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A NOTE ON THE ALGORITHM OF RESOLUTION OF A CAPACITATED TRANSPORTATION PROBLEM WITH FOUR SUBSCRIPTS

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Abstract

We present some modifications of the algorithm proposed in [3], by us in kybernetes review. The degeneracy problem is successfully treated and the presentation is considerably improved.

1. Introduction

As in [3], we consider the following capacitated transportation problem with four subscripts which we denote (T_C)

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} c_{ijkl}x_{ijkl} \tag{1}$$

subject to the constraints:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \text{ for all } i = 1, \ldots, m \tag{2}$$

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$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \text{ for all } j = 1, \ldots, n \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \text{ for all } k = 1, \ldots, p \tag{4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \text{ for all } l = 1, \ldots, q \tag{5}$$

$$0 \le x_{ijkl} \le d_{ijkl} \text{ for all } (i, j, k, l). \tag{6}$$

In this problem, α_i, β_j, γ_k, δ_l, d_{ijkl} and c_{ijkl} are given and are such that for all i, j, k, l, we have $\alpha_i > 0$, $\beta_j > 0$, $\gamma_k > 0$, $\delta_l > 0$, $d_{ijkl} > 0$ and $c_{ijkl} \ge 0$.

Recall that a program $x = (x_{ijkl})$ of problem (T_C) is degenerate if the number of columns corresponding to variables x_{ijkl} satisfying $0 < x_{ijkl} < d_{ijkl}$ is strictly less than $rank(A)$.

It is useful to present the data of the problem thanks to the following transportation table. It consists of an array of M rows and N columns, three additional rows and an additional column. The entries of these N columns of the first, second, and third additional rows are reserved for the data of the quantities d_{ijkl}, c_{ijkl}, and x_{ijkl}, respectively. The additional column is for the data of quantities α_i, β_j, γ_k, and δ_l, respectively. Finally, the entry of the array on the line corresponding to α'_i and the column P_{ijkl} is 1 if $i = i'$ and 0 otherwise. Same things for $\beta_{j'}$, $\gamma_{k'}$, and $\delta_{l'}$. We give an example below.

d_{1111} .	d_{1211} .	$\cdots$	d_{mnpq} .
c_{1111} .	c_{1211} .	$\cdots$	c_{mnpq} .

x_{1111} .	x_{1211} .	$\cdots$	x_{mnpq} .	
1	1	$\cdots$	0	α_1 .
:	:	$\cdots$	:	:
0	0	$\cdots$	1	α_m .
1	0	$\cdots$	0	β_1 .
0	1	$\cdots$	0	β_2 .
:	:	$\cdots$	:	:
0	0	$\cdots$	1	β_n .
1	1	$\cdots$	0	γ_1 .
:	:	$\cdots$	:	:
0	0	$\cdots$	1	γ_p .
1	1	$\cdots$	0	δ_1 .
:	:	$\cdots$	:	:
0	0	$\cdots$	1	δ_q .

The algorithm described in [3, Section 5], is not immunized against degenerateness. This aim is confirmed by our numerical experimentations. We propose here some modifications (essentially in Step 1 of Phase 1 and in $a/$ and $b/$ of Phase 2) to overcome this inconvenient. Furthermore, this new presentation is more convenient to apprehend and to implement. Note that the squares (i, j, k, l) are arranged in a similar way as variables x_{ijkl} (see Section 4 of [3]) and the subscript r which indicate the iteration number is replaced by the superscript (r). In Section 2, we describe the new version of our algorithm.

2. Algorithm

Phase 1 (It finds a basic program or says that (T_C) is not solvable)

Step 1.

Initialization. For all (i, j, k, l), $\hat{\alpha}_i = \alpha_i$, $\hat{\beta}_j = \beta_j$, $\hat{\gamma}_k = \gamma_k$, $\hat{\delta}_l = \delta_l$ and $b_{ijkl} = 0$, (b_{ijkl} is a Boolean variable equal to 1 if x_{ijkl} has already been determined and 0 if not yet),

$$E = \{(i, j, k, l), \text{ such that } b_{ijkl} = 0\}.$$

Iteration.

