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# THE FUZZY PRIME RADICAL OF A $k$-SEMIRING

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**Abstract**

We define and study the fuzzy prime radical of a $k$-semiring. Also it is shown that the prime radical of the quotient semiring $R/\mathrm{FPR}(R)$ of a $k$-semiring by the fuzzy prime radical $\mathrm{FPR}(R)$ is zero. Algebraic properties of the fuzzy ideals $\mathrm{FPR}(R)$ and $\mathrm{FPR}(S)$ under a homomorphism from $R$ onto $S$ are also discussed.



## 1. Introduction

Chun et al. [2] constructed an extension of a $k$-semiring and studied a $k$-ideal of a $k$-semiring. Chun et al. [3] constructed the quotient semiring of a $k$-semiring by a $k$-ideal. Liu [15] introduced and studied the notion of fuzzy ideal of a ring. Following Liu, Mukherjee and Sen [18] defined and examined fuzzy prime ideals of a ring. Kumbhojkar and Bapat [12, 13] defined studied the ring $R/J$ of the cosets of the fuzzy ideal $J$.

Kumar [7-11] extended the concept of fuzzy ideal to fuzzy semiprimary (semiprime, primary, prime, maximal) ideals in a ring. Also Malik and Mordeson [16] gave the necessary and sufficient conditions for a fuzzy subring or a fuzzy ideal $A$ of a commutative ring $R$ to be extended to one $A^e$ of a commutative ring $S$ containing $R$ as a subring.

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In particular, Kuraoki and Kuroki [14] defined fuzzy quotient rings and gave homomorphism theorems and isomorphism theorem as to fuzzy ideals.

Kim and Park [6] defined and studied the notion of the $k$-fuzzy ideal in a semiring, and they also introduced and studied the quotient semiring $R/A$ of a $k$-semiring $R$ by a $k$-fuzzy ideal $A$.

Kim [4, 5] defined and investigated a fuzzy maximal ideal of a $k$-semiring and also characterized the quotient $k$-semiring $R/A$ of a $k$-semiring $R$ by a fuzzy maximal ideal $A$. Also he defined and studied the fuzzy Jacobson radical of a $k$-semirings, and obtained some algebraic properties of the fuzzy Jacobson radical of $k$-semirings.

Furthermore, Kumar [9] defined and studied the fuzzy Jacobson radical $\mathrm{FJR}(R)$ and the fuzzy prime radical $\mathrm{FPR}(R)$ of a ring $R$.

The purpose of this paper is to define and study the fuzzy prime radical $\mathrm{FPR}(R)$ of a $k$-semiring $R$. In particular, we show that the prime radical of the quotient semiring $R/\mathrm{FPR}(R)$ of a $k$-semiring $R$ by the fuzzy prime radical $\mathrm{FPR}(R)$ is zero and investigate the algebraic properties of the fuzzy prime radical $\mathrm{FPR}(R)$ of $k$-semiring $R$.



## 2. Preliminaries

In this section, we review some definitions and some results which will be used in the later section.

**Definition 2.1** [2]. A set $R$ together with associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) is called a *semiring* provided:

(1) addition is a commutative operation,

(2) there exists $0 \in R$ such that $x + 0 = x$ and $x0 = 0x = 0$ for each $x \in R$, and

(3) multiplication distributes over addition both from the left and the right.

**Definition 2.2** [2]**.** A semiring $R$ is called a *$k$-semiring* if for any $a, b \in R$ there exists a unique element $c$ in $R$ such that either $b = a + c$ or $a = b + c$ but not both.

**Definition 2.3** [3]**.** A non-empty subset $I$ of a semiring $R$ is called a *subsemiring* if $I$ is itself a semiring with respect to the binary operations defined in $R$. A subsemiring $I$ is called an *ideal* of $R$ if $r \in R$, $a \in I$ imply $ar$ and $ra \in I$.

**Definition 2.4** [3]**.** An ideal $I$ of a semiring $R$ is called a *$k$-ideal* if $r + a \in I$ implies $r \in I$ for each $r \in R$ and each $a \in I$.

