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CONTINUOUS DEFORMATIONS OF C^*-ALGEBRA EXTENSIONS

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Abstract

We study continuous deformations of C^*-algebra extensions.

Introduction

A continuous deformation from a C^*-algebra $\mathfrak{A}$ to another $\mathfrak{B}$ is a continuous field C^*-algebra on the closed interval $[0, 1]$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_0 = \mathfrak{B}$ and $\mathfrak{A}_t = \mathfrak{A}$ for $0 < t \leq 1$, and we denote it by $E_{\mathfrak{A}} = \Gamma([0, 1], \{\mathfrak{A}_t\}_{t \in [0,1]})$ (see Dixmier [2] and Blackadar [1]). See also [3].

In this paper we consider continuous deformations for C^*-algebra extensions in some cases. Our first intention for this is to study relative continuous deformations for C^*-algebras with closed (two-sided) ideals. It seems to have not been considered in the literature. More cases could be considered.

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Continuous Deformations of C^*-algebra Extensions

We say that a *continuous deformation* from a C^*-algebra $\mathfrak{A}$ to another $\mathfrak{B}$ is internal if $\mathfrak{B}$ is a C^*-subalgebra of $\mathfrak{A}$. Denote by $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ a continuous field C^*-algebra on the base space X a locally compact Hausdorff space with fibers $\mathfrak{A}_t$ (vanishing at infinity). Replace $\Gamma_0(\cdot)$ with $\Gamma(\cdot)$ if X is compact.

Theorem 1.1. *Let $\mathfrak{A}$ be a C^*-algebra with continuous trace and $\mathfrak{I}$ be its closed ideal. If $\mathfrak{I}$ has an internal continuous deformation $E_{\mathfrak{I}}$ to its C^*-subalgebra $\mathfrak{I}_0$, then it induces an internal continuous deformation $E_{\mathfrak{A}}$ from $\mathfrak{A}$ to its C^*-subalgebra $\mathfrak{A}_0$ containing $\mathfrak{I}_0$ as a closed ideal, preserving $E_{\mathfrak{I}}$.*

Proof. Since $\mathfrak{A}$ has continuous trace, it can be viewed as a continuous field C^*-algebra on its Hausdorff spectrum $\mathfrak{A}^\wedge$ consisting of equivalence classes of irreducible representations of $\mathfrak{A}$, i.e., $\mathfrak{A} \cong \Gamma_0(\mathfrak{A}^\wedge, \{\pi(\mathfrak{A})\}_{\pi\in\mathfrak{A}^\wedge})$, where an irreducible representation π of $\mathfrak{A}$ is identified with its class in $\mathfrak{A}^\wedge$ (see [2]). Also, we have $\mathfrak{I} \cong \Gamma_0(\mathfrak{I}^\wedge, \{\pi(\mathfrak{I})\}_{\pi\in\mathfrak{I}^\wedge})$. Thus, we have the following exact sequence:

$$0 \to \Gamma_0(\mathfrak{I}^\wedge, \{\pi(\mathfrak{I})\}_{\pi\in\mathfrak{I}^\wedge}) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \cong \Gamma_0(\mathfrak{A}^\wedge\setminus\mathfrak{I}^\wedge, \{\pi(\mathfrak{I})\}_{\pi\in\mathfrak{A}^\wedge\setminus\mathfrak{I}^\wedge}) \to 0,$$

where $\mathfrak{A}^\wedge\setminus\mathfrak{I}^\wedge$ is the complement of $\mathfrak{I}^\wedge$ in $\mathfrak{A}^\wedge$. Our claim follows from the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{A}/\mathfrak{I} & \longrightarrow & 0, \\
 & & E_{\mathfrak{I}} \downarrow & & \downarrow E_{\mathfrak{A}} & & \| & & \\
0 & \longrightarrow & \mathfrak{I}_0 & \longrightarrow & \mathfrak{A}_0 & \longrightarrow & \mathfrak{A}/\mathfrak{I} & \longrightarrow & 0,
\end{array}$$

where the vertical arrows mean continuous deformations. □

Example 1.2. Let H_3 be the real 3-dimensional Heisenberg Lie group and $C^*(H_3)$ be its group C^*-algebra. The group C^*-algebra $C^*(H_3)$ can be viewed as a continuous field C^*-algebra $\mathfrak{A} = \Gamma_0(\mathbb{R}, \{\mathfrak{A}_t\}_{t\in\mathbb{R}})$ with fibers $\mathfrak{A}_0 = C_0(\mathbb{R}^2)$ and $\mathfrak{A}_t = \mathbb{K}$ for $t \in \mathbb{R}\setminus\{0\}$ (see [2]). Thus, we have the following exact sequence and its continuous deformation:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{J} = C_0(\mathbb{R}\setminus\{0\}) \otimes \mathbb{K} & \longrightarrow & \mathfrak{A} & \longrightarrow & C_0(\mathbb{R}^2) & \longrightarrow & 0, \\ & & E_{\mathfrak{J}} \downarrow & & \downarrow E_{\mathfrak{A}} & & \| & & \\ 0 & \longrightarrow & C_0(\mathbb{R}\setminus\{0\}) \otimes M_n(\mathbb{C}) & \longrightarrow & \mathfrak{A}_0 & \longrightarrow & C_0(\mathbb{R}^2) & \longrightarrow & 0, \end{array}$$

where $E_{\mathfrak{J}}$ is induced from the canonical internal continuous deformation from $\mathbb{K}$ to $M_n(\mathbb{C})$ (any $n \geq 1$).