While $E \neq \phi$ do

- Choose a 4-tuple $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) \in E$, such that $c_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min_{(i, j, k, l) \in E} c_{ijkl}$,

  (see **Remark 1**, below)

- Take $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min(\hat{\alpha}_{\bar{i}}, \hat{\beta}_{\bar{j}}, \hat{\gamma}_{\bar{k}}, \hat{\delta}_{\bar{l}}, d_{\bar{i}\bar{j}\bar{k}\bar{l}})$, and $b_{\bar{i}\bar{j}\bar{k}\bar{l}} = 1$,

  (i.e., $x_{\bar{i}\bar{j}\bar{k}\bar{l}}$ is determined),

- Update $\hat{\alpha}_{\bar{i}}$, $\hat{\beta}_{\bar{j}}$, $\hat{\gamma}_{\bar{k}}$, and $\hat{\delta}_{\bar{l}}$ as follows:

(1) $\hat{\alpha}_{\bar{i}} = \hat{\alpha}_{\bar{i}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\alpha}_{\bar{i}} = 0$, then take $x_{\bar{i}jkl} = 0$ for all $(j, k, l) \neq (\bar{j}, \bar{k}, \bar{l})$ and $b_{\bar{i}jkl} = 1$ for all (j, k, l),

(2) $\hat{\beta}_{\bar{j}} = \hat{\beta}_{\bar{j}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\beta}_{\bar{j}} = 0$, then take $x_{i\bar{j}kl} = 0$ for all $(i, k, l) \neq (\bar{i}, \bar{k}, \bar{l})$ and $b_{i\bar{j}kl} = 1$ for all (i, k, l),

(3) $\hat{\gamma}_{\bar{k}} = \hat{\gamma}_{\bar{k}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\gamma}_{\bar{k}} = 0$, then take $x_{ij\bar{k}l} = 0$ for all $(i, j, l) \neq (\bar{i}, \bar{j}, \bar{l})$ and $b_{ij\bar{k}l} = 1$ for all (i, j, l),

(4) $\hat{\delta}_{\bar{l}} = \hat{\delta}_{\bar{l}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$,

if $\hat{\delta}_{\bar{l}} = 0$, then take $x_{ijk\bar{l}} = 0$ for all $(i, j, k) \neq (\bar{i}, \bar{j}, \bar{k})$ and $b_{ijk\bar{l}} = 1$ for all (i, j, k).

Step 2.

(a) Take

$$\varepsilon = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{k=1}^{p} e_k = \sum_{l=1}^{q} f_l,$$

such that:

$$a_i = \alpha_i - \sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} \text{ with } i = 1, \ldots, m,$$

$$b_j = \beta_j - \sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} \text{ with } j = 1, \ldots, n,$$

$$e_k = \gamma_k - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} \text{ with } k = 1, \ldots, p,$$

$$f_l = \delta_l - \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} \text{ with } l = 1, \ldots, p.$$

Note that the numbers a_i, b_j, e_k and f_l are nonnegative.

If $\varepsilon = 0$, then $x = (x_{ijkl})$ is an **initial basic program** for the problem (T_C), we denote it by $x^{(0)}$. **Go to Phase 2**.

Else, Construct a problem $T_C(\widetilde{M})$ by the procedure described in (P1) below, and find an initial basic program $\overline{x}^{(0)}$ for the problem $T_C(\widetilde{M})$, as in Step 1.

$(\bar{x}^{(0)} = (\bar{x}_{ijkl}^{(0)})$, with $i = 1, \ldots, m+1$, $j = 1, \ldots, n+1$, $k = 1, \ldots, p+1$ and $l = 1, \ldots, q+1)$.

Then, according to **Remark 1** below, $\bar{x}_{m+1, n+1, p+1, q+1}^{(0)} = 0$.

If $\bar{x}^{(0)}$ is optimal, then **the problem** (T_C) **is not solvable**. **Stop**.

(b) Improvement of a basic program for $T_C(\widetilde{M})$.

Initialization. $r = 1$, $\varepsilon > 0$ is known,

(1) Determine $\bar{x}^{(r)}$ as in Phase 2.