Let $R$ be a $k$-semiring. Let $R'$ be a set of the same cardinality with $R - \{0\}$ such that $R \cap R' = \varnothing$ and let denote the image of $a \in R - \{0\}$ under a given bijection by $a'$. Let $\oplus$ and $\odot$ denote addition and multiplication respectively on a set $\overline{R} = R \cup R'$ as follows:

$$a \oplus b = \begin{cases} a + b & \text{if } \ a, b \in R \\ (x + y)' & \text{if } \ a = x', b = y' \in R' \\ c & \text{if } \ a \in R, b = y' \in R', a = y + c \\ c' & \text{if } \ a \in R, b = y' \in R', a + c = y, \end{cases}$$

where $c$ is the unique element in $R$ such that either $a = y + c$ or $a + c = y$ but not both, and

$$a \odot b = \begin{cases} ab & \text{if } \ a, b \in R \\ xy & \text{if } \ a = x', b = y' \in R' \\ (ay)' & \text{if } \ a \in R, b = y' \in R' \\ (xb)' & \text{if } \ a = x' \in R', b \in R. \end{cases}$$

It can be shown that these operations are well defined and thus if $R$ is a $k$-semiring, then $(\overline{R}, \oplus, \odot)$ is a ring, called the *extension ring* of $R$.

**Remark.** Let $\ominus a$ denote the additive inverse of any element $a \in R$ and write $a \oplus (\ominus b)$ simply as $a \ominus b$. Then it is clear that $a' = \ominus a$ and $a = \ominus a'$ for all $a \in R$.

Note that if $R$ is a $k$-semiring with identity, then $\overline{R}$ is a ring with identity.

**Theorem 2.5** [2]. *Let $R$ be a $k$-semiring, $I$ be an ideal, and $I' = \{a' \in R' \mid a \in I\}$. Then $I$ is a $k$-ideal of $R$ if and only if $\bar{I} = I \cup I'$ is an ideal of the extension ring $\overline{R}$, called the extension ideal of $I$.*

Note that if $R$ is a $k$-semiring and $\overline{R}$ is the extension ring of $R$, then each ideal of $\overline{R}$ is the extension ideal of a $k$-ideal of $R$ and each $k$-ideal of $R$ is the intersection of its extension ideal and $R$ (see [2]).

Let $R$ be a $k$-semiring and $\overline{R}$ be its extension ring. Let $I$ be a $k$-ideal of $R$ and $\bar{I}$ be its extension ideal of $\overline{R}$. Define a relation $a \equiv b$ by $a \oplus b' \in \bar{I}$, where $a, b \in R$. Then this relation is an equivalence relation on $R$. Let $a \oplus I$ be the equivalence class containing $a \in R$ determined by $\equiv$. Let $R/I = \{a \oplus I \mid a \in R\}$ be the set of all equivalence classes determined by $\equiv$. Then $R/I = \{a \oplus I \mid a \in R\}$ is a $k$-semiring under the two operations $(a \oplus I) \oplus (b \oplus I) = (a + b) \oplus I$ and $(a \oplus I) \odot (b \oplus I) = (ab) \oplus I$ (see [3]).



**Definition 2.6** [3]. A mapping $f$ from a $k$-semiring $R$ into a $k$-semiring $S$ is called a *homomorphism* if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

**Theorem 2.7** [3]. *Let $f : R \to S$ be a $k$-semiring homomorphism. Let $\overline{R}$ and $\overline{S}$ be extension rings of $R$ and $S$, respectively. Define a map $\bar{f} : \overline{R} \to \overline{S}$ by*

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in R \\ f(x')' & \text{if } x \in R'. \end{cases}$$

*Then $\bar{f}$ is a ring homomorphism, called extension ring homomorphism of $f$.*

**Theorem 2.8** [2]. *Let $\overline{R}$ be the extension ring of a commutative $k$-semiring with identity, $I$ be a $k$-ideal of $R$ and $\bar{I}$ be the extension ideal of $I$ in $\overline{R}$. Then $I$ is prime $k$-ideal of $R$ iff $\bar{I}$ is prime ideal of $\overline{R}$.*

**Definition 2.9** [9]**.** A $k$-ideal $I$ of a $k$-semiring $R$ is a *prime $k$-ideal* if for all $a, b \in R$, $ab \in I$ implies that either $a \in I$ or $b \in I$.

**Definition 2.10.** Let $R$ be a $k$-semiring and $\overline{R}$ be the extension ring of $R$. The intersection of all prime $k$-ideals of $R$ is called the *prime radical* of $R$, denoted by $PR(R)$.

