$,

$$P(X_1 = x_1, \ldots, X_m = x_m)$$

$$= \sum_{\sum_j jx_{ij} = x_i} \frac{n}{n + \sum_i \sum_j (1 + \mu_i) x_{ij}} \binom{n + \sum_i \sum_j (1 + \mu_i) x_{ij}}{x_{11}, \ldots, x_{mk}, n + \sum_i \sum_j \mu_i x_{ij}}$$

$$\times (p_1 \cdots p_k)^{n + \sum_i \sum_j \mu_i x_{ij}} \prod_i \prod_j q_{ij}^{x_{ij}} \prod_{s=1}^{k-1} p_s^{\sum_i \sum_{j=s+1}^{k} x_{ij}},$$

where $p_j = 1 - \sum_i q_{ij}$ $(1 \le j \le k)$.

Proof. For any fixed non-negative integers $x_1, \ldots, x_m$ a typical element of the event $(X_1 = x_1, \ldots, X_m = x_m)$ is a sequence of $\sum_i X_i + k(n + \sum_i \sum_j \mu_i X_{ij})$ outcomes of the letters $F_1, \ldots, F_m$ and S, such that the event e_{ij} appears x_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ times and the event $\tilde{e}_k$ appears $n + \sum_i \sum_j \mu_i x_{ij}$ times, satisfying the condition A and $\sum_j jx_{ij} = x_i$ $(1 \le i \le m)$.

Fix x_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ (n and μ_i $(1 \le i \le m)$ are fixed) and denote the event e_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ by a step in Z_{ij} direction and the event $\tilde{e}_k$ by a step in Z_0 direction. Therefore, we represent a sequence of x_{ij} events e_{ij} $(1 \le i \le m$ and $1 \le j \le k)$ and $n + \sum_i \sum_j \mu_i x_{ij}$ events $\tilde{e}_k$ by an $(mk + 1)$-dimensional lattice path from the origin to $(n + \sum_i \sum_j \mu_i x_{ij}, x_{11}, \ldots, x_{mk})$, which does not touch the hyperplane $z_0 = n + \sum_i \sum_j \mu_i x_{ij}$ except at the point $(n + \sum_i \sum_j \mu_i x_{ij}, x_{11}, \ldots, x_{mk})$. Then the number of such lattice paths is

$$\frac{n}{n + \sum_i \sum_j (1 + \mu_i) x_{ij}} \binom{n + \sum_i \sum_j (1 + \mu_i) x_{ij}}{x_{11}, \ldots, x_{mk}, n + \sum_i \sum_j \mu_i x_{ij}}$$

(see Sen and Jain [16]) and each one of them has probability

$$(p_1 \cdots p_k)^{n + \sum_i \sum_j \mu_i x_{ij}} \prod_i \prod_j q_{ij}^{x_{ij}} \prod_{s=1}^{k-1} p_s^{\sum_i \sum_{j=s+1}^{k} x_{ij}}.$$

The theorem then follows, since the non-negative integers $x_{i1}, \ldots, x_{ik}$ $(1 \le i \le m)$ may vary subject to $\sum_j jx_{ij} = x_i$ $(1 \le i \le m)$.

By means of the transformations $x_{ij} = n_{ij}$ and $x_i = n_i + \sum_j (j-1)n_{ij}$ $(1 \le i \le m$ and $1 \le j \le k)$, the multinomial theorem and relation (10) of Sen and Jain [16], it may be seen that the above derived probability function is a proper probability distribution. Even though n is an integer, the argument still holds true for any positive real number n.

For $k = 1$, this distribution reduces to the multivariate generalized negative binomial distribution (see Jain [3]), and, for $\mu_i = 0$ $(1 \le i \le m)$, it reduces to the multivariate extended negative binomial distribution of order k (see Philippou and Antzoulakos [11]). Furthermore, for $p_1 = \cdots = p_k = p$, it reduces to the multivariate generalized negative binomial distribution of order k, type I (see Tripsiannis et al. [19]). We therefore introduce the following.