(2) If $\bar{x}_{m+1, n+1, p+1, q+1}^{(r)} = \varepsilon$, then $x^{(r)} = (x_{ijkl}^{(r)})$ with $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, p$, and $l = 1, \ldots, q$, is an **initial basic program** for the problem (T_C). **Go to Phase 2**.

(3) If $\bar{x}^{(r)}$ is optimal (Phase 2), then **the problem** (T_C) **is not solvable**. **Stop**.

(4) Do $r = r + 1$ and repeat (1), to (3).

Next, we describe the second phase.

Phase 2 (Research of an optimal program for (T_C))

When Phase 2 starts, we know an initial basic program $x^{(0)}$. Take $r = 0$.

(a) Determine the set $I^{(r)}$ of the interesting 4-tuples (i, j, k, l),

(see **Remark 2** below).

(b) For all $(i, j, k, l) \in I^{(r)}$, solve the linear system

$$u_i^{(r)} + v_j^{(r)} + w_k^{(r)} + t_l^{(r)} = c_{ijkl}.$$

(c) For all $(i, j, k, l) \notin I^{(r)}$, determine

$$\Delta_{ijkl}^{(r)} = c_{ijkl} - (u_i^{(r)} + v_j^{(r)} + w_k^{(r)} + t_l^{(r)}).$$

Take $\Gamma_0^{(r)}$ and $\Gamma_d^{(r)}$ as two tables such that

$$\Gamma_0^{(r)} = \{\Delta_{ijkl}^{(r)} \text{ such that } x_{ijkl}^{(r)} = 0\},$$

$$\Gamma_d^{(r)} = \{\Delta_{ijkl}^{(r)} \text{ such that } x_{ijkl}^{(r)} = d_{ijkl}\},$$

and elements $\Delta_{ijkl}^{(r)}$ are represented as variables x_{ijkl} in the transportation table.

By going from the left to the right in $\Gamma_0^{(r)}$, choose

$$\Delta_{i_0 j_0 k_0 l_0}^{(r)} \text{ as the first element } \Delta_{ijkl}^{(r)} < 0 \text{ found,}$$

if all elements of $\Gamma_0^{(r)}$ are nonnegative, then choose in $\Gamma_d^{(r)}$ similarly,

$$\Delta_{i_0 j_0 k_0 l_0}^{(r)} \text{ as the first element } \Delta_{ijkl}^{(r)} > 0 \text{ found.}$$

If all elements of $\Gamma_d^{(r)}$ are nonpositive, then **the program** $x^{(r)}$ **is optimal**. **Stop**.

(d) Construct via the procedure described in (P2) below, a cycle $\mu^{(r)}$ containing some interesting 4-tuples (i, j, k, l) and the uninteresting 4-tuple (i_0, j_0, k_0, l_0) corresponding to $\Delta_{i_0 j_0 k_0 l_0}^{(r)}$.

Take

$\sigma^{(r)} = \{(i, j, k, l)$ such that (i, j, k, l) is a 4-tuple forming the cycle $\mu^{(r)}\}$

$\sigma^{(r)-} = \{(i, j, k, l)$ such that $(i, j, k, l) \in \sigma^{(r)}$, with $\alpha_{ijkl} < 0\}$

$\sigma^{(r)+} = \{(i, j, k, l)$ such that $(i, j, k, l) \in \sigma^{(r)}$, with $\alpha_{ijkl} > 0\}$.

If $\Delta^{(r)}_{i_0 j_0 k_0 l_0} \in \Gamma^{(r)}_0$, then determine

$$\theta^{(r)}_1 = \min_{(i, j, k, l) \in \sigma^{(r)-}} (x^{(r)}_{ijkl} / -\alpha_{ijkl}),$$

$$\theta^{(r)}_2 = \min_{(i, j, k, l) \in \sigma^{(r)+}} ((d_{ijkl} - x^{(r)}_{ijkl}) / \alpha_{ijkl}),$$

$$\theta^{(r)} = \min(\theta^{(r)}_1, \theta^{(r)}_2).$$

Next, take

$$x^{(r+1)} = \{x^{(r)}_{ijkl} + \alpha_{ijkl}\theta^{(r)}, (i, j, k, l) \in \sigma^{(r)}\} \cup \{x^{(r)}_{ijkl}, (i, j, k, l) \notin \sigma^{(r)}\}.$$