**Theorem 2.11.** *Let $R$ be a $k$-semiring and $\overline{R}$ be the extension ring of $R$. Then $PR(R) = PR(\overline{R}) \cap R$ is an ideal of all nilpotent elements of $R$.*

**Proof.**

$$\begin{aligned} PR(R) &= \bigcap\{P_i \mid P_i \text{ is a prime } k\text{-ideal of } R\} \\ &= \bigcap\{\overline{P_i} \cap R \mid \overline{P_i} \text{ is a prime ideal of } \overline{R}\} \\ &= \bigcap\{\overline{P_i} \mid \overline{P_i} \text{ is a prime ideal of } \overline{R}\} \cap R \\ &= PR(\overline{R}) \cap R. \end{aligned}$$

Since $PR(\overline{R})$ is the ideal of all nilpotent elements of $\overline{R}$ by ring theory. Thus $PR(R)$ is the ideal of all nilpotent elements of $R$. This completes the proof.



## 3. Fuzzy Ideal of a $k$-semiring

In this section, we have some properties of the fuzzy ideals of commutative $k$-semirings with identity. Throughout this paper unless otherwise all semirings are commutative $k$-semirings with identity.

**Definition 3.1** [6]**.** A *fuzzy ideal* of a semiring $R$ is a function $A : R \to [0, 1]$ satisfying the following conditions:

(1) $A(x + y) \geq \min\{A(x), A(y)\}$

(2) $A(xy) \geq \max\{A(x), A(y)\}$ for all $x, y \in R$.

**Theorem 3.2** [6]**.** *Let $A$ be a fuzzy ideal of a semiring $R$. Then $A(x) \leq A(0)$ for all $x \in R$.*

**Definition 3.3** [6]. Let $A$ be a fuzzy subset of a semiring $R$. Then the set $A_t = \{x \in R \mid A(x) \geq t\}$ $(t \in [0, 1])$ is called the *level subset* of $R$ with respect to $A$.

**Theorem 3.4** [6]. *Let $A$ be a fuzzy ideal of a semiring $R$. Then the level subset $A_t$ $(t \leq A(0))$ is the ideal of $R$.*

In general, it is not true that if $A$ is a fuzzy ideal of a semiring $R$, then $A_t$ $(t \leq A(0))$ is $k$-ideal of $R$, for we have the following example.

**Example 3.5** [6]. Let $R = Z^*$, be the set of nonnegative integers and let $I = (2, 3)$ be an ideal of $R$ generated by 2 and 3. Define a fuzzy subset $A$ of $R$ by

$$A(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$$

Then $A$ is a fuzzy ideal but $A_R = I$ is not $k$-ideal of $R$.

**Definition 3.6** [16]. Let $f : R \to S$ be a homomorphism of semirings and $B$ be a fuzzy subset of $S$. We define a *fuzzy subset* $f^{-1}B$ of $R$ by $f^{-1}B(x) = B(f(x))$ for all $x \in R$.



**Definition 3.7** [20]. Let $f : R \to S$ be a homomorphism of semirings and $A$ be a fuzzy subset of $R$. We define a *fuzzy subset* $f(A)$ of $S$ by

$$f(A)(y) = \begin{cases} \operatorname{Sup}\{A(t) \mid t \in R,\ f(t) = y\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing. \end{cases}$$

**Definition 3.8** [15]. A *fuzzy ideal* of a ring $R$ is a function $A : R \to [0, 1]$ satisfying the following axioms

(1) $A(x + y) \geq \min\{A(x), A(y)\}$

(2) $A(xy) \geq \max\{A(x), A(y)\}$

(3) $A(-x) = A(x)$.

Let $R$ be a commutative $k$-semiring and $\overline{R}$ be its extension ring. If $A$ is a fuzzy ideal of $R$ such that all its level subsets are $k$-ideals of $R$, then

$R = \bigcup_{t \in \mathrm{Im} A} A_t$, $\overline{R} = \bigcup_{t \in \mathrm{Im} A} \overline{A}_t$ and $s > t$ if and only if $A_s \subset A_t$ if and only if $\overline{A}_s \subset \overline{A}_t$. Thus we have the following theorem.