Definition 2.2. A random vector (rv) $\mathbf{X} = (X_1, \ldots, X_m)$ is said to have the *multivariate extended generalized negative binomial distribution of order* k, with parameters n, $\mu_1, \ldots, \mu_m$, $q_{11}, \ldots, q_{mk}$ $(n > 0, \mu_i \ge -1$ $(1 \le i \le m)$ all integers, $0 < q_{ij} < 1$ $(1 \le i \le m)$, $\sum_i q_{ij} < 1$ $(1 \le j \le k)$ and $p_j = 1 - \sum_i q_{ij})$, to be denoted by $MEGN\,B_k(n; \mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk})$, if, for $x_i = 0, 1, \ldots$ $(1 \le i \le m)$,

$$P(\mathbf{X} = \mathbf{x}) = \sum_{\sum_j jx_{ij} = x_i} \frac{n\Gamma(n + \sum_i \sum_j (1+\mu_i)x_{ij})}{\Gamma(n + \sum_i \sum_j \mu_i x_{ij} + 1)\prod_i \prod_j x_{ij}!} (p_1 \cdots p_k)^{n + \sum_i \sum_j \mu_i x_{ij}}$$

$$\times \prod_i \prod_j q_{ij}^{x_{ij}} \prod_{s=1}^{k-1} p_s^{\sum_i \sum_{j=s+1}^{k} x_{ij}}.$$

For $m = 1$, this distribution reduces to a new distribution of order k, which we call *extended generalized negative binomial distribution of order* k with parameters n, μ, $p_1, \ldots, p_k$ $(n > 0, \mu \ge -1$ is an integer, $0 < p_j < 1$ and $q_j = 1 - p_j$ $(1 \le j \le k))$ and we denote it by $EGNB_k(n; \mu; p_1, \ldots, p_k)$, since, for $\mu = 0$, this distribution reduces to the (shifted) extended negative binomial distribution of order k of Aki [1] and, for $k = 1$, it reduces to the generalized negative binomial distribution (see Jain and Consul [4]), with $\beta = \mu + 1$. Furthermore, for $p_1 = \cdots = p_k = p$, it reduces to the (shifted) generalized negative binomial distribution of order k, type I, of Tripsiannis et al. [17].

It is well known that the multivariate generalized logarithmic series distribution may be obtained as a limit of the multivariate generalized negative binomial distribution (see Jain [3]). We shall extend this result to the multivariate extended generalized negative binomial distribution of order k, and we shall name the limit accordingly.

Proposition 2.1. *Let* $\mathbf{X}_n$ $(n > 0)$ *be* $m \times 1$ *rv's distributed as* $MEGNB_k(n; \mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk})$, *set* $a = -[\log(p_1 \cdots p_k)]^{-1}$ *and assume that* $n \to 0$. *Then, for* $x_i = 0, 1, \ldots,$ $(1 \leq i \leq m)$ *and* $\sum_i x_i > 0$, *we have*

$$P(X_{n1} = x_1, \ldots, X_{nm} = x_m \mid \textstyle\sum_i X_{ni} > 0)$$

$$\to a \sum_{\sum_j j x_{ij} = x_i} \frac{1}{\sum_i \sum_j (1 + \mu_i) x_{ij}} \binom{\sum_i \sum_j (1 + \mu_i) x_{ij}}{x_{11}, \ldots, x_{mk}, \sum_i \sum_j \mu_i x_{ij}} (p_1 \cdots p_k)^{\sum_i \sum_j \mu_i x_{ij}}$$

$$\times \prod_i \prod_j q_{ij}^{x_{ij}} \prod_{s=1}^{k-1} p_s^{\sum_i \sum_{j=s+1}^{k} x_{ij}}.$$

For $k = 1$, this distribution reduces to the usual multivariate generalized logarithmic series distribution (see Jain [3]), and, for $\mu_i = 0$ $(1 \leq i \leq m)$, it reduces to the multivariate extended logarithmic series distribution of order k (see Philippou and Antzoulakos [11]). Furthermore, for $p_1 = \cdots = p_k = p$, it reduces to the multivariate generalized logarithmic series distribution of order k, type I (see Tripsiannis et al. [19]). We therefore introduce the following definition.