Else $(\Delta^{(r)}_{i_0 j_0 k_0 l_0} \in \Gamma^{(r)}_d)$, determine

$$\theta^{(r)}_1 = \min_{(i, j, k, l) \in \sigma^{(r)+}} (x^{(r)}_{ijkl} / \alpha_{ijkl}),$$

$$\theta^{(r)}_2 = \min_{(i, j, k, l) \in \sigma^{(r)-}} ((d_{ijkl} - x^{(r)}_{ijkl}) / -\alpha_{ijkl}),$$

$$\theta^{(r)} = \min(\theta^{(r)}_1, \theta^{(r)}_2).$$

Next, take

$$x^{(r+1)} = \{x^{(r)}_{ijkl} - \alpha_{ijkl}\theta^{(r)}, (i, j, k, l) \in \sigma^{(r)}\} \cup \{x^{(r)}_{ijkl}, (i, j, k, l) \notin \sigma^{(r)}\}.$$

(e) Do $r = r + 1$ and repeat (a) to (d) until the optimality condition holds.

In the description of the Algorithm, we have made reference to the two following remarks:

Remark 1. If there are several elements corresponding to the minimum of c_{ijkl}, then we choose one, for instance the first found in the transportation table by going from the left to the right.

Remark 2. If the program is degenerate (i.e., the number of columns of A_x is strictly less than $rank(A)$, then we complete A_x with additional columns so that we obtain a matrix having $rank(A)$ linearly independent columns. Next $I^{(r)}$ can be determined.

Also, the algorithm makes appeal to the two following procedures:

(P1) -**construction of a problem** $T_C(\widetilde{M})$:

The problem $T_C(\widetilde{M})$ is obtained from problem (T_C) by adding four fictitious points with indices $m + 1$, $n + 1$, $p + 1$, and $q + 1$ such that:

$$c_{m+1, n+1, p+1, q+1} = 0,$$

and

$$c_{m+1, jkl} = c_{i, n+1, kl} = c_{ij, p+1, l} = c_{ijk, q+1} = \widetilde{M},$$

where $\widetilde{M}$ is a very large number and there are no limitation on the capacities for the paths involving a fictitious point.

(P2) -**determination of cycles:**

A cycle $\mu^{(r)}$ is determined by solving the linear system

$$\sum_{(i, j, k, l) \in I^{(r)}} \alpha_{ijkl} P_{ijkl} = -P_{i_0 j_0 k_0 l_0}.$$

The non-null solutions α_{ijkl} are called *coefficients* of the cycle $\mu^{(r)}$.

Appendix

(1) **Notations.**

Let:

- $\mathbf{E}_b$: The set of vectors corresponding to variables $x_{ijkl}^{(r)}$ verifying

$$0 < x_{ijkl}^{(r)} < d_{ijkl}.$$

- N_b : The number of elements of $\mathbf{E}_b$.

- $\mathbf{E}_h$: The set of vectors corresponding to variables $x_{ijkl}^{(r)}$ verifying

$$x_{ijkl}^{(r)} = 0 \text{ or } x_{ijkl}^{(r)} = d_{ijkl}.$$

- N_h : The number of elements of $\mathbf{E}_h$, $N_h = mnpq - N_b$.

- $\mathbf{E}_s$: The set of first s elements in $\mathbf{E}_h$ by going from the left to the right, such that $s = rank(A) - N_b$.

(2) **A procedure for determining** $I^{(r)}$:

When this procedure starts, we know a set $\mathbf{E}_s$,

(i) **If** $\mathbf{E}_s \cup \mathbf{E}_b$ is free, then a base $I^{(r)}$ is determined **Stop**.

(ii) Replace the first (the second, ..., or the s^{ieme}) element in E_s by the $(s+1)^{ieme}$ one in E_h, or take an another unspecified subset of E_h, and repeat (i) until determining a base $I^{(r)}$ **Stop**.

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