**Theorem 3.9** [6]**.** *Let $R$ be a commutative $k$-semiring and $\overline{R}$ be its extension ring. Let $A$ be a fuzzy ideal of $R$ such that all its level subsets are $k$-ideals of $R$. Define the fuzzy subset $\overline{A}$ of $\overline{R}$ by for all $x \in \overline{R}$, $\overline{A}(x) = \sup\{t \mid x \in \overline{A}_t, t \in \mathrm{Im} A\}$. Then $\overline{A}$ is a fuzzy ideal of $\overline{R}$.*

**Theorem 3.10** [6]**.** *Let $A$ be as in Theorem 3.9. Then $\overline{A}$ is an extension of $A$.*

**Definition 3.11** [6]**.** Let $A$ be as in Theorem 3.9 and let $\overline{A}$ be the extension ideal of $A$. The fuzzy subset $x + A : R \to [0, 1]$ defined by $(x + A)(z) = \overline{A}(z \oplus x')$ is called a *coset* of the fuzzy ideal $A$.

**Theorem 3.12** [6]**.** *Let $R$, $\overline{R}$, $A$ and $\overline{A}$ be as in Theorem 3.9. Then $x + A = y + A$ $(x, y \in R)$ if and only if $\overline{A}(x \oplus y') = A(0)$.*

**Theorem 3.13** [6]**.** *Let $A$ be as in Theorem 3.9 and $\overline{A}$ be the extension of $A$. If $x + A = u + A$ and $y + A = v + A$ $(x, y, u, v \in R)$, then*

(1) $x + y + A = u + v + A$

(2) $xy + A = uv + A$.

Theorem 3.13 allows us to define two binary operation "+" and "·" on the set $R/A$ of cosets of the fuzzy ideal $A$ as follows:

$$(x + A) + (y + A) = x + y + A$$

and

$$(x + A) \cdot (y + A) = xy + A.$$

It is easy to show that $R/A$ is a $k$-semiring under these well-defined binary operations with additive identity $A$ and multiplicative identity $1 + A$. In this case, the semiring $R/A$ is called the *factor semiring* or the *quotient semiring* of $R$ by $A$.

**Definition 3.14** [11]**.** A fuzzy ideal $A$ of $R$ is called a *fuzzy prime* if $\forall a, b \in R$, either $A(ab) = A(a)$ or else $A(ab) = A(b)$.

**Definition 3.15** [11]**.** A fuzzy ideal $A$ of $R$ is called a *fuzzy semiprime* if $A(a^m) = A(a)$, $\forall a \in R$ and $\forall m \in \mathbb{Z}_+$.

In the following theorem, we have the relation between the fuzzy prime ideal of a $k$-semiring and the fuzzy prime ideal of a ring.

**Theorem 3.16.** *Let $A$ be a fuzzy ideal of a $k$-semiring $R$ such that all its level subsets are $k$-ideals of $R$, $\overline{A}$ be its extension and $\overline{R}$ be the extension ring of $R$. Then $A$ is a fuzzy prime ideal of $R$ iff $\overline{A}$ is a fuzzy prime ideal of $\overline{R}$.*

**Proof.** It is straightforward.

**Theorem 3.17** [4]**.** *Let $f : R \to S$ be an epimorphism of $k$-semirings and $B$ be a fuzzy ideal of $S$. Then $B$ is a fuzzy prime ideal of $S$ iff $f^{-1}B$ is a fuzzy prime ideal of $R$.*

**Definition 3.18** [4]**.** Let $A$ be a fuzzy ideal of a $k$-semiring $R$ such that all its level subsets are $k$-ideals of $R$. A fuzzy ideal $A$ of $R$ is called a *fuzzy maximal* if (i) $A(0) = 1$; (ii) $A(e) < A(0)$; and (iii) whenever $A(b) < A(0)$ for some $b \in R$, then $\overline{A}(e_R \oplus (rb)') = A(0)$ for some $r \in R$, where $e_R$ is identity of $R$.

Let $R = Z^*$, be the set of nonnegative integers. Define a fuzzy subset $\alpha$ of $R$ by

$$\alpha(x) = \begin{cases} 1 & \text{if } x \in (2) \\ s & \text{if } x \notin (2) \text{ for } s \in [0, 1). \end{cases}$$

Then $\alpha$ is a fuzzy maximal ideal and a fuzzy prime ideal of $R$.