Definition 2.3. A rv $\mathbf{X} = (X_1, \ldots, X_m)$ is said to have the *multivariate extended generalized logarithmic series distribution of order* k, with parameters $\mu_1, \ldots, \mu_m$, $q_{11}, \ldots, q_{mk}$ $(\mu_i \geq -1$ $(1 \leq i \leq m)$ all integers, $0 < q_{ij} < 1$ $(1 \leq i \leq m$ and $1 \leq j \leq k)$, $\sum_i q_{ij} < 1$ $(1 \leq j \leq k)$ and $p_j = 1 - \sum_i q_{ij})$, to be denoted by $MEGL\,S_k(\mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk})$, if, for $x_i = 0, 1, \ldots,$ $(1 \leq i \leq m)$ and $\sum_i x_i > 0$, $P(\mathbf{X} = \mathbf{x})$ is equal to the limit given in Proposition 2.1.

For $m = 1$, this distribution reduces to a new distribution of order k, which we call *extended generalized logarithmic series distribution of order* k with parameters μ, $p_1, \ldots, p_k$ ($\mu \geq -1$ is an integer, $0 < p_j < 1$ and $q_j = 1 - p_j$ $(1 \leq j \leq k)$) and we denote by $EGLS_k(\mu;\ p_1, \ldots, p_k)$, since, for $\mu = 0$, this distribution reduces to the extended logarithmic series distribution of order k of Aki [1] and, for $k = 1$, it reduces to the generalized logarithmic series distribution (see Jain and Gupta [5]), with $\beta = \mu + 1$. Also, for $p_1 = \cdots = p_k = p$, it reduces to the generalized logarithmic series distribution of order k, type I, of Tripsiannis et al. [17].

It is well known that the multivariate generalized Poisson distribution may be obtained as a limit of the multivariate generalized negative binomial distribution (see Jain [3]). This result readily extends as follows.

Proposition 2.2. *Let* $\mathbf{X}_n$ $(n > 0)$ *be an* $m \times 1$ *rv's distributed as* $MEGN\ B_k(\mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk})$, *and assume that* $nq_{ij} \to \theta_{ij}$ $(\theta_{ij} > 0)$ *and* $\mu_\nu q_{ij} \to \lambda_{ij\nu}(|\lambda_{ij\nu}| < 1)$ *for* $1 \leq i,\ \nu \leq m$ *and* $1 \leq j \leq k$, *as* $q_{ij} \to 0$ $(1 \leq i \leq m$ *and* $1 \leq j \leq k)$, $\mu_i \to \infty$ *and* $n \to \infty$. *Then, for* $x_i = 0, 1, \ldots,$ $(1 \leq i \leq m)$, *we have*

$$P(\mathbf{X}_n = \mathbf{x})$$

$$\to \sum_{\sum_j jx_{ij} = x_i} \frac{\theta_{11}}{\prod_i \prod_j x_{ij}!} (\theta_{11} + \textstyle\sum_\nu \lambda_{11\nu} \sum_j x_{\nu j})^{x_{11}-1} (\theta_{12} + \sum_\nu \lambda_{12\nu} \sum_j x_{\nu j})^{x_{12}}$$

$$\times \cdots \times (\theta_{mk} + \textstyle\sum_\nu \lambda_{mk\nu} \sum_j x_{\nu j})^{x_{mk}} e^{-\sum_i \sum_j \theta_{ij} - \sum_\nu [(\sum_i \sum_j \lambda_{ij\nu})(\sum_j x_{\nu j})]}.$$

For $k = 1$, this distribution reduces to the multivariate generalized Poisson distribution (see Jain [3]) and for $\lambda_{ij\nu} = 0$ $(1 \leq i, \nu \leq m$ and $1 \leq j \leq k)$, it reduces to the multivariate Poisson distribution or multivariate extended Poisson distribution of order k (see Philippou et al. [13]). We therefore introduce the following definition.

Definition 2.4. A rv $\mathbf{X} = (X_1, \ldots, X_m)$ is said to have the *multivariate extended generalized Poisson distribution of order* k, with parameters $\theta_{11}, ..., \theta_{mk}$, $\lambda_{111}, ..., \lambda_{mkm}$ $(\theta_{ij} > 0$ and $|\lambda_{ij\nu}| < 1)$ $(1 \le i, \nu \le m$ and $1 \le j \le k)$, to be denoted by $MEGP_k(\theta_{11}, ..., \theta_{mk}; \lambda_{111}, ..., \lambda_{mkm})$ if, for $x_i = 0, 1, ...,$ $(1 \le i \le m)$, $P(\mathbf{X} = \mathbf{x})$ is equal to the limit given in Proposition 2.2.

