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GEOMETRIC PROPERTIES OF SHAPE FUNCTIONS OF SELF-SIMILAR SOLUTIONS

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Abstract

We study some geometric properties of shape functions of self-similar solutions to the initial-boundary value problem with homogeneous Neumann boundary condition for the semi-linear parabolic equation: $u_t = u_{xx} + (|u|^{q-1}u)_x - |u|^{p-1}u$, where p, q are positive numbers. These shapes of the solutions of the corresponding nonlinear ordinary differential equation are of very different nature. The properties usually depend on the critical value $q = 1, 2;\ p = 1, 3$ and initial data as usual.

1. Introduction

In this paper, we consider a semi-linear parabolic equation:

$$u_t = u_{xx} + (|u|^{q-1}u)_x - |u|^{p-1}u \quad \text{in } Q \tag{1.1}$$

with homogeneous Neumann boundary condition

$$u_x(0, t) = 0, \tag{1.2}$$

where $Q = \{(x, t) | x > 0, t > 0\}$.

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Equation (1.1) appears in the description of a continuous medium for which the constitutive relation for the stress contains a large linear proportional to the strain, a small term which is quadratic in strain, and a small dissipative term proportional to the strain rate. The inviscid form of (1.1) arises in several applications, which includes nonlinear acoustic propagation [19], the Gunn effect in semiconductors [17], rotating thin liquid films [18], chloride concentration in kidney [16] and flow of petroleum in underground reservoirs [5, 10]. This has also been considered by Bukiet et al. [3] in the inviscid limit. Moreover, equation (1.2) is appear in heat conduction and filtration process. One says that the term u_{xx} is diffusion term, $(|u|^{q-1}u)_x$ is convection term, $|u|^{p-1}u$ is absorption term, see [22]. The actual physical model involving (1.1) which is subject to random initial data and through analytical understanding is beyond our ability. Thus, in this paper we consider simple case, where the initial value $u(x, 0) = u_0(x)$ is a deterministic function.

We are interested in nonnegative solutions of the nonlinear heat equation (1.1) having the form

$$u(x, t) = t^{-\alpha} g(xt^{-\beta}) := t^{-\alpha} g(\xi), \tag{1.3}$$

where α, β are some real numbers and $(x, t) \in Q$.

For equation (1.1), we substitute (1.3) into (1.1) and obtain

$$\alpha := \frac{1}{2(q-1)} = \frac{1}{p-1}, \quad \beta := \frac{1}{2}, \quad p = 2q - 1 \tag{1.4}$$

and g as a function of $\xi = xt^{-\beta}$, is defined on $[0, \infty)$, solves of ODE:

$$g'' + \frac{\xi}{2} g' + \alpha g + q|g|^{q-1} g' = |g|^{p-1} g. \tag{1.5}$$

The self-similar solutions play an important role in the large time behavior of solutions of problem (1.1) with (1.2) and initial data u_0. We observe that if $u(x, t)$ solves (1.1), then the *rescaled functions*

$$u_\rho(x, t) = \rho^{(2-q)/(q-1)} u(\rho x, \rho^2 t), \quad \rho > 0 \tag{1.6}$$

define a one parameter family of solutions to (1.1). A solution $u(x, t)$ is said to be *self-similar* when $u_\rho(x, t) = u(x, t)$ for every $\rho > 0$. It can be easily verified that $u(x, t)$ is a self-similar solution to (1.1) if and only if u has the form (1.3), where g satisfies (1.5), $\xi = xt^{-\beta}$ and α, β are as above.

In this note we shall classify completely positive solutions of (1.5) according to the parameters *p, q*. In the mean time, we shall find conditions on *p, q* which ensure the existence of the so-called very singular (self-similar) solutions for (1.1). The very singular solution has a stronger singularity at the origin than the singular solution of that equation.

By a *singular solution* we mean a nonnegative and nontrivial solution which satisfies the equation and initial-boundary conditions on $\overline{Q}\setminus(0, 0)$ and satisfy

$$\lim_{t \searrow 0} \sup_{x > \varepsilon} u(x, t) = 0 \quad \forall \varepsilon > 0. \tag{1.7}$$

A singular solution is called a *very singular solution* if

$$\lim_{t \searrow 0} \int_{x \le \varepsilon} u(x, t)\, dx = \infty. \tag{1.8}$$

Note that condition (1.7) is equivalent to, if u is given by (1.3),

$$\lim_{\xi \to \infty} \xi^{\alpha/\beta} g(\xi) = 0. \tag{1.9}$$

Furthermore, if $0 < \beta < \alpha$ and the solution g of (1.5) satisfies (1.9), then $u(x, t)$ given explicitly by (1.3) satisfies (1.7) and (1.8), i.e., it is a very singular self-similar solution of (1.1).

Here, we apply a shooting method and replace the condition at infinity by the one at the origin. We thus study an initial value problem (1.5) for $\xi > 0$ with:

$$g'(0) = 0, \quad g(0) = \lambda, \tag{1.10}$$

here λ may be any positive number. This initial value problem (1.5) with (1.10) for each $\lambda > 0$ a unique solution which we shall denote by $g(\xi, \lambda)$.

