Far East J. Math. Sci. (FJMS) **26**(3) (2007), 697-704
This paper is available online at http://www.pphmj.com

A NOTE ON LEFT(RIGHT) SEMIREGULAR *po*-SEMIGROUPS

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Abstract

In this note, we give a characterizations of left(right) semiregular *po*-semigroups.

1. Introduction

The concept of the left(right) semiregular *po*-semigroups has been introduced in [3] and extends concept of regular semigroups in case of ordered semigroups. In this note, we give the characterizations of left(right) semiregular *po*-semigroups.

A *po*-semigroup (: ordered semigroup) is an ordered set $(S, \leq)$ at the same time a semigroup such that:

$$a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx, \quad \forall a, b, x \in S.$$

Definition 1.1 [3]**.** An element a of a *po*-semigroup S is a *left* (resp.

2000 Mathematics Subject Classification: 20M10, 06F05.

Keywords and phrases: *po*-semigroup, left(right) semiregular, left weakly prime, left weakly semiprime, left ideal, right ideal, ideal, bi-ideal.

Received April 23, 2007

right) *semiregular element* if $a \leq xaya$ (resp. $a \leq ax'ay'$) for some $x, y, x', y' \in S$.

S is called *left* (resp. *right*) *semiregular* if all elements of S are left (resp. right) semiregular.

Equivalent definition. $a \in (aSaS]$ (resp. $a \in (SaSa]$) for every $a \in S$.

Definition 1.2. Let S be a *po*-semigroup and $\varnothing \neq A \subseteq S$. Then A is called a *left* (resp. *right*) *ideal* of S if

(1) $SA \subseteq A$.

(2) If $a \in A$, $b \leq a$ with $b \in S$, then $b \in A$.

A is called an *ideal* of S if A is both a left and right ideal of S.

Definition 1.3 [4]**.** Let S be a *po*-semigroup and $\varnothing \neq B \subseteq S$. Then B is called a *bi-ideal* of S if

(1) $BSB \subseteq B$.

(2) If $a \in B$, $b \leq a$ with $b \in S$, then $b \in B$.

Every ideal and left(right) ideal is a bi-ideal.

Definition 1.4 [2]**.** Let S be a *po*-semigroup and $\varnothing \neq Q \subseteq S$. Then Q is called a *bi-ideal* of S if

(1) $QS \cap SQ \subseteq Q$.

(2) If $a \in Q$, $b \leq a$ with $b \in S$, then $b \in Q$.

For $H \subseteq S$, we denote $(H] := \{t \in S \mid t \leq h$ for some $h \in H\}$. For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$ $(a \in S)$. We denote by $L(a)$, $R(a)$, $I(a)$, $B(a)$ and $Q(a)$ the left ideal, right ideal, ideal, bi-ideal and quasi-ideal of S generated by a $(a \in S)$. For a *po*-semigroup S, one can easily prove that:

$$L(a) = (a \cup aS], \quad R(a) = (a \cup Sa], \quad I(a) = (a \cup aS \cup Sa \cup SaS],$$

$$B(a) = (a \cup a^2 \cup aSa]\ [4], \qquad Q(a) = (a \cup (aS \cap Sa)]\ [2].$$

Definition 1.5 [3]. Let L be a left ideal of a *po*-semigroup S. Then L is called *left* (resp. *right*) *weakly prime* if for any left (resp. right) ideals A, B of S such that $AB \subseteq L$, we have $A \subseteq L$ or $B \subseteq L$.

Definition 1.6. Let L be a left ideal of a *po*-semigroup S. Then L is called *left* (resp. *right*) *weakly semiprime* if for any left (resp. right) ideals A of S such that $A^2 \subseteq L$, we have $A \subseteq L$.

Lemma 1.1 [1]. *Let S be a po-semigroup. Then*

(1) $A \subseteq (A]$, $\forall A \subseteq S$.

(2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.

(3) $(A](B] \subseteq (AB]$, $\forall A, B \subseteq S$.

(4) $((A]] = (A]$, $\forall A \subseteq S$.

(5) *For every right ideal, left ideal, ideal and bi-ideal T of S, we have* $(T] = T$.

(6) *If A, B are ideals of S, then* $(AB]$, $A \cap B$, $A \cup B$ *are ideals of S.*

(7) $(Sa]$ (*resp.* $(aS]$) *is a left* (*resp. right*) *ideal of S,* $(aSa]$ *is a bi-ideal of S for every* $a \in S$.

2. Main Results

Theorem 2.1. *Let S be a po-semigroup. Then the following conditions are equivalent*:

(1) *S is left semiregular.*

(2) *Every ideal of S is a left semiregular subsemigroup of S.*

(3) $(L^2] = L$ *for every left ideal L of S.*

(4) $I \cap L = (IL]$ *for every ideal I and every left ideal L of S.*

(5) $I \cap B \subseteq (IB]$ *for every ideal I and every bi-ideal B of S.*

(6) $L_1 \cap L_2 \subseteq (L_1 L_2]$ *for every left ideals* L_1, L_2.