Kumar [9] defined the fuzzy maximal ideal of a ring as follow:

**Definition 3.19** [8]**.** A fuzzy ideal $A$ of a ring $R$ is called *fuzzy maximal* if (i) $A(0) = 1$; (ii) $A(e_R) < A(0)$; and (iii) whenever $A(b) < A(0)$ for some $b \in R$, then $A(e_R - rb) = A(0)$ for some $r \in R$.

In the following theorem, we have the relation between the fuzzy maximal ideal of a $k$-semiring and the fuzzy maximal ideal of a ring.

**Theorem 3.20** [4]**.** *Let $A$ be as in Definition* 3.18, $\overline{A}$ *be its extension and* $\overline{R}$ *be the extension ring of $R$. Then $A$ is a fuzzy maximal ideal of $R$ iff* $\overline{A}$ *is a fuzzy maximal ideal of* $\overline{R}$.

**Theorem 3.21** [4]**.** *Let $f : R \to S$ be an epimorphism of $k$-semirings and $B$ be a fuzzy ideal of $S$. Then $B$ is a fuzzy maximal ideal of $S$ iff $f^{-1}B$ is a fuzzy maximal ideal of $R$.*

**Theorem 3.22.** *Every fuzzy maximal ideal of a $k$-semiring is fuzzy prime.*

**Proof.** Let $\alpha$ be any fuzzy maximal ideal of a $k$-semiring $R$. Then by Theorem 3.20, $\overline{\alpha}$ is a fuzzy maximal ideal of a ring $\overline{R}$, so that $\overline{\alpha}$ is a fuzzy prime ideal of $\overline{R}$. Thus $\alpha$ is a fuzzy prime ideal of $R$. This completes the proof.

## 4. The Fuzzy Prime Radical of a $k$-semiring



In this section, we define the fuzzy prime radical $\mathrm{FPR}(R)$ of a $k$-semiring $R$ and have some properties of the quotient ring $R/\mathrm{FPR}(R)$ of $R$ by the fuzzy prime radical $\mathrm{FPR}(R)$, and obtain some algebraic properties of $\mathrm{FPR}(R)$ and $\mathrm{FPR}(S)$ under a homomorphism from a $k$-semiring $R$ onto a $k$-semiring $S$.

Kumar [9] defined the fuzzy prime radical $\mathrm{FPR}(R)$ of a ring $R$ as follows: $\mathrm{FPR}(R) = \bigcap\{\theta \mid \theta$ is a fuzzy prime ideal of $R\}$. Similarly, we define a fuzzy prime radical of a $k$-semiring.

**Definition 4.1.** Let $R$ be a $k$-semiring. The intersection of all fuzzy prime ideals of $R$ is called the *fuzzy prime radical* of $R$, denoted by $\mathrm{FPR}(R)$.

**Definition 4.2** [5]**.** Let $R$ be a $k$-semiring. The intersection of all fuzzy maximal ideals of $R$ is called the *fuzzy Jacobson radical* of $R$, denoted by $\mathrm{FJR}(R)$.

From Theorem 3.22, we have the following theorem.

**Theorem 4.3.** *Let $R$ be a $k$-semiring. Then* $\mathrm{FPR}(R) \subseteq \mathrm{FJR}(R)$.

**Proof.**

$$\begin{aligned}\mathrm{FJR}(R) &= \bigcap\{\alpha \mid \alpha \text{ is a fuzzy maximal ideal of } R\}\\ &\supseteq \bigcap\{\alpha \mid \alpha \text{ is a fuzzy prime ideal of } R\}\\ &= \mathrm{FPR}(R).\end{aligned}$$

**Notation.** We denote the collection of all fuzzy prime ideals of $R$ by $\mathrm{FPI}(R)$ and denote the collection of all fuzzy prime ideals of $R$ such that all its level subsets are $k$-ideals of $R$ by $\mathrm{FPI}^*(R)$. The intersection of all fuzzy prime ideals in $\mathrm{FPI}^*(R)$ is denoted by $\mathrm{FPR}^*(R)$.