In many cases, it turns out that the limit

$$L(\lambda) = \lim_{\xi \to \infty} \xi^{2\alpha} g(\xi) \text{ exists,} \tag{1.11}$$

and we distinguish between fast and slow orbits according to whether $L(\lambda) = 0$ or not respectively. As mentioned before the fast orbit brings out a very singular solution of (1.1).

Our goal is to find values q, p and initial data λ which ensure that $g(\cdot, \lambda)$ is either sign changing solution or a global positive decaying solution to ODE (1.5) with condition (1.10) (this is certainly the case $\alpha > 0$) and to give asymptotic behavior of solutions at near infinity. Moreover, in the case $\alpha < 0$, we shall describe which values q, p and λ give a global positive solutions blowing up to infinity and find its blow-up rate at infinity. In the first case, the asymptotic behavior of solutions are classified by either fast or slow orbits.

There have been many works on the deals with the existence, uniqueness and geometric properties of shape functions of self-similar solutions to a class of parabolic equations with absorption (or source, convection) term. For instance, it is thoroughly treated on the porous medium equation with absorption term;

$$u_t = \Delta u^m - u^q$$

with $m > 0$, $q > 1$. For $m = 1$ (linear diffusion case), see [2], [6], for $m > 1$ (slow diffusion case), see [21] and for $0 < m < 1$ (fast diffusion case), see [13], [14]. Recently some papers studied for a class of convection-diffusion equation with absorption term;

$$u_t = \Delta u \pm a \cdot \nabla(|u|^{q-1} u) - |u|^{p-1} u,$$

where $p, q > 1$. The key to prove their results is to deduce new equation by rotating the coordinate axes and rescaling. They also derived some estimates and used a convergence of rescaled solutions to self-similar (non-radial) ones and thus concluded that the asymptotic of general solutions is self-similar, see [15], [14], et al. Similar argument is used in a convection-diffusion equation without absorption term, see [7], [8], for details. In addition, in papers [1], [11], [20], the very singular self-similar solutions are considered on half line because there does not exists any radial symmetric solution. Here, we use a simple method (via shooting) to investigate the self-similar solutions under the homogeneous Neumann boundary problem in one dimension and classify all the self-similar solutions completely for $q > 0$, $q = 2p - 1$. Moreover, in each cases, we may find rich geometric properties appeared before.

The plan of the paper is the following. In Section 2, we obtain basic properties of g which will be useful in the proof of the main results. In Section 3, we study the existence, nonexistence of global positive solutions and find decay rates of global positive decay solutions in the nice case $\alpha > 0$. In Section 4, we study the existence and find blow-up rate of infinite blow-up solutions in case $\alpha < 0$.

2. Preliminary Results

In this section we shall derive some basic properties of g which will be useful in the proof of the main results. In fact, we first show that $g'(\xi)$ has one sign depending on the sign of α and size of λ.

Lemma 2.1. *Assume that* $\alpha > 0$ $(p > 1, q > 1)$, $\lambda > 0$. *Let* g *be a solution of* (1.5), (1.10) *such that* $g > 0$ *on* $[0, \xi_0)$.

Then

(i) $g'(\xi) < 0$ *for all* $\xi \in (0, \xi_0)$ *and* $0 < \lambda < \lambda^*$,

(ii) $g'(\xi) = 0$ *for all* $\xi \in (0, \xi_0)$ *and* $\lambda = \lambda^*$,

(iii) $g'(\xi) > 0$ *for all* $\xi \in (0, \xi_0)$ *and* $\lambda > \lambda^*$,

where $\lambda^* = \left(\frac{1}{|p-1|}\right)^{\frac{1}{p-1}}$.

Proof. By (1.5), (1.10) we obtain $g''(0) = \lambda^p - \alpha\lambda < 0$ and $g'(0) = 0$, the function g is strictly decreasing for small ξ. Suppose that there exists ξ_1 such that $g'(\xi) < 0$ on $(0, \xi_1)$ and $g'(\xi_1) = 0$. Using (1.5) one sees

$g''(\xi_1) < 0$. This contradicts to the left hand side of point ξ_1. By similar method we easily prove (ii) and (iii). □

Lemma 2.2. *Assume that* $\alpha < 0$ $(0 < q < 1, 0 < p < 1)$, $\lambda > 0$. *Let* g *be a solution of* (1.5), (1.10) *such that* $g > 0$ *on* $[0, \xi_0)$.

Then $g'(\xi) > 0$ *for all* $\xi \in (0, \xi_0)$ *and* $\lambda > 0$.

Proof. The proof is similar to Lemma 2.1. □

Next, we show that there does not exist any zero of $g(\xi)$ for $\alpha \leq \frac{1}{2}$.

Lemma 2.3. *Assume that* $\alpha \leq \frac{1}{2}$ $(p \geq 3, \quad q \geq 2$ *or* $0 < p < 1$, $0 < q < 1)$. *Let g be a solution of* (1.5), (1.10).