(7) $L(a) \cap L(b) \subseteq (L(a)L(b)]$, $\forall a, b \in S$.

(8) $L(a) = ((L(a))^2]$, $\forall a \in S$.

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of S. Then $I^2 \subseteq SI \subseteq I$. Thus I is a subsemigroup of S.

Let $a \in I$. Then there exist $x, y \in S$ such that

$$a \leq xaya \leq x(xaya)y(xaya) = (xxay)a(yxay)a.$$

Since

$$xxay, \ yxay \in SSIS \subseteq I,$$

we have $a \in (IaIa]$. Thus I is a left semiregular subsemigroup.

(2) $\Rightarrow$ (3) Let L be a left ideal of S. Let $x \in (L^2]$. Then $x \leq ab$ for some $a, b \in L$. Since L is a left ideal of S, we have $ab \in L$ and $x \in L$. Let $a \in L$. Since S is an ideal of S itself. By hypothesis, a is semiregular element of S. We have $a \leq taha$ for some $t, h \in S$. Since $taha \in SLSL \subseteq L^2 \subseteq (L^2]$, we have $a \in (L^2]$.

(3) $\Rightarrow$ (4) For every ideal I and every left ideal of S. Then $(IL] \subseteq (SL] \subseteq (L] = L$, $(IL] \subseteq (IS] \subseteq (I] = I$, i.e., $(IL] \subseteq I \cap L$. Since $I \cap L$ is a left ideal of S. By hypothesis, we have

$$I \cap L = ((I \cap L)^2] = ((I \cap L)(I \cap L)] \subseteq (IL].$$

(4) $\Rightarrow$ (5) Let $a \in B \cap I$. By hypothesis, we have

$$a \in I \cap L(a) = (IL(a)] = (I(a \cup Sa]] \subseteq (Ia \cup ISa] \subseteq (IB \cup ISB] \subseteq (IB].$$

(5) $\Rightarrow$ (6) For any $a \in L_1 \cap L_2$, by hypothesis, we have

$$\begin{aligned} a \in I(a) \cap B(a) = (I(a)B(a)] &= (a \cup aS \cup Sa \cup SaS](a \cup a^2 \cup aSa] \\ &\subseteq (L_1 \cup L_1S \cup SL_1 \cup SL_1S](L_2 \cup L_2L_2 \cup L_2SL_2] \\ &\subseteq (L_1 \cup L_1S](L_2] \subseteq (L_1L_2 \cup L_1SL_2] \subseteq (L_1L_2]. \end{aligned}$$

$(6) \Rightarrow (7)$ It is obvious.

$(7) \Rightarrow (8)$ It is obvious.

$(8) \Rightarrow (1)$ Let $a \in S$. Since $L(a) = ((L(a))^2]$, we have

$$(L(a))^2 = (((L(a))^2]((L(a))^2]] \subseteq ((L(a))^4]$$

$$\Rightarrow L(a) = ((L(a))^2] \subseteq (((L(a))^4]] = (L(a))^4]$$

$$\subseteq (SL(a)] \subseteq (L(a)] = L(a)$$

$$\Rightarrow L(a) = ((L(a))^4].$$

On the other hand,

$$((L(a))^4] = (a \cup Sa]^4$$

$$\subseteq ((a \cup Sa)^2](a \cup Sa]^2 \subseteq (a^2 \cup aSa \cup Sa^2 \cup SaSa](a \cup Sa]^2$$

$$\subseteq (Sa](a \cup Sa]^2 \subseteq (Sa](Sa] \subseteq (SaSa].$$

Therefore,

$$a \in L(a) = ((L(a))^4] \subseteq ((SaSa]] = (SaSa].$$

Theorem 2.2. *A po-semigroup S is left semiregular if and only if for every ideal I, every bi-ideal B, every left ideal L and every right ideal R of S,*

$$L \cap I \cap B \subseteq (LIB] \ (resp.\ R \cap I \cap B \subseteq (IRB]).$$

Proof. Let I be an ideal, B be a bi-ideal and L be a left ideal of S. Let $a \in L \cap I \cap B$. Since S is left semiregular, there exists $x, y \in S$ such that $a \leq xaya$. Then we have

$$a \leq xaya \leq x(xaya)ya \leq x(xaya)y(xaya)$$

$$= (xxa)yayx(aya) \in (SL)(SISS)(BSB) \subseteq LIB.$$

Thus $a \in (LIB]$. If R is a right ideal of S and $a \in B \cap I \cap R$, then

$$a \leq xaya \leq x(xaya)ya \leq x(xaya)y(xaya)$$

$$= (xxay)(ayx)(aya) \in (SIS)(RS)(BSB) \subseteq IRB.$$

Conversely, let $a \in S$. We consider the ideal $I(a)$, bi-ideal $B(a)$ and left ideal $L(a)$ of S generated by a. By hypothesis, we have