**Theorem 4.4** [5]**.** *Let $\alpha$ and $\beta$ be fuzzy ideals of a $k$-semiring $R$ such that all its level subsets are $k$-ideals of $R$. Then $\alpha \cap \beta$ is a fuzzy ideal of $R$ such that all its level subsets are $k$-ideals of $R$.*

**Theorem 4.5** [5]**.** *Let $\alpha$ and $\beta$ be fuzzy ideals of a $k$-semiring $R$ such that all its level subsets are $k$-ideals of $R$. Then $\overline{\alpha \cap \beta} = \overline{\alpha} \cap \overline{\beta}$.*

**Theorem 4.6.** *Let $R$ be a $k$-semiring and $\overline{R}$ be the extension ring of $R$. If $A$ be a fuzzy prime ideal of $\overline{R}$, then there exists a fuzzy prime ideal $\alpha$ of $R$ such that all its level subsets are $k$-ideals of $R$ and $\overline{\alpha} = A$.*

**Proof.** Let $\alpha$ be the restriction $A \mid R$ of $A$ to $R$. For each $t \in [0, 1]$, let $x + y \in \alpha_t$, and $y \in \alpha_t$, then $\alpha(x + y) \geq t$ and $\alpha(y) \geq t$. So $A(x + y) \geq t$ and $A(y) \geq t$, which implies that $x + y \in A_t$ and $y \in A_t$. Since $A_t$ is an ideal of $\overline{R}$, $x \in A_t$. Thus $A(x) \geq t$ and thus $\alpha(x) \geq t$. So $x \in \alpha_t$. Hence $\alpha_t$ is a $k$-ideal of $R$ for all $t \in [0, 1]$ and hence $\overline{\alpha} = A$. On the other hand, by Theorem 3.16, $\alpha$ is a fuzzy prime ideal of $R$. This completes the proof.

**Lemma 4.7.** *Let $R$ be a $k$-semiring and $\overline{R}$ be the extension ring of $R$. Then a mapping $f : \mathrm{FPI}^*(R) \to \mathrm{FPI}^*(\overline{R})$ defined by $f(\alpha) = \overline{\alpha}$ is bijective.*

**Proof.** Let $f(\alpha) = f(\beta)$. Then $\overline{\alpha} = \overline{\beta}$ and $\alpha = \beta$. Thus $f$ is one-one. Let $A$ be any element of $\mathrm{FPI}^*(\overline{R})$. Then by Theorem 4.6, there exists an element $\alpha \in \mathrm{FPI}^*(R)$ such that $\overline{\alpha} = A$. Thus $f(\alpha) = \overline{\alpha} = A$. Hence $f$ is onto. This completes the proof.

**Theorem 4.8.** *Let $R$ be a $k$-semiring and $\overline{R}$ be the extension ring of $R$. Then* $\overline{\mathrm{FPR}^*(R)} = \mathrm{FPR}^*(\overline{R})$.

**Proof.** By Theorem 4.5 and Lemma 4.7,

$$\overline{\mathrm{FPR}^*(R)} = \overline{\bigcap \{\alpha \mid \alpha \in \mathrm{FPI}^*(R)\}}$$

$$= \bigcap \{\overline{\alpha} \mid \overline{\alpha} \in \mathrm{FPI}^*(\overline{R})\}$$

$$= \mathrm{FPR}^*(\overline{R}).$$

**Theorem 4.9** [9]**.** *Let $R$ be a commutative ring with identity and let* $\mu = \mathrm{FPR}(R)$. *Then $R/\mu$ is zero.*

**Remark.** Clearly a fuzzy prime ideal of a $k$-semiring $R$ is fuzzy semiprime.

**Theorem 4.10** [5]**.** *If $\alpha$ and $\beta$ be fuzzy semiprime ideals of $R$, then $\alpha \cap \beta$ is a fuzzy semiprime ideal of $R$.*

From the above Remark and Theorem 4.10, we have the following.

**Corollary 4.11.** *If $R$ is a $k$-semiring, then* $\mathrm{FPR}(R)$ *is fuzzy semiprime.*

**Theorem 4.12.** *Let $R$ be a $k$-semiring and let* $\mu = \mathrm{FPR}(R)$. *Then $R/\mu$ is zero prime radical.*

**Proof.** Since $PR(R/\mu) = PR(\overline{R/\mu}) \cap R/\mu$ is the ideal of all nilpotent elements of $R/\mu$ by Theorem 2.11, we must show that every nilpotent element of $R/\mu$ is zero. Let $x + \mu$ be any element of $R/\mu$. Then $x^n + \mu = 0 + \mu$ for some $n \in \mathbb{Z}_+$, so that $\overline{\mu}(x^n \oplus 0') = \mu(0)$. Since $\overline{\mu}(x^n \oplus 0') = \overline{\mu}(x^n) = \mu(x^n) = \mu(x)$ by Corollary 4.11, $\mu(x) = \mu(0)$ and so $\overline{\mu}(x \oplus 0') =$

$\mu(0)$, which implies that $x + \mu = 0 + \mu$ by Theorem 3.12. Hence $PR(R/\mu)$ is zero prime radical.