Then $g > 0$ *for any* $\xi > 0$, $\lambda > 0$.

Proof. Suppose that g has first zero ξ_1 and such that $g(\xi_1) = 0$, $g'(\xi_1) < 0$, $g(\xi) > 0$ on $\xi \in (0, \xi_1)$.

By integrating from 0 to ξ_1, we obtain

$$g'(\xi_1) = \left(\frac{1}{2} - \alpha\right) \int_0^{\xi_1} g(s)\, ds + \int_0^{\xi_1} g^p(s)\, ds + \lambda^q.$$

Since the sign of left hand side is negative and the sign of right hand side is positive, this is impossible. □

3. The Case $\alpha > 0$ $(q > 1, p > 1)$

3.1. The case $0 < \alpha \leq \frac{1}{2}$ $(q \geq 2, p \geq 3)$

In this subsection, we show that there does not exist any fast orbit if α satisfies $0 < \alpha \leq \frac{1}{2}$.

We denote by $(0, \xi_{\max})$ the maximal existence interval of positive solution. By Lemma 2.1, $g' < 0$ in $(0, \xi_{\max})$ for any $\lambda \in (0, \lambda^*)$ and either

(1) $\xi_{\max} = \infty$ and $\lim_{\xi \to \infty} g(\xi; \lambda) = 0$, or

(2) $\xi_{\max} < \infty$ and $g(\xi_{\max}; \lambda) = 0$.

Note that we easily see that the problem (1.5), (1.10) has global positive decay solution for $\lambda \in (0, \lambda^*)$ by Lemmas 2.1 and 2.3.

Theorem 3.1. *Assume* $0 < \alpha \leq \frac{1}{2}$. *For each* $\lambda \in (0, \lambda^*)$, *let* $g(\xi; \lambda)$ *be the solution of* (1.5), (1.10).

Then $g > 0$, *and* $g' < 0$ *in* $(0, \infty)$ *and* $\liminf_{\xi \to \infty} \xi^{2\alpha} g(\xi; \lambda) > 0$.

Proof. Multiplying (1.5) by $\xi^{2\alpha - 1}$ we have, for $\xi \in (0, \xi_{\max})$,

$$\left(\xi^{2\alpha-1} g' + \frac{1}{2}\xi^{2\alpha} g\right)' = \xi^{2\alpha-1}\left(g'' + \frac{2\alpha - 1}{\xi} g' + \alpha g + \frac{\xi}{2} g'\right).$$

By (1.5), we get

$$\left(\xi^{2\alpha-1} g' + \frac{1}{2}\xi^{2\alpha} g\right)' = \xi^{2\alpha-1}\left(| g' |^q + \frac{2\alpha - 1}{\xi} g' - q g^{q-1} g'\right) > 0$$

by Lemma 2.1.

Define the function $F(\xi) := \xi^{2\alpha-1} g' + \frac{1}{2}\xi^{2\alpha} g$ then $F(\xi)$ is strictly increasing in $(0, \xi_{\max})$. Note that $\lim_{\xi \to 0} F(\xi) = 0$, we have $F > 0$ in $(0, \xi_{\max})$. Since $g' < 0$, we conclude that $\xi_{\max} = \infty$ and g decays to 0 as ξ tends to infinite. Moreover, $\xi^{2\alpha} g(\xi; \lambda) \geq 2F(\xi)$ and $F(\xi)' > 0$, hence $\liminf_{\xi \to \infty} \xi^{2\alpha} g(\xi; \lambda) > 0$. □

We shall see later that the limit $\lim_{\xi \to \infty} \xi^{2\alpha} g(\xi; \lambda)$ exists for each $\lambda \in (0, \lambda^*)$. Thus we may conclude together with Theorem 3.1 that there exist slow orbits only.

3.2. The case $\alpha > \frac{1}{2}$ $(1 < q < 2, 1 < p < 3)$

In this subsection, we first show that the solution changes sign for small λ and we next show that the solution becomes a positive global solution decaying to zero for suitably large λ. We then prove that these solutions are either slow or fast orbits. The slow orbits are ordered and the minimal one becomes the fast orbit as we have seen in many cases, see [3], [12] for example.

By Lemma 2.1, if g is a solution of problem (1.5), (1.10), then g decreases as long as positive for any $0 < \lambda < \lambda^*$ and we first show that

Theorem 3.2. *Assume that* $\alpha > \frac{1}{2}$.

Then $g(\xi, \lambda)$ *changes the sign for sufficiently small* $\lambda > 0$. *That is,* $\xi_{\max} < \infty$.

