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ON VERTEX NUMBERING OF CERTAIN CLASSES OF GRAPHS

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Abstract

A labeling or numbering of a graph G is an assignment f of labels to the vertices of G that induces for each edge uv a labeling depending on the vertex labels $f(u)$ and $f(v)$. In this paper, we study some classes of graphs and their corresponding labelings or numbering.

1. Introduction

Unless mentioned or otherwise, a graph in this paper shall mean a simple finite graph without isolated vertices. For all terminology and notations in Graph Theory, we follow [3], and all terminology regarding to labeling, we follow [4]. In [6], we suggested a new labeling known as *odd sequential labeling*. Let G be a (p, q) graph. Let $G = (V, E)$. Consider an injective function $f : V(G) \to X$, where $X = \{0, 1, 2, \ldots, 2q - 1\}$ if G is a tree and $X = \{0, 1, 2, \ldots, q\}$ otherwise. Define the function $f^* : E(G) \to N$, the set of all natural numbers such that $f^*(uv) = f(u) + f(v)$ for all edges uv. If the set of induced edges labels of $f^*(uv)$ is of the form

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$\{1, 3, 5, ..., 2q-1\