**Lemma 4.13** [5]**.** *Let* $\varphi : R \to S$ *be an epimorphism from a k-semiring* $R$ *onto a k-semiring* $S$. *Let* $\beta_1$ *and* $\beta_2$ *be fuzzy ideals of* $S$. *Then* $\varphi^{-1}(\beta_1 \cap \beta_2) = \varphi^{-1}\beta_1 \cap \varphi^{-1}\beta_2$.

**Lemma 4.14** [5]**.** *Let* $\alpha$ *and* $\beta$ *be fuzzy ideals of a k-semiring* $R$. *Then* $(\alpha \cap \beta)_t = \alpha_t \cap \beta_t$ *for all* $t \in [0, 1]$.

**Lemma 4.15** [5]**.** *Let* $\alpha$ *and* $\beta$ *be fuzzy ideals of a k-semiring* $R$. *If* $\alpha_t$ *and* $\beta_t$ *are k-ideals of* $R$ *for all* $t \in [0, 1]$, *then* $\alpha_t \cap \beta_t$ *is a k-ideal of* $R$.

**Theorem 4.16.** *Let* $\varphi : R \to S$ *be an epimorphism from a k-semiring* $R$ *onto a k-semiring* $S$. *Then* $\varphi^{-1}(\mathrm{FPR}(S)) \supseteq \mathrm{FPR}(R)$.

**Proof.**

$$\begin{aligned}\varphi^{-1}(\mathrm{FPR}(S)) &= \varphi^{-1}(\cap\{\beta \mid \beta \text{ is a fuzzy prime ideal of } S\}) \\ &= \cap\{\varphi^{-1}(\beta) \mid \beta \text{ is a fuzzy prime ideal of } S\} \\ &\supseteq \mathrm{FPR}(R).\end{aligned}$$

From Theorem 4.16, we have the following.

**Corollary 4.17.** *Let* $\varphi : R \to S$ *be an epimorphism from a k-semiring* $R$ *onto a k-semiring* $S$. *Then* $\varphi(\mathrm{FPR}(R)) \subseteq \mathrm{FPR}(S)$.

**Proof.** By Theorem 4.16, we have

$$\varphi(\mathrm{FPR}(R)) \subseteq \varphi(\varphi^{-1}(\mathrm{FPR}(S))) = \mathrm{FPR}(S).$$

**Definition 4.18** [7]**.** Let $R$ and $S$ be any sets and let $f : R \to S$ be a function. A fuzzy subset $A$ of $R$ is called *f-invariant* if $f(x) = f(y)$ implies $A(x) = A(y)$, where $x, y \in R$.

**Lemma 4.19** [4]**.** *Let* $\varphi : R \to S$ *be an epimorphism from a k-semiring* $R$ *onto a k-semiring* $S$ *and let* $\alpha$ *be* $\varphi$*-invariant fuzzy ideal of* $R$. *Then* $\alpha$ *is a fuzzy prime ideal of* $R$ *iff* $\varphi(\alpha)$ *is a fuzzy prime ideal of* $S$.

**Lemma 4.20** [4]. *Let* $\varphi : R \to S$ *be an epimorphism from a* $k$*-semiring* $R$ *onto a* $k$*-semiring* $S$ *and let* $\alpha$ *be* $\varphi$*-invariant fuzzy ideal of* $R$ *such that all its level subsets are* $k$*-ideals of* $R$. *Then* $\overline{\alpha}$ *is* $\overline{\varphi}$*-invariant.*

**Lemma 4.21** [9]. *If* $\mu$ *is any fuzzy ideal of a commutative ring with identity* $R$, *then* $\mu(x - y) = \mu(0) \Leftrightarrow x + \mu = y + \mu$ *for any* $x, y \in R$.