Proof. We choose $\lambda = \varepsilon > 0$ is sufficiently small and let $g_\varepsilon = \frac{g}{\varepsilon}$. Thus, g_ε such that in the following equation

$$\begin{cases} g_\varepsilon'' + \frac{\xi}{2} g_\varepsilon' + \alpha g_\varepsilon + \varepsilon^{q-1} | g_\varepsilon |^{q-1} g_\varepsilon' = \varepsilon^{p-1} | g_\varepsilon |^{p-1} g_\varepsilon \text{ in } \xi > 0 \\ g_\varepsilon'(0) = 0, \quad g_\varepsilon(0) = 1. \end{cases}$$

Define the energy function $E(g_\varepsilon) := (g_\varepsilon')^2 + \alpha g_\varepsilon^2 - \frac{2}{p+1} \varepsilon^{p-1} g_\varepsilon^{p+1}$. By differentiating we have $\frac{d}{d\xi} E(g_\varepsilon) = -2(g_\varepsilon')^2 \left(\frac{\xi}{2} + q\varepsilon^{q-1} | g_\varepsilon |^{q-1} \right) < 0$. Hence $E(g_\varepsilon)$ is uniformly bounded by $\alpha - \frac{2}{p-1} = \frac{3-p}{p^2-1}$ and $(g_\varepsilon')^2 + \frac{3-p}{p^2-1} \varepsilon^{p-1} g_\varepsilon^2 \leq E(g_\varepsilon) := (g_\varepsilon')^2 + g_\varepsilon^2 \left(\alpha - \frac{2}{p+1} \varepsilon^{p-1} g_\varepsilon^{p-1} \right)$, both g_ε and g_ε' are uniformly bounded with respect to $\xi \geq 0$ and $\varepsilon > 0$. Therefore, it follows by standard continuity arguments that

$$g_\varepsilon \to g_0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad C^2([0, R])$$

for any $R > 0$, where g_0 is the solution of the reduced problem

$$\begin{cases} g'' + \frac{\xi}{2} g' + \alpha g = 0 \ \text{ in } \ \xi > 0 \\ g'(0) = 0, \quad g(0) = 1. \end{cases} \tag{3.2.1}$$

We claim that g_0 has a zero when $\alpha > \frac{1}{2}$. By contradiction suppose that $g_0(\xi) > 0$ for $\xi > 0$. Thus, $g_0'(\xi) < 0$ for any $\xi > 0$ and so $g_0(\xi) \le 1$ for $\xi \ge 0$. Integrating (3.2.1) over $(0, \xi)$, we obtain

$$g_0'(\xi) + \frac{1}{2}\xi g_0(\xi) = \left(\frac{1}{2} - \alpha\right)\int_0^\xi g_0(s)\,ds < -A \text{ for large } \xi$$

for some positive number A. This is impossible. Hence, g_0 has a zero. Since $g_0' < 0$ at the first zero of g_0, it follows that for ε sufficiently small, g_ε has a zero as well. □

Next, we prove the existence of positive global decay solutions for suitable large $\lambda \in (0, \lambda^*)$. Indeed, they have either fast or slow orbits and so we obtain asymptotic behavior of solutions of the initial boundary problem for ξ tends to infinity. We need the following lemmas

Lemma 3.3. *Assume that* $\alpha > 0$ $(q > 1,\ p > 1)$. *Let g be a solution of* (1.5), (1.10) *such that* $g(\xi) > 0$ *for* $\xi > 0$.

Then $\lim_{\xi\to\infty} g(\xi) = 0$, $\lim_{\xi\to\infty} g'(\xi) = 0$.

Proof. By Lemma 2.1, $g' < 0$ and g is bounded below by 0. Thus, $g \to g_0 < \infty$ as $\xi \to \infty$. First there exists sequence ξ_n such that $g'(\xi_n) \to 0$ as $\xi_n \to \infty$.

Now define the energy function $E(g) := g'^2 + \alpha g^2 - \frac{2}{p+1} g^{p+1}$. By differentiating we have $\frac{d}{d\xi}E(g) = -2(g')^2\left(\frac{\xi}{2} + q|g|^{q-1}\right) < 0$. Hence $E(g)$ is monotone decreasing for any ξ and so one deduces that $g'(\xi) \to 0$ as $\xi \to \infty$. We claim that g_0 vanishes. By contradiction, we suppose that

$g_0 > 0$. The equation (3.2.1) gives

$$g'' + \left(\frac{\xi}{2} + q\lambda^{q-1}\right)g' = -\alpha g + q(\lambda^{q-1} - g^{q-1})g' + g^p$$

$$\leq -\alpha g + g^p \leq (-\alpha + \lambda^{p-1})g_0.$$

Define the functions $a(\xi) := \frac{1}{2}\xi + q\lambda^{q-1}$, $A(\xi) := \frac{\xi^2}{4} + q\lambda^{q-1}\xi$. Multiplying this by $e^{A(\xi)}$ and integrating we obtain

$$g'(\xi) < (\lambda^{p-1} - \alpha)g_0 e^{-A(\xi)}\int_0^\xi e^{A(s)}ds.$$

Since

$$\lim_{\xi\to\infty}\frac{\int_0^\xi e^{A(s)}ds}{\frac{1}{\xi}e^{A(\xi)}} = 2,$$

we infer

$$g'(\xi) < -\frac{C}{\xi} \text{ for large } \xi$$

for some positive number C, which implies that $g(\xi) \to -\infty$ as $\xi \to +\infty$. This is impossible which completes the proof. □

We rewrite (1.5), (1.10) as

$$\begin{cases} g' = h \\ h' = -\frac{\xi}{2}h - \alpha g - qg^{q-1}h + g^p. \end{cases} \tag{3.2.2}$$

The initial condition now becomes

$$g(0) = \lambda, \quad h(0) = 0,$$

that is, the orbits start on the positive g-axis. Since this axis consists of regular points, the existence of a local solution (g, h) is ensured for every $\lambda > 0$. All along g-axis, the vector field points into the fourth quadrant. Hence, as long as the orbit exists and $g > 0$, we have $h < 0$. Note that this system has only two critical points $(0, 0)$, $(0, \lambda^*)$.