For $m = 1$, this distribution reduces to a new distribution of order k, which we call *extended generalized Poisson distribution of order k* with parameters $\theta_1, ..., \theta_k$, $\lambda_1, ..., \lambda_k$ $(\theta_j > 0$ and $|\lambda_j| < 1)$ $(1 \le j \le k)$, to be denoted by $EGP_k(\theta_1, ..., \theta_k; \lambda_1, ..., \lambda_k)$, since, for $k = 1$, this distribution reduces to the generalized Poisson distribution (see Consul and Jain [2]), and for $\theta_j = \theta$ and $\lambda_j = \lambda$ $(1 \le j \le k)$, it reduces to the generalized Poisson distribution of order k, type I, of Tripsiannis et al. [17]. Also, for $\lambda_j = 0$ $(1 \le j \le k)$, it reduces to the multiparameter or extended Poisson distribution of order k (see Aki [1] and Philippou [10]).

3. Characteristics of the Multivariate Extended Generalized Distributions of Order k

In this section, we obtain the means and variances-covariances of the multivariate extended generalized distributions of order k, treated in Section 2. We first recall the following definitions from the work of Tripsiannis et al. [18].

Definition 3.1. A rv $\mathbf{X} = (X_1, ..., X_m)$ is said to have the *multivariate generalized negative binomial distribution of order k*, with parameters n, $\mu_1, ..., \mu_m$, $Q_{11}, ..., Q_{mk}$ $(n > 0,\ \mu_i \ge -1\ (1 \le i \le m)$ all integers, $0 < Q_{ij} < 1$ $(1 \le i \le m$ and $1 \le j \le k)$, $\sum_i \sum_j Q_{ij} < 1$ and $P = 1 - \sum_i \sum_j Q_{ij})$, to be denoted by $MGNB_k(n; \mu_1, ..., \mu_m; Q_{11}, ..., Q_{mk})$ if, for $x_i = 0, 1, ...,$ $(1 \le i \le m)$,

$$P(\mathbf{X} = \mathbf{x})$$

$$= \sum_{\sum_j jx_{ij} = x_i} \frac{n\Gamma(n + \sum_i \sum_j (1 + \mu_i) x_{ij})}{\Gamma(n + \sum_i \sum_j \mu_i x_{ij} + 1) \prod_i \prod_j x_{ij}!} P^{n + \sum_i \sum_j \mu_i x_{ij}} \prod_i \prod_j Q_{ij}^{x_{ij}}.$$

Definition 3.2. A rv $\mathbf{X} = (X_1, \ldots, X_m)$ is said to have the *multivariate generalized logarithmic series distribution of order k*, with parameters $\mu_1, \ldots, \mu_m$, $Q_{11}, \ldots, Q_{mk}$ ($\mu_i \geq -1$ $(1 \leq i \leq m)$ all integers, $0 < Q_{ij} < 1$ $(1 \leq i \leq m$ and $1 \leq j \leq k)$, $\sum_i \sum_j Q_{ij} < 1$ and $P = 1 - \sum_i \sum_j Q_{ij}$), to be denoted by $MGLS_k(\mu_1, \ldots, \mu_m; Q_{11}, \ldots, Q_{mk})$, if, for $x_i = 0, 1, \ldots$ $(1 \leq i \leq m)$ and $\sum_i x_i > 0$, where $a = -(\log P)^{-1}$,

$$P(\mathbf{X} = \mathbf{x})$$

$$= a \sum_{\sum_j j x_{ij} = x_i} \frac{1}{\sum_i \sum_j (1 + \mu_i) x_{ij}} \binom{\sum_i \sum_j (1 + \mu_i) x_{ij}}{x_{11}, \ldots, x_{mk}, \sum_i \sum_j \mu_i x_{ij}} P^{\sum_i \sum_j \mu_i x_{ij}} \prod_i \prod_j Q_{ij}^{x_{ij}}.$$