However, it is not always true that $E_{\mathfrak{J}}$ induces $E_{\mathfrak{A}}$.

Example 1.3. Let A_2 be the real 2-dimensional $ax + b$ group and $C^*(A_2)$ be its group C^*-algebra. The group C^*-algebra $C^*(A_2)$ is decomposed into

$$0 \to \mathfrak{J} = \mathbb{K} \oplus \mathbb{K} \to \mathfrak{A} = C^*(A_2) \to C_0(\mathbb{R}) \to 0.$$

As for $E_{\mathfrak{J}}$, we consider the continuous deformation from $\mathbb{K}$ to its diagonal $C_0(\mathbb{N})$. If $E_{\mathfrak{J}}$ induces $E_{\mathfrak{A}}$ from $\mathfrak{A}$ to a C^*-algebra extension $\mathfrak{A}_0$ of $C_0(\mathbb{N}) \oplus C_0(\mathbb{N})$ by $C_0(\mathbb{R})$, then $\mathfrak{A}_0$ must be commutative. But this is impossible since $\mathfrak{A}$ is noncommutative so that there exist $a, b \in \mathfrak{A}$ such that $[a, b] = ab - ba \neq 0$, and so the norm $\|[a, b]\| \neq 0$ but $\|[a_0, b_0]\| = 0$ for $a_0, b_0 \in \mathfrak{A}_0$, where a, b go to a_0, b_0 under $E_{\mathfrak{A}}$ respectively. Note that $C^*(A_2)$ is of type I but not CCR so that it is not of continuous trace, and it never be written as a continuous field C^*-algebra.

Even if $\mathfrak{A}$ is not of continuous trace, $E_{\mathfrak{J}}$ may induce $E_{\mathfrak{A}}$.

Example 1.4. Let $\mathfrak{A} = \mathfrak{J} \oplus \mathfrak{B}$ be the direct sum of C^*-algebras. Assume that $\mathfrak{J}$ has continuous trace and $\mathfrak{B}$ is not of continuous trace, and so is not $\mathfrak{A}$. Then some $E_{\mathfrak{J}}$ induces $E_{\mathfrak{A}}$.

In a general situation,

Theorem 1.5. *Let* $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ *be an extension of*

C^*-algebras. Assume that $\mathfrak{I}$ is stable, i.e., $\mathfrak{I} \cong \mathfrak{I} \otimes \mathbb{K}$. Then an internal continuous deformation of $\mathfrak{I}$ with respect to that of $\mathbb{K}$ induces a continuous deformation from $\mathfrak{A}$.

Proof. Note that

$$\mathfrak{I} \cong \mathfrak{I} \otimes \mathbb{K} \cong \mathfrak{I} \otimes \mathbb{K} \otimes M_n(\mathbb{C}) \cong \mathfrak{I} \otimes \mathbb{K} \otimes \mathbb{K}.$$

An extension $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{B} \to 0$ of C^*-algebras can be identified with the Busby map τ from $\mathfrak{B}$ to the corona $M(\mathfrak{I})/\mathfrak{I}$, the quotient of the multiplier algebra $M(\mathfrak{I})$ of $\mathfrak{I}$ (see [4]). As for $E_{\mathfrak{I}}$, we consider a continuous deformation from $\mathfrak{I} \otimes \mathbb{K} \otimes M_n(\mathbb{C})$ to $\mathfrak{I} \otimes \mathbb{K} \otimes \mathbb{C}^n \cong \oplus^n \mathfrak{I}$. Then $M(\mathfrak{I} \otimes \mathbb{K} \otimes \mathbb{C}^n) \cong \oplus^n M(\mathfrak{I})$. Thus, we obtain a continuous deformation from τ to the sum $\tau \oplus \cdots \oplus \tau$ (n times) from $\mathfrak{B}$ to $\oplus^n (M(\mathfrak{I})/\mathfrak{I})$.

As for $E_{\mathfrak{I}}$, if we take another one from $\mathfrak{I} \otimes \mathbb{K} \otimes \mathbb{K}$ to $\mathfrak{I} \otimes \mathbb{K} \otimes C_0(\mathbb{N})$, then it induces a continuous deformation from τ to the sequence $(\tau)_{j=1}^{\infty}$. Note that $M(\mathfrak{I} \otimes \mathbb{K} \otimes C_0(\mathbb{N}))$ contains $M(\mathfrak{I}) \otimes M(C_0(\mathbb{N}))$, and $M(C_0(\mathbb{N})) \cong C(\beta\mathbb{N})$ containing the constant sequences, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$ (for instance, see [4]).

More generally, note that a C^*-subalgebra of $\mathbb{K}$ can be viewed as a finite or infinite direct sum of $\mathbb{K}$ or $M_n(\mathbb{C})$. Thus, an internal continuous deformation of $\mathbb{K}$ to such a direct sum also induces a continuous deformation from $\mathfrak{A}$. □

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