Given any $\delta > 0$, we denote

$$\mathcal{L}_\delta = \{(g, h) : 0 < g < \lambda^*, 0 > h > -\delta g\}.$$

By a similar argument as in [2] we obtain the following lemmas.

Lemma 3.4. *For given* $\delta > 0$, *there exists* $\xi_\delta := 2\left(\delta + \frac{\alpha}{\delta}\right)$ *such that* $\mathcal{L}_\delta$ *is positively invariant for* $\xi > \xi_\delta$. *That is,* $(g(\xi_\delta), h(\xi_\delta)) \in \mathcal{L}_\delta$ *implies that the orbit* $(g(\xi), h(\xi))$ *of* (3.2.2) *remains in* $\mathcal{L}_\delta$ *for all* $\xi \geq \xi_\delta$.

Remark 3.5. Indeed, if $\delta = 1$, then $0 < g < \lambda^*$, $g' + g > 0$ when $\xi < \xi_{\max}$ and ξ close to $\xi_{\max}$. Thus $\xi_{\max} = \infty$.

Remark 3.6. If λ is sufficiently close to λ^*, then $g(\xi, \lambda) > 0$ for all $\xi \geq 0$. The proof is similar to Lemma 4 in reference [2], which implies that the existence of the global positive decay solution for suitably large $\lambda < \lambda^*$.

According to Lemmas 3.3 and 3.4 we have

Lemma 3.7. *Suppose* $g(\xi, \lambda) > 0$ *for all* $\xi > 0$.

Then either

$$\lim_{\xi \to \infty} \frac{g'}{g} = 0$$

or

$$\lim_{\xi \to \infty} \frac{g'}{g} = -\infty.$$

The proof is similar to the proof in paper [2]. For simplicity, we omit the details.

Now we are going to deal with the behavior of positive solutions g of (1.5), (1.10) as ξ goes to infinity. We know that if g remains positive it must go to 0. It turns out that (g, g') can approach $(0, 0)$ along two directions on phase plane only. Moreover, we know that the existence of the strictly decaying positive solution by Remark 3.6 and then we may suppose the existence of the global, positive solution.

Proposition 3.8. *Let* g *be a solution of* (1.5), (1.10) *such that* $g > 0$ *for any* $\xi > 0$.

Then the limit $L(\lambda) = \lim_{\xi\to\infty} \xi^{2\alpha} g(\xi)$ *exists in* $[0, +\infty)$ *and we have*

$$\lim_{\xi\to\infty} \frac{g'}{g} = -\infty \Rightarrow L(\lambda) = 0,$$

$$\lim_{\xi\to\infty} \frac{g'}{g} = 0 \Rightarrow L(\lambda) > 0.$$

Proof. If $\lim_{\xi\to\infty} \frac{g'}{g} = -\infty$, then $g(\xi) = O(e^{-k\xi})$ as $\xi \to \infty$. And so $\xi^{2\alpha} g(\xi) \to 0$ as $\xi \to \infty$. Thus $L(\lambda) = 0$.

Now suppose that

$$\lim_{\xi\to\infty} \frac{g'}{g} = 0.$$

Set $u(\xi) = \frac{g'}{g}$, then u such that

$$u' + \frac{\xi}{2} u = -\alpha + \varphi(\xi), \quad u(0) = 0,$$

here $\varphi(\xi) = -qg^{q-1}u - u^2 + g^{p-1}$. Note that $\varphi(\xi) \to 0$ as $\xi \to \infty$. By simple calculation,

$$u(\xi) = e^{-\frac{\xi^2}{4}} \int_0^\xi \{-\alpha + \varphi(s)\} e^{\frac{s^2}{4}} ds,$$

for any $\xi > 0$.

And by the L'Höpital's rule, we obtain

$$\lim_{\xi\to\infty} \xi u(\xi) = \lim_{\xi\to\infty} \frac{\int_0^\xi \{-\alpha + \varphi(s)\} e^{\frac{s^2}{4}} ds}{\frac{1}{\xi} e^{\frac{\xi^2}{4}}} = -2\alpha.$$

Thus, $u = \dfrac{g'}{g}$ satisfies

$$u(\xi) = \frac{-2\alpha}{\xi} + \frac{o(\xi)}{\xi} \text{ for large } \xi,$$

which leads to

$$g(\xi) = \mathrm{L}(\lambda)\,\xi^{-2\alpha}\{1 + o(1)\}, \quad \mathrm{L}(\lambda) > 0. \qquad \square$$

We also give the asymptotic behavior of $g(\xi, \lambda)$ for suitably large $\lambda < \lambda^*$.