By setting $Q_{i1} = q_{i1}$ and $Q_{ij} = p_1 \cdots p_{j-1} q_{ij}$ $(0 < q_{ij} < 1,\ 0 < \sum_i q_{ij} < 1$ and $p_j = 1 - \sum_i q_{ij})$ for $1 \leq i \leq m$ and $1 \leq j \leq k$, which imply $P = p_1 \cdots p_k$, we observe that

$$MGN\,B_k(n; \mu_1, \ldots, \mu_m; Q_{11}, \ldots, Q_{mk}) = MEGN\,B_k(n; \mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk})$$

and

$$MGLS_k(\mu_1, \ldots, \mu_m; Q_{11}, \ldots, Q_{mk}) = MEGLS_k(\mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk}).$$

Then, the following two propositions are direct consequences of Propositions 2.2 and 4.4 of Tripsiannis et al. [18].

Proposition 3.1. *Let* $\mathbf{X} = (X_1, \ldots, X_m)$ *be a rv following the multivariate extended generalized negative binomial distribution of order k. Then, the mean and variance-covariance are given by*

(i) $$E(X_i) = \frac{n}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}} \sum_j j p_1 \cdots p_{j-1} q_{ij}, \quad (1 \leq i \leq m),$$

(ii) $$Var(X_i) = \frac{n}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}}$$
$$\left(\sum_j j^2 p_1 \cdots p_{j-1} q_{ij} + \frac{1}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}} \right.$$
$$\times \left[2\mu_i + 1 + \frac{\sum_i \sum_j (1 + \mu_i) \mu_i p_1 \cdots p_{j-1} q_{ij}}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}} \right]$$
$$\left. (\sum_j j p_1 \cdots p_{j-1} q_{ij})^2 \right), \quad (1 \leq i \leq m),$$

$$\text{(iii)}\ \operatorname{Cov}(X_i,\ X_s) = \frac{n}{(p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij})^2}$$

$$\left(\mu_i + \mu_s + 1 + \frac{\sum_i \sum_j (1+\mu_i)\mu_i p_1 \cdots p_{j-1} q_{ij}}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}}\right)$$

$$\times (\textstyle\sum_j j p_1 \cdots p_{j-1} q_{ij})(\sum_j j p_1 \cdots p_{j-1} q_{sj}),\ (1 \le i \ne s \le m).$$

Proposition 3.1 reduces to the mean and variance-covariance of the (i) multivariate generalized negative binomial distribution (see Sen and Jain [16]), for $k = 1$, (ii) multivariate extended negative binomial distribution of order k (see Philippou and Antzoulakos [11]), for $\mu_i = 0\ (1 \le i \le m)$ and (iii) multivariate generalized negative binomial distribution of order k, type I (see Tripsiannis et al. [19]), for $p_1 = \cdots = p_k = p$.

Proposition 3.2. *Let* $\mathbf{X} = (X_1, \ldots, X_m)$ *be a rv following the multivariate extended generalized logarithmic series distribution of order k. Then, the mean and variance-covariance are given by*

$$\text{(i)}\ E(X_i) = \frac{n}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}} \textstyle\sum_j j p_1 \cdots p_{j-1} q_{ij},\ (1 \le i \le m),$$

$$\text{(ii)}\ \operatorname{Var}(X_i) = \frac{a}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}}$$

$$\left(\textstyle\sum_j j^2 p_1 \cdots p_{j-1} q_{ij} + \dfrac{1}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}}\right.$$

$$\times \left[2\mu_i + 1 + a + \frac{\sum_i \sum_j (1+\mu_i)\mu_i p_1 \cdots p_{j-1} q_{ij}}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}}\right]$$

$$\left.(\textstyle\sum_j j p_1 \cdots p_{j-1} q_{ij})^2\right),\ (1 \le i \le m),$$

$$\text{(iii)}\ \operatorname{Cov}(X_i,\ X_s) = \frac{a}{(p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij})^2}$$

$$\left(\mu_i + \mu_s + 1 + a + \frac{\sum_i \sum_j (1+\mu_i)\mu_i p_1 \cdots p_{j-1} q_{ij}}{p_1 \cdots p_k - \sum_i \sum_j \mu_i p_1 \cdots p_{j-1} q_{ij}}\right)$$

$$\times (\textstyle\sum_j j p_1 \cdots p_{j-1} q_{ij})(\sum_j j p_1 \cdots p_{j-1} q_{sj}),\ (1 \le i \ne s \le m).$$

