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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS FOR OPERATOR ON HILBERT SPACE

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Abstract

In this paper, we introduce a class $\mathcal{M}_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$, defined by a generalised Ruscheweyh derivative operator introduced by authors in [1]. Further, we defined this class using the notion of operator on Hilbert space. Some properties related to the aforementioned class are obtained.

1. Introduction

Let $\mathcal{A}$ be denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the unit disk $\mathbb{U} = \{z : |z| < 1\}$.

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Definition 1.1 [1]**.** We define the operator D_λ^n by:

$$D_\lambda^0 f(z) = (1-\lambda)f(z) + \lambda z f'(z) = D_\lambda f(z), \quad \lambda \geq 0,$$

$$D_\lambda^1 f(z) = (1-\lambda)zf'(z) + \lambda z(zf'(z))',$$

$$D_\lambda^n f(z) = D_\lambda\left(\frac{z(z^{n-1}f(z))^n}{n!}\right), \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

If f is given by (1.1), then we can write

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty}[1 + \lambda(k-1)]C(n, k)a_k z^k,$$

where

$$C(n, k) = \binom{k+n-1}{n} = \frac{\prod_{j=1}^{k-1}(j+n)}{(k-1)!}, \quad k \geq 2.$$

Definition 1.2. Let $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq a < b \leq 1$, $b > 0$, $0 < \gamma \leq b/(b-a)$, $n \in \mathbb{N}_0$ and $\lambda \geq 0$ we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma)$ if and only if

$$\left|\frac{\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1}{(b-a)\gamma\left[\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - \alpha\right] - b\left[\frac{z(D_\lambda^n f(z))'}{D_\lambda^n f(z)} - 1\right]}\right| < \beta, \quad z \in \mathbb{U}. \tag{1.2}$$

Further, let $\mathcal{T}$ be denote the subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.3}$$

Let us define

$$\mathcal{M}\mathcal{T}_\lambda^n(a, b, \alpha, \beta, \gamma) = \mathcal{M}_\lambda^n(a, b, \alpha, \beta, \gamma) \cap \mathcal{T}.$$

Perhaps, Silverman [5] was the responsible person to introduce the class $\mathcal{T}$.

Note that the class $\mathcal{M}T_0^0(-1, 1, 0, 1, 1/2)$ reduces to the class of functions $f \in \mathcal{A}$ that satisfies $|zf'(z)/f(z) - 1| < 1$, namely the functions are starlike (see [5]). In [2], it was proven that $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma) \subset \mathcal{M}T_0^0(-1, 1, 0, 1, 1/2)$, namely all function in $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma)$ are starlike.

Let H be a Hilbert space on the complex field. Let A be an operator on H. For a complex analytic function f on the unit disc $\mathbb{U}$, we denote by $f(A)$, the operator on H defined by Riesz-Dunford integral [3]

$$f(A) = \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1} dz,$$

where I is the identity operator on H, C is a positively oriented simple closed rectifiable contour lying in $\mathbb{U}$ and containing the spectrum of A in its interior domain [4].

Let us begin with our definition which shall be used throughout the paper.

Definition 1.3. Let $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le a < b \le 1$, $b > 0$, $0 < \gamma \le b/(b - a)$, $n \in \mathbb{N}_0$, $\lambda \ge 0$ and for all operator A with $\| A \| < 1$ and $A \ne \Theta$ (Θ is the zero operator on H) we say that a function $f \in \mathcal{T}$ is in the class $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$ if it satisfies the condition:

$$\| A(D_\lambda^n f(A))' - D_\lambda^n f(A) \| < \beta \| (b - a)\gamma [A(D_\lambda^n f(A))' - \alpha D_\lambda^n f(A)] - b[A(D_\lambda^n f(A))' - D_\lambda^n f(A)] \|.$$

2. Coefficient Estimates

Now we begin with our first result concerning the necessary and sufficient condition for functions f, to be in the class $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$.

Theorem 2.1. *Let the function* f *be defined by* (1.3). *Then* $f \in \mathcal{MT}_\lambda^n(a, b, \alpha, \beta, \gamma; A)$ *for all proper contraction* A *with* $A \neq \Theta$ *if and only if*

$$\sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma) a_k \leq \beta\gamma(b-a)(1-\alpha), \tag{2.1}$$

where

$$\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)$$
$$= \{k - 1 + \beta[b(k+1) - (b-a)\gamma(k+\alpha)]\}[1 + \lambda(k-1)]C(n, k),$$

for $0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ -1 \leq a < b \leq 1,\ b > 0,\ 0 < \gamma \leq b/(b-a),\ n \in \mathbb{N}_0$, $\lambda \geq 0$.