Theorem 3.9. *Let* g *be a solution of* (1.5), (1.10) *such that* $g > 0$ *for any* $\xi > 0$.

1. *If* $\mathrm{L}(\lambda) = 0$, *then there exists* $A > 0$ *such that*

$$g(\xi, \lambda) = A\xi^{2\alpha-1}e^{-\frac{\xi^2}{4}}\{1 - b\xi^{-2} + o(\xi^{-2})\},$$

as $\xi \to \infty$.

2. *If* $\mathrm{L}(\lambda) > 0$, *then*

$$g(\xi, \lambda) = \mathrm{L}(\lambda)\,\xi^{-2\alpha}\{1 - c\xi^{-2} + o(\xi^{-2})\},$$

as $\xi \to \infty$, *where* $b := \dfrac{(2q-3)(q-2)}{(q-1)^2}$, $c := -2\alpha(1 + 2\alpha) + 2\alpha q[\mathrm{L}(\lambda)]^{q-1}$ $+ [\mathrm{L}(\lambda)]^{p-1}$.

Proof. 1. Define

$$Q(\xi) := \xi h + \frac{1}{2}\xi^2 g(\xi)$$

$$G := \xi^2 Q(\xi) - (2\alpha - 1)\xi^2 g(\xi).$$

By direct calculation, we obtain

$$\lim_{\xi\to\infty}\frac{h}{\xi g} = -\frac{1}{2}, \ \lim_{\xi\to\infty}\frac{Q}{g} = 2\alpha - 1,$$

$$\lim_{\xi\to\infty}\frac{G}{g} = 4(\alpha-1)(2\alpha-1) := 2b.$$

Therefore, there exists positive number A such that

$$g(\xi, \lambda) = A\xi^{2\alpha-1}e^{-\frac{\xi^2}{4}}\{1 - b\xi^{-2} + o(\xi^{-2})\},$$

as $\xi \to \infty$.

2. When $\mathrm{L}(\lambda) > 0$, we deduce

$$\begin{aligned}\lim_{\xi\to\infty}\xi^2(\xi u(\xi) + 2\alpha) &= 2\lim_{\xi\to\infty}\xi^2\varphi(\xi) - 4\alpha\\ &= -8\alpha^2 - 4\alpha + 2\lim_{\xi\to\infty}(\xi^2 g^{p-1} - q\xi^2 g^{q-1}u),\end{aligned}$$

where $\varphi(\xi) = -g^{q-1}u - u^2 + g^{p-1}$.

Since

$$\begin{aligned}\lim_{\xi\to\infty}(\xi^2 g^{p-1} - q\xi^2 g^{q-1}u) &= \lim_{\xi\to\infty}[(\xi^{\frac{2}{p-1}}g)^{p-1} - q(\xi u)(\xi^{\frac{1}{q-1}}g)^{q-1}]\\ &= (\mathrm{L}(\lambda))^{p-1} + 2\alpha q(\mathrm{L}(\lambda))^{q-1},\end{aligned}$$

we have

$$\lim_{\xi\to\infty}\xi^2(\xi u(\xi) + 2\alpha) = -8\alpha^2 - 4\alpha + 2(\mathrm{L}(\lambda))^{p-1} + 4\alpha q(\mathrm{L}(\lambda))^{q-1} := 2c.$$

Thus, $u = \frac{g'}{g}$ satisfies

$$u(\xi) = \frac{-2\alpha}{\xi} + \frac{2c}{\xi^3} + \frac{o(\xi)}{\xi^3} \quad \text{for large } \xi,$$

which leads to

$$g(\xi, \lambda) = \mathrm{L}(\lambda)\xi^{-2\alpha}\{1 - c\xi^{-2} + o(\xi^{-2})\},$$

as $\xi \to \infty$. □

Finally, we show that positive solutions are ordered and cannot intersect each other.

Theorem 3.10. *Assume that* $\alpha > 0$ *and* g_i *are solutions of problem* (1.5), (1.10) *with initial data* $g_i(0) = \lambda_i > 0$, *such that* $g_i(\xi) > 0$ *for* $\xi > 0$ *and* $\lambda_2 > \lambda_1$ *for* $\lambda_i \in (0, \lambda^*)$, $i = 1, 2$.

Then $g_2(\xi) > g_1(\xi)$, $g_2'(\xi) < g_1'(\xi)$ *for any* $\xi > 0$.

Proof. Since $g_i''(0) = -\alpha\lambda_i < 0$, $g_i'(0) = 0$ and $\lambda_2 > \lambda_1$, there exists sufficiently small $\xi_0 > 0$ such that $g_2(\xi) > g_1(\xi)$, $g_2'(\xi) < g_1'(\xi)$ for any $0 \le \xi < \xi_0$.