**Theorem 4.22.** *Let* $\varphi : R \to S$ *be an epimorphism from a* $k$*-semiring* $R$ *onto a* $k$*-semiring* $S$, $\overline{\varphi}$ *be the extension of* $\varphi$ *and let* $\mu = \mathrm{FPR}^*(R)$. *If* $\mu$ *is* $\varphi$*-invariant, then every element of* $\mathrm{FPI}^*(R)$ *is* $\varphi$*-invariant.*

**Proof.** Let $\varphi(x) = \varphi(y)$ $(x, y \in R)$. Then $\overline{\varphi}(x) = \overline{\varphi}(y)$. Since $\mu$ is $\varphi$-invariant, by Lemma 4.20, $\overline{\mu}$ is $\overline{\varphi}$-invariant. Thus $\overline{\mu}(x) = \overline{\mu}(y)$, so that $\overline{\alpha}(x - y) = \overline{\alpha}(0)$ for each $\alpha \in \mathrm{FPI}^*(R)$. Thus $\overline{\alpha}(x) = \overline{\alpha}(y)$ and thus $\alpha(x) = \alpha(y)$ for each $\alpha \in \mathrm{FPI}^*(R)$. This completes the proof.

From Lemma 4.19 and Theorem 4.22, we have the following.

**Theorem 4.23.** *Let* $\varphi : R \to S$ *be an epimorphism from a* $k$*-semiring* $R$ *onto a* $k$*-semiring* $S$ *and let* $\mu \in \mathrm{FPR}^*(R)$ *be* $\varphi$*-invariant. Then if* $\alpha \in \mathrm{FPI}^*(R)$, *then* $\varphi(\alpha) \in \mathrm{FPI}^*(S)$.

**Lemma 4.24.** *Let* $\varphi$ *and* $\mu$ *be as Theorem* 4.23. *Then* $f : \mathrm{FPI}^*(R) \to \mathrm{FPI}^*(S)$ *defined by* $f(\alpha) = \varphi(\alpha)$ *is bijective.*

**Proof.** By Theorem 4.23, $f$ is well defined. Let $f(\alpha) = f(\beta)$. Then $\varphi(\alpha) = \varphi(\beta)$. Since $\mu$ is $\varphi$-invariant, by Theorem 4.22, $\alpha$ and $\beta$ are $\varphi$-invariant, so that $\alpha = \beta$. Thus $f$ is one-one. Let $\beta$ be any element of $\mathrm{FPI}^*(S)$. Then by Theorem 3.17, there exists $\alpha = \varphi^{-1}(\beta) \in \mathrm{FPI}^*(R)$ such that $f(\alpha) = \varphi(\alpha) = \varphi(\varphi^{-1}(\beta)) = \beta$. Thus $f$ is onto. This completes the proof.

By Lemma 4.24, we have $\mathrm{FPI}^*(R) = \{\varphi^{-1}(\beta) \mid \beta \in \mathrm{FPI}^*(S)\}$. So we obtain the following theorem.

**Theorem 4.25.** *Let* $\varphi$ *and* $\mu$ *be as Theorem* 4.23 *and let* $\nu = \mathrm{FPR}^*(S)$. *Then* $\varphi^{-1}(\nu) = \mu$.

**Proof.**

$$\varphi^{-1}(\nu) = \varphi^{-1}(\bigcap \{\beta \mid \beta \in \mathrm{FPI}^*(S)\}) = \bigcap\{\varphi^{-1}(\beta) \mid \beta \in \mathrm{FPI}^*(S)\} = \mu.$$

**Corollary 4.26.** *Let* $\varphi$ *and* $\mu$ *be as Theorem* 4.23 *and let* $\nu = \mathrm{FPR}^*(S)$. *Then* $\varphi(\mu) = \nu$.

**Proof.** By Theorem 4.25, $\varphi(\mu) = \varphi(\varphi^{-1}(\nu)) = \nu$.

From Theorems 4.8, 4.25 and Corollary 4.26, we have the following corollary.

**Corollary 4.27.** *Let* $\varphi$ *and* $\mu$ *be as Theorem* 4.23 *and let* $\nu = \mathrm{FPR}^*(S)$. *Let* $\overline{R}$ *and* $\overline{S}$ *be the extension rings of* $R$ *and* $S$, *respectively*. *Then*

(1) $\overline{\varphi^{-1}(\nu)} = \mathrm{FPR}^*(\overline{R})$.

(2) $\overline{\varphi(\mu)} = \mathrm{FPR}^*(\overline{S})$.



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