For $k = 1$, Proposition 3.2 reduces to the mean and variance-covariance of the multivariate generalized logarithmic series distribution (see Proposition 4.4 of Tripsiannis et al. [18], for $k = 1$), and for $\mu_i = 0$ $(1 \leq i \leq m)$, it reduces to the characteristics of the multivariate extended logarithmic series distribution of order k (see Philippou and Antzoulakos [11]). Furthermore, for $p_1 = \cdots = p_k = p$, it provides the characteristics of the multivariate generalized logarithmic series distribution of order k, type I (see Tripsiannis et al. [19]).

The mean and variance-covariance of the multivariate extended generalized Poisson distribution of order k can be easily obtained as a limit of the respective characteristics of the multivariate extended generalized negative binomial distribution of the same order.

Proposition 3.3. *Let* $\mathbf{X} = (X_1, \ldots, X_m)$ *be a rv following the multivariate extended generalized Poisson distribution of order k. Then, the mean and variance-covariance are given by*

(i) $E(X_i) = \dfrac{\sum_j j\theta_{ij}}{1 - \sum_i \sum_j \lambda_{iji}}$, $(1 \leq i \leq m)$,

(ii) $$\mathrm{Var}(X_i) = \frac{1}{1 - \sum_i \sum_j \lambda_{iji}} \left[\sum_j j^2\theta_{ij} + \frac{\sum_j j\theta_{ij}}{1 - \sum_i \sum_j \lambda_{iji}} \left(2\sum_j j\lambda_{iji} + \frac{(\sum_j j\theta_{ij})(\sum_i \sum_j \lambda_{iji}^2/\theta_{ij})}{1 - \sum_i \sum_j \lambda_{iji}} \right) \right], \quad (1 \leq i \leq m),$$

(iii) $$\mathrm{Cov}(X_i, X_s) = \frac{(\sum_j j\theta_{ij})(\sum_j j\theta_{sj})}{(1 - \sum_i \sum_j \lambda_{iji})^2} \left(\frac{\lambda_{iji}}{\theta_{ij}} + \frac{\lambda_{sjs}}{\theta_{sj}} + \frac{\sum_i \sum_j \lambda_{iji}^2/\theta_{ij}}{1 - \sum_i \sum_j \lambda_{iji}} \right), \quad (1 \leq i \neq s \leq m).$$

For $k = 1$, Proposition 3.3 reduces to the mean and variance-covariance of the multivariate generalized Poisson distribution (see Proposition 4.5 of Tripsiannis et al. [18], for $k = 1$). Also, for $\lambda_{ij\nu} = 0$ $(1 \leq i, \nu \leq m$ and $1 \leq j \leq k)$, it reduces to the characteristics of the multivariate Poisson distribution (see Philippou et al. [13]).

For $m = 1$, Propositions 3.1-3.3 provide the means and variances of the respective univariate distributions of order k.

4. Special and Limiting Cases of $MEGNB_k(n;\ \mu_1,\ \dots,\ \mu_m;\ q_{11},\ \dots,\ q_{mk})$

In the present section, we note that the multivariate extended generalized negative binomial distribution of order k reduces to six new multivariate extended distributions of order k for appropriate choices of its parameters.

Case I. The $MEGNB_k(n; \mu_1, \dots, \mu_m; q_{11}, \dots, q_{mk})$, for $\mu_i = 1$ $(1 \leq i \leq m)$ and $q_{ij}/p_j = P_{ij}$ $(1 \leq i \leq m$ and $1 \leq j \leq k)$ so that $q_{ij} = P_{ij}/Q_j$ and $p_j = 1/Q_j$, where $Q_j = 1 + \sum_i P_{ij}$, and n is replaced by nmk and x_{ij} by $x_{ij} - n$, reduces to a new multivariate distribution which we call *multivariate extended Haight distribution of order k* with parameters n, $P_{11}, \dots, P_{mk}$.