Let $R :=$ supremum of all such ξ_0's, we claim that $R = \infty$. If it is finite, let

$$H(\xi) := \frac{g_2}{g_1}.$$

We know that

$$H(0) > 1, \quad H'(0) = 0.$$

Let $\xi_1 > 0$ be such that $H(\xi) > 1$ for $0 \le \xi < \xi_1$ and $H(\xi_1) = 1$ (such an ξ_1 would exist because we are assuming $R < \infty$).

But $H'(\xi) = W(\xi)/q_1(\xi)^2$, where W is the *wronskian*

$$W(\xi) = g_2'(\xi)\, g_1(\xi) - g_2(\xi)\, g_1'(\xi).$$

Using equation (3.2.1), one sees that W satisfies the differential equation

$$\frac{d}{d\xi}(W(\xi)\, e^{h(\xi)}) = e^{h(\xi)}(\frac{\xi}{2} W + qg_1^{q-1}W + g_2''g_1 - g_1''g_2)$$

$$= e^{h(\xi)} g_1(g_2^{q-1} - g_1^{q-1})[g_2(g_1^{q-1} + g_2^{q-1}) - qg_2'],$$

where $h(\xi) := \frac{\xi^2}{4} + \int_0^\xi qg_1^{q-1}ds.$

Since the right hand side is strictly positive for $0 \le \xi < \xi_1$ this implies that $W(\xi) > 0$ and $H'(\xi) > 0$ for $0 < \xi < \xi_1$. This contradicts to $H(0) > 1 = H(\xi_1)$. This completes the proof. □

Remark 3.11. Indeed, by Theorem 3.10 we may find the fast orbit as a monotone limit of slow orbits and we may easily prove the uniqueness of fast and slow orbits, c.f. see [20].

4. The Case $\alpha < 0$ $(0 < q < 1,\ 0 < p < 1)$

The standard theory of initial value problems imply the existence and uniqueness of solutions in a neighborhood of the origin. At $\xi = 0$, $g''(0) = -\alpha\lambda > 0$. So in a small neighborhood of origin g is increasing and positive. Lemma 2.2 implies that the solution is blow-up.

We first show that the solution does not blow-up in finite time and then we show that there are only one type of positive solutions, that is, slow orbits.

Lemma 4.1. *Let g be a solution of problem* (1.5), (1.10).

Then $g'(\xi) > 0$ *for any* $\xi > 0$ *and* $\lambda > 0$, *moreover g cannot blow-up for finite* ξ.

Proof. Let $\xi_0 > 0$ be the first zero for g'. We have $g''(\xi_0) > 0$. This is impossible. Thus, $g'(\xi) > 0$ for any $\xi > 0$. (Indeed, by Lemmas 2.2 and 2.3.)

Suppose that g is blow-up at $\overline{\xi} < \infty$. Set

$$E := (g')^2 + \alpha g^2 - \frac{2}{p+1} g^{p+1}. \tag{4.1}$$

Using (4.1), (1.5) one sees that $E'(\xi) = -2(g')^2\left[\frac{\xi}{2} + qg^{q-1}\right] < 0$, for all $\xi \in (0, \overline{\xi})$. Hence $E(g) \le \alpha\lambda^2 - \frac{2}{p+1}\lambda^{p+1} < 0$. Since $0 < q < 1$, we have

$$(g')^2 + \alpha g^2 - \frac{2}{p+1} g^2\lambda^{p-1} \le E < 0,$$

for all $\xi \in (0, \overline{\xi})$.

This means that

$$(g')^2 < \left(-\alpha + \frac{2}{p+1}\lambda^{p-1}\right)g^2$$

on the same interval so $g(\xi) < \exp\left(\sqrt{-\alpha + \frac{2}{p+1}\lambda^{p-1}\xi}\right)$. This contradicts to the assumption. Thus, g cannot blow-up for finite ξ. □

Lemma 4.2.

$$\lim_{\xi \to \infty} g(\xi) = +\infty.$$

Proof. Suppose that g is bounded. Since $g' > 0$, we have $g \to g_0 < \infty$ as $\xi \to \infty$. First there exists sequence ξ_n such that $g'(\xi_n) \to 0$ as $\xi_n \to \infty$.

Now define the energy function $E(g) := (g')^2 + \alpha g^2 - \frac{2}{p+1} g^{p+1}$. By differentiating we have $\frac{d}{d\xi}E(g) = -2(g')^2\left(\frac{\xi}{2} + q|g|^{q-1}\right) < 0$. Hence $E(g)$ is monotone decreasing for any ξ and so deduce that $g'(\xi) \to 0$ as $\xi \to \infty$.

By equation (1.5) we get

$$\lim_{\xi\to\infty}\left(g'' + \frac{\xi}{2}g'\right) = \lim_{\xi\to\infty}(-\alpha g - qg^{q-1}g' + g^p)$$

$$= -\alpha g_0 + g_0^p > \frac{1}{2}(-\alpha g_0 + g_0^p) > 0.$$

And so

$$g'' + \frac{\xi}{2}g' > \frac{1}{2}(-\alpha g_0 + g_0^p),$$

for large $\xi > 0$.