Proof. Assume that the inequality (2.1) holds. Then we have

$$\| A(D_\lambda^n f(A))' - D_\lambda^n f(A) \|$$
$$< \beta \| (b-a)\gamma[A(D_\lambda^n f(A))' - \alpha D_\lambda^n f(A)] - b[A(D_\lambda^n f(A))' - D_\lambda^n f(A)] \|$$
$$= \| A(D_\lambda^n f(A))' - D_\lambda^n f(A) \|$$
$$- \beta \| [(b-a)\gamma - b] A(D_\lambda^n f(A))' + [b - (b-a)\gamma\alpha] D_\lambda^n f(A) \|$$
$$= \left\| \sum_{k=2}^{\infty} (1-k)[1 + \lambda(k-1)]C(n, k) a_k A^k \right\|$$
$$- \beta \Bigg\| [(b-a)\gamma - b]A - [(b-a)\gamma - b] \sum_{k=2}^{\infty} k[1 + \lambda(k-1)]C(n, k) a_k A^k$$
$$+ [b - (b-a)\gamma\alpha]A - [b - (b-a)\gamma\alpha] \sum_{k=2}^{\infty} [1 + \lambda(k-1)]C(n, k) a_k A^k \Bigg\|$$
$$\leq \sum_{k=2}^{\infty} \{(k-1) + \beta[b - (b-a)\gamma]k + \beta[b - (b-a)\gamma\alpha]\}[1 + \lambda(k-1)]C(n, k) a_k$$
$$- \beta\gamma(b-a)(1-\alpha) \leq 0.$$

Hence f is in the class $\mathcal{MT}_\lambda^n(a, b, \alpha, \beta, \gamma; A)$.

Conversely, suppose that

$$\| A(D_\lambda^n f(A))' - D_\lambda^n f(A) \| < \beta \| (b-a)\gamma[A(D_\lambda^n f(A))' - \alpha D_\lambda^n f(A)] - b[A(D_\lambda^n f(A))' - D_\lambda^n f(A)] \|$$

so that

$$\left\| \sum_{k=2}^{\infty} -(k-1)[1+\lambda(k-1)]C(n, k)a_k A^k \right\|$$

$$< \beta \left\| (b-a)\gamma(1-\alpha)A - [(b-a)\gamma - b]\sum_{k=2}^{\infty} k[1+\lambda(k-1)]C(n, k)a_k A^k - [b-(b-a)\gamma\alpha]\sum_{k=2}^{\infty}[1+\lambda(k-1)]C(n, k)a_k A^k \right\|.$$

Selecting $A = eI$ $(0 < e < 1)$ in above inequality, we have

$$\frac{\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]C(n,k)a_k e^k}{(b-a)(1-\alpha)\gamma e - \sum_{k=2}^{\infty}\{[b-(b-a)\gamma]k + [b-(b-a)\gamma\alpha]\}[1+\lambda(k-1)]C(n,k)a_k e^k} < \beta. \tag{2.2}$$

Upon clearing denominator in (2.2) and letting $e \to 1$ $(0 < e < 1)$, we get

$$\sum_{k=2}^{\infty}(k-1)[1+\lambda(k-1)]C(n, k)a_k$$

$$\leq \beta(b-a)(1-\alpha)\gamma$$

$$-\beta\sum_{k=2}^{\infty}\{[b-(b-a)\gamma]k + [b-(b-a)\gamma\alpha]\}[1+\lambda(k-1)]C(n, k)a_k$$

which implies that

$$\sum_{k=2}^{\infty}\{k-1+\beta[b(k+1)-(b-a)\gamma(k+\alpha)]\}[1+\lambda(k-1)]C(n, k)a_k$$

$$\leq \beta\gamma(b-a)(1-\alpha).$$

Finally, the result is sharp for the function

$$f(z) = z - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)} z^k, \quad (k \geq 2).$$

Thus the theorem is complete. □

Corollary 2.2. *Let the function f defined by* (1.3) *be in the class* $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$. *Then we have*

$$a_k \leq \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)}, \quad (k \geq 2).$$

3. Distortion Theorem

A distortion property for function f to be in the class $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$ is given as follows.

Theorem 3.1. *Let the function f defined by* (1.3) *be in the class* $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$. *Then for* $\| A \| < 1$ *and* $A \neq \Theta$, *we have*

$$\| A \| - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} \| A \|^2 \leq \| f(A) \|$$

$$\leq \| A \| + \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} \| A \|^2.$$

The result is sharp for function f given by

$$f(z) = z - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} z^2.$$

Proof. In view of Theorem 2.1, we have

$$\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma) a_k$$

$$\leq \beta(b-a)(1-\alpha)\gamma,$$

which gives

$$\sum_{k=2}^{\infty} a_k \leq \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)}.$$

Hence we have

$$\| f(A) \| \geq \| A \| - \| A \|^2 \sum_{k=2}^{\infty} a_k$$

$$\geq \| A \| - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} \| A \|^2,$$

and

$$\| f(A) \| \leq \| A \| + \| A \|^2 \sum_{k=2}^{\infty} a_k$$

$$\leq \| A \| + \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(2, a, b, \alpha, \beta, \gamma)} \| A \|^2.$$

4. Closure Theorems

In this section we prove the following closure theorems for the $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$.