Case II. The $MEGNB_k(n; \mu_1, \dots, \mu_m; q_{11}, \dots, q_{mk})$, for $q_{ij}/p_j = P_{ij}$ $(1 \leq i \leq m$ and $1 \leq j \leq k)$ so that $q_{ij} = P_{ij}/Q_j$ and $p_j = 1/Q_j$, where $Q_j = 1 + \sum_i P_{ij}$, and n is replaced by $k \sum_i \mu_i$ and x_{ij} by $x_{ij} - 1$, reduces to a new multivariate distribution which we call *multivariate extended Takács distribution of order k* with parameters $\mu_1, \dots, \mu_m$, $P_{11}, \dots, P_{mk}$.

Case III. The $MEGNB_k(n; \mu_1, \dots, \mu_m; q_{11}, \dots, q_{mk})$, for $\mu_i = d_i - 1$, and n is replaced by $nk \sum_i d_i$ and x_{ij} by $x_{ij} - n$, reduces to a new multivariate distribution which we call *multivariate extended binomial-delta distribution of order k* with parameters n, $d_1, \dots, d_m$, $p_1, \dots, p_k$.

Case IV. The $MEGNB_k(n; \mu_1, \dots, \mu_m; q_{11}, \dots, q_{mk})$, for $q_{ij}/p_j = P_{ij}$ $(1 \leq i \leq m$ and $1 \leq j \leq k)$ so that $q_{ij} = P_{ij}/Q_j$ and $p_j = 1/Q_j$, where $Q_j = 1 + \sum_i P_{ij}$, and n is replaced by $nk \sum_i \mu_i$ and x_{ij} by $x_{ij} - n$, reduces to a new multivariate distribution which we call *multivariate extended negative binomial-delta distribution of order k* with parameters n, $\mu_1, \dots, \mu_m$, $P_{11}, \dots, P_{mk}$.

Case V. The $MEGNB_k(n; \mu_1, \dots, \mu_m; q_{11}, \dots, q_{mk})$, for $q_{ij}/p_j = P_{ij}$ $(1 \leq i \leq m$ and $1 \leq j \leq k)$ so that $q_{ij} = P_{ij}/Q_j$ and $p_j = 1/Q_j$, where $Q_j = 1 + \sum_i P_{ij}$, reduces to a new multivariate distribution which we call *multivariate extended negative binomial-negative binomial distribution of order k* with parameters n, $\mu_1, \dots, \mu_m$, $P_{11}, \dots, P_{mk}$.

For $k = 1,$ the above five new multivariate distributions of order k reduce to the corresponding usual multivariate distributions (see Sen and Jain [16]) and for $P_{ij} = P_i$ $(1 \le j \le k)$ $(p_j = p$ $(1 \le j \le k)$, for Case III), they reduce to the corresponding, type I, multivariate distributions of order k of Tripsiannis et al. [19]. Furthermore, for $m = 1,$ they reduce to new univariate extended distributions of order k.

Case VI. In $MEGNB_k(n; \mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk}),$ let $\mu_i = -1$ $(1 \le i \le m)$ and interchange p_j and q_j $(1 \le j \le k)$. Then, for $x_i = 0, 1, \ldots, kn,$ $(1 \le i \le m)$

$$P(\mathbf{X} = \mathbf{x}) = \sum_{\sum_j jx_{ij} = x_i} \binom{n}{x_{11}, \ldots, x_{mk}, n - \sum_i \sum_j x_{ij}} (q_1 \cdots q_k)^{n - \sum_i \sum_j x_{ij}}$$

$$\times \prod_i \prod_j p_{ij}^{x_{ij}} \prod_{s=1}^{k-1} q_s^{\sum_i \sum_{j=s+1}^k x_{ij}},$$

which reduces to the usual multinomial distribution with parameters n, $p_1, \ldots, p_k,$ (see Patil et al. [8, p. 14]), for $k = 1,$ and to the multinomial distribution of order k, type I, of Tripsiannis et al. [19], for $p_{ij} = p_i$ $(1 \le j \le k)$. We say that the rv $\mathbf{X}$ has the *extended multinomial distribution of order k*, with parameters $n,$ $p_{11}, \ldots, p_{mk}$ and denote it by $EM_k^*(n; p_{11}, \ldots, p_{mk})$. Also, for $m = 1,$ it reduces to a new distribution of order k, which we call *extended binomial distribution of order k*, with parameters $n,$ $p_1, \ldots, p_k$ and denote it by $EB_k^*(n; p_1, \ldots, p_k)$.