Integrating we obtain

$$g'(\xi) > \frac{1}{2}(-\alpha g_0 + g_0^p)e^{-\frac{\xi^2}{4}}\int_0^{\xi} e^{\frac{s^2}{4}}ds.$$

Since

$$\lim_{\xi\to\infty}\frac{\int_0^{\xi} e^{\frac{s^2}{4}}\,ds}{\frac{1}{\xi}e^{\frac{\xi^2}{4}}} = 2,$$

we infer

$$g'(\xi) > \frac{C}{\xi} \quad \text{for large } \xi$$

for some positive number C, which implies that $g(\xi) \to +\infty$ as $\xi \to +\infty$. This is impossible which completes the proof. □

We next investigate the behavior of g for large ξ. Recall that u is bounded by $\sqrt{-\alpha + \frac{2}{p+1}\lambda^{p-1}}$ as in the proof of Lemma 4.1.

Set $u(\xi) = \frac{g'}{g}$, then u such that

$$u' + \frac{\xi}{2}u = -\alpha + \varphi(\xi), \quad u(0) = 0$$

here $\varphi(\xi) = -qg^{q-1}u - u^2 + g^{p-1}$.

By calculation, we know that

$$u(\xi) = e^{-\frac{\xi^2}{4}}\int_0^{\xi}\{-\alpha + \varphi(s)\}\,e^{\frac{s^2}{4}}\,ds,$$

for any $\xi > 0$, and by the L'Höpital's rule, we obtain

$$\lim_{\xi\to\infty}\xi u(\xi) = \lim_{\xi\to\infty}\frac{\int_0^{\xi}\{-\alpha + \varphi(s)\}\,e^{\frac{s^2}{4}}\,ds}{\frac{1}{\xi}e^{\frac{\xi^2}{4}}} = \lim_{\xi\to\infty}(-2\alpha + 2\varphi(\xi)).$$

The last inequality yields that $u(\xi)$ at ∞ behaves like $(-2\alpha + 2\varphi(\xi))/\xi$.

On the other hand, it is easy to see that

$$|\,\varphi(\xi)\,| \leq q\sqrt{-\alpha + \frac{2}{p+1}\lambda^{p-1}}\,g^{q-1} + \left(\sqrt{-\alpha + \frac{2}{p+1}\lambda^{p-1}}\right)^2 + g^{p-1}$$

so it is clearly implies that $u(\xi) \to 0$ as $\xi \to \infty$, and then $\varphi(\xi) \to 0$.

Theorem 4.3. *Assume that* $\alpha < 0$ $(0 < q < 1,\ 0 < p < 1)$. *Let* g *be a solution of problem* (1.5), (1.10).

Then there exists $\mathrm{L}(\lambda) > 0$ *such that*

$$g(\xi, \lambda) = \mathrm{L}(\lambda)\,\xi^{-2\alpha}\{1 - c\xi^{-2} + o(\xi^{-2})\},$$

as $\xi \to \infty$, *where* $c := -2\alpha(1 + 2\alpha) + 2\alpha q[\mathrm{L}(\lambda)]^{q-1} + [\mathrm{L}(\lambda)]^{p-1}$.

Proof. Since

$$\lim_{\xi\to\infty} \frac{g'}{g} = 0,$$

$$\lim_{\xi\to\infty} \varphi(\xi) = 0,$$

we have $\lim_{\xi\to\infty} \xi u(\xi) = -2\alpha$ and

$$\lim_{\xi\to\infty} \xi^2(\xi u(\xi) + 2\alpha) = 2\lim_{\xi\to\infty} \xi^2\varphi(\xi) - 4\alpha$$

$$= -8\alpha^2 - 4\alpha + 2\lim_{\xi\to\infty}(\xi^2 g^{p-1} - q\xi^2 g^{q-1}u),$$

where $\varphi(\xi) = -g^{q-1}u - u^2 + g^{p-1}$.

By simple calculation, we obtain

$$\lim_{\xi\to\infty}(\xi^2 g^{p-1} - q\xi^2 g^{q-1}u) = \lim_{\xi\to\infty}[(\xi^{\frac{2}{p-1}}g)^{p-1} - q(\xi u)(\xi^{\frac{1}{q-1}}g)^{q-1}]$$

$$= (\mathrm{L}(\lambda))^{p-1} + 2\alpha q(\mathrm{L}(\lambda))^{q-1},$$

and so

$$\lim_{\xi\to\infty} \xi^2(\xi u(\xi) + 2\alpha) = -8\alpha^2 - 4\alpha + 2(\mathrm{L}(\lambda))^{p-1} + 4\alpha q(\mathrm{L}(\lambda))^{q-1} := 2c.$$

Thus, $u = \dfrac{g'}{g}$ satisfies

$$u(\xi) = \frac{-2\alpha}{\xi} + \frac{2c}{\xi^3} + \frac{o(\xi)}{\xi^3} \text{ for large } \xi,$$

which leads to

$$g(\xi, \lambda) = \mathrm{L}(\lambda)\,\xi^{-2\alpha}\{1 - c\xi^{-2} + o(\xi^{-2})\},$$

as $\xi \to \infty$.

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