First, let the function $f_j(z)$ will be defined for $j = 1, 2, \ldots, m,$ by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} a_k, \quad z \in \mathbb{U}, \quad (a_{k,j} \geq 0). \tag{4.1}$$

Theorem 4.1 *Let the functions* $f_j(z)$ *defined by* (4.1) *be in the class* $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$ *for every* $j = 1, 2, \ldots, m.$ *Then the functions* $h(z)$ *defined by*

$$h(z) = \sum_{j=1}^{m} c_j f_j(z), \quad (c_j \geq 0) \tag{4.2}$$

is also in the class $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A),$ *where* $\sum_{j=1}^{m} c_j = 1.$

Proof. According to the definition of h, we can write

$$h(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j a_{k,j}\right) z^k. \tag{4.3}$$

Further, since functions $f_j(z)$ are in $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$ for every $j = 1, 2, \ldots, m$ we get

$$\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma) a_{k,j} \leq \beta(b-a)(1-\alpha)\gamma$$

for every $j = 1, 2, \ldots, m$. We can see that

$$\Rightarrow \sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma)\left(\sum_{j=1}^{m} c_j a_{k,j}\right)$$

$$\Rightarrow \sum_{j=1}^{m} c_j\left(\sum_{k=2}^{\infty} \Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma) a_{k,j}\right)$$

$$\leq \left(\sum_{j=1}^{\infty} c_j\right)\beta(b-a)(1-\alpha)\gamma = \beta(b-a)(1-\alpha)\gamma.$$

Hence the theorem follows.

Corollary 4.2. *The class* $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$ *is close under convex linear combination.*

Proof. Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$. Then the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \leq \mu \leq 1)$$

is in the class $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$. Also, by taking $m = 2$, $c_1 = \mu$ and $c_2 = (1-\mu)$ in Theorem 4.1, we have the corollary.

As a consequence of Corollary 4.2, there exist the extreme points of the class $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$.

Theorem 4.3. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)} z^k, \quad (k \geq 2;\ n = 1, 2, 3, \ldots).$$

Then f *is in the class* $\mathcal{M}T_\lambda^n(a, b, \alpha, \beta, \gamma; A)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z),$$

where $\mu_k \geq 0$ $(k \geq 1)$ *and* $\sum_{k=1}^{\infty} \mu_k = 1.$

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$$

$$= \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z)$$

$$= \mu_1 z + \sum_{k=2}^{\infty} \mu_k \left\{ z - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)} z^k \right\}$$

$$= \mu_1 z + \sum_{k=2}^{\infty} \mu_k z - \sum_{k=2}^{\infty} \mu_k \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)} z^k$$

$$= z - \sum_{k=2}^{\infty} \mu_k \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)} z^k.$$

Thus

$$= \sum_{k=2}^{\infty} \mu_k \left(\frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)} \right)$$

$$\left(\frac{\Phi_\lambda^n(k, a, b, \alpha, \beta, \gamma)}{\beta(b-a)(1-\alpha)\gamma} \right)$$

$$= \sum_{k=2}^{\infty} \mu_k = \sum_{k=1}^{\infty} \mu_k - \mu_1 = 1 - \mu_1 \leq 1.$$

So by Theorem 2.1, f is in the class $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$.

Conversely, we suppose f is in the class $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$. Since

$$a_k \leq \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma)}, \quad (k \geq 2).$$

We may set

$$\mu_k = \frac{\Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma)}{\beta(b-a)(1-\alpha)\gamma} a_k, \quad (k \geq 2)$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

Then

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} a_k z^k \\ &= z - \sum_{k=2}^{\infty} \mu_k \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma)} z^k \\ &= z - \sum_{k=2}^{\infty} \mu_k [z - f_k(z)] = z - \sum_{k=2}^{\infty} \mu_k z + \sum_{k=2}^{\infty} \mu_k f_k(z) \\ &= \left(1 - \sum_{k=2}^{\infty} \mu_k\right) z + \sum_{k=2}^{\infty} \mu_k f_k(z) = \mu_1 z + \sum_{k=2}^{\infty} \mu_k f_k(z) \\ &= \sum_{k=1}^{\infty} \mu_k f_k(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z). \end{aligned}$$

Thus the proof is complete.

Corollary 4.4. *The extreme points of* $\mathcal{M}T_{\lambda}^{n}(a, b, \alpha, \beta, \gamma; A)$ *are the functions* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{\beta(b-a)(1-\alpha)\gamma}{\Phi_{\lambda}^{n}(k, a, b, \alpha, \beta, \gamma)} z^k, \quad (k \geq 2).$$

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