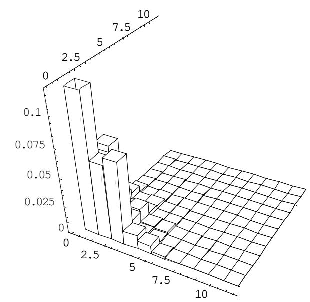
Next, we establish a proposition which relates asymptotically $MEGNB_k(\cdot)$ to the multivariate Poisson $(MP_k(\lambda_{11}, \ldots, \lambda_{mk}))$ distribution of order k.

Proposition 4.1. *Let* $\mathbf{X}_n$ $(n > 0)$ *and* $\mathbf{X}$ *be two rv's distributed as* $MEGNB_k(n; \mu_1, \ldots, \mu_m; q_{11}, \ldots, q_{mk})$ *and* $MP_k(\lambda_{11}, \ldots, \lambda_{mk})$, *respectively, and assume that* $q_{ij} \to 0$ $(1 \le i \le m$ *and* $1 \le j \le k)$ *and* $nq_{ij} \to \lambda_{ij}$ $(\lambda_{ij} > 0,$ $1 \le i \le m$ *and* $1 \le j \le k)$, *as* $n \to \infty$. *Then, for* $x_i = 0, 1, \ldots$ $(1 \le i \le m)$, *we have*

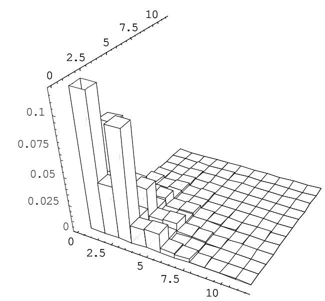
$$P(\mathbf{X}_n = \mathbf{x}) \to P(\mathbf{X} = \mathbf{x}).$$

Figure 4.1 presents the graphs of $MEGNB_k(\cdot)$, for selected values of its parameters.

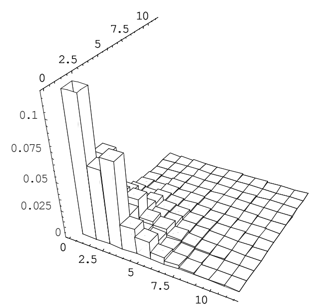
$q_{11} = 0.025$, $q_{12} = 0.025$
$q_{21} = 0.025$, $q_{22} = 0.025$



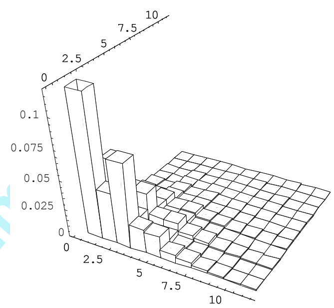
$q_{11} = 0.025$, $q_{12} = 0.05$
$q_{21} = 0.025$, $q_{22} = 0.05$



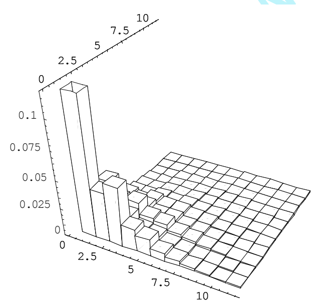
$q_{11} = 0.05$, $q_{12} = 0.05$
$q_{21} = 0.05$, $q_{22} = 0.05$



$q_{11} = 0.05$, $q_{12} = 0.075$
$q_{21} = 0.05$, $q_{22} = 0.075$



$q_{11} = 0.075$, $q_{12} = 0.075$
$q_{21} = 0.075$, $q_{22} = 0.075$



$q_{11} = 0.075$, $q_{12} = 0.1$
$q_{21} = 0.075$, $q_{22} = 0.1$

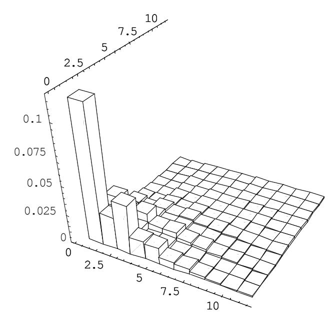


Figure 4.1. Bivariate extended negative binomial distribution of order 2, for $n = 5$, $\mu_1 = 1$ and $\mu_2 = 2$.

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