$$a \in B(a) \cap I(a) \cap L(a)$$

$$= (L(a)I(a)B(a)] \subseteq (SI(a)B(a)] \subseteq (I(a)B(a)]$$

$$= (a \cup aS \cup Sa \cup SaS](a \cup a^2 \cup aSa]$$

$$\subseteq (a^2 \cup a^3 \cup a^2Sa \cup aSa \cup aSa^2 \cup aSaSa \cup SaSa \cup SaSa^2 \cup SaSaSa]$$

$$\subseteq (a^2 \cup a^3 \cup aSa \cup SaSa].$$

Then $a \leq t$ for some $t \in a^2 \cup a^3 \cup aSa \cup SaSa$. If $t = a^2$, then $a \leq a^2 \leq a^4 \in SaSa$, i.e., $a \in (SaSa]$. If $t = a^3$, then $a \leq a^3 \leq a^5 \in SaSa$. If $t = axa$ for some $x \in S$, then $a \leq axa \leq axaxa \in SaSa$. If $t \in SaSa$, then $a \in SaSa$. Thus S is left semiregular.

Suppose now that $R \cap I \cap B \subseteq ((IRB])$. Then

$$a \in B(a) \cap I(a) \cap R(a) = (I(a)R(a)B(a)] \subseteq (I(a)SB(a)] \subseteq (I(a)B(a)].$$

Therefore S is left semiregular (cf. the previous case).

Using the method of the proof of Theorem 2.2, we can also get the following theorem.

Theorem 2.3. *A po-semigroup S is left semiregular if and only if for every ideal I, every quasi-ideal Q, every left ideal L and every right ideal R of S,*

$$L \cap I \cap Q \subseteq (IQL] \quad (\textit{resp. } R \cap I \cap Q \subseteq (IRQ]).$$

Following we characterize the semiregular semigroups by means of left weakly prime and left weakly semiprime.

Theorem 2.4. *Let S be a po-semigroup. Then the following conditions are equivalent*:

(1) *S is left semiregular.*

(2) *Every left ideal of S is a left weakly semiprime.*

(3) *Every left ideal of S is the intersection of all left weakly prime left ideals of S containing it.*

Proof. (1) $\Rightarrow$ (2) Let L, A be left ideals of S such that $A^2 \subseteq L$ and let $a \in A$. Since a is left semiregular, we have $a \in (SaSa] \subseteq (SASA] \subseteq (A^2] \subseteq (L] = L$.

(2) $\Rightarrow$ (3) Let L be a left ideal of S. Let

$$\mathcal{L} = \{A \mid A \text{ left weakly prime left ideals of } S \text{ and } A \supseteq L\}.$$

Let $P = \bigcap_{A \in \mathcal{L}} A$. Since $A \supseteq L$ for any $A \in \mathcal{L}$, we have $L \subseteq P$. Let $a \in P$ and $a \notin L$. Let $\mathcal{W}$ be the class of left ideals of S containing L but not containing a. Applying Zorn's lemma to $\mathcal{W}$, there exists the maximal left ideal M not containing a but $M \supseteq L$. Then M is a left weakly prime left ideal of S. In fact, let $L_1 L_2 \subseteq M$, $L_1 \not\subseteq M$ and $L_2 \not\subseteq M$. Since $L_1 \cup M \supset M$ and $L_2 \cup M \supset M$ are left ideals of S, by the maximality of M, we have $L_1 \cup M = S$, $L_2 \cup M = S$. Hence $a \in L_1 \cap L_2$. By hypothesis, M is a left weakly semiprime, we have

$$L(a)^2 = L(a)L(a) \subseteq L_1 L_2 \subseteq M \Rightarrow L(a) \subseteq M.$$

Hence $a \in M$, which is a contradiction. Therefore M is a left weakly prime left ideal and $M \in \mathcal{L}$. Thus we have $a \in P \subseteq M$, it is impossible. Therefore, $L = P$.

(3) $\Rightarrow$ (1) Let L be a left ideal of S. By (3), we have $(L^2] = \bigcap_{A \supseteq (L^2]} A$ and A is a left ideal of S. Since $L^2 \subseteq (L^2] \subseteq A$, by (3), we have $L \subseteq A$. Hence $L \subseteq \bigcap_{A \supseteq (L^2]} A = (L^2]$. This implies that $L = (L^2]$. By Theorem 2.1, S is left semiregular.

Remark. Using the method of the proofs of Theorems 2.1 and 2.4, we can get the dualities for right semiregular, right weakly prime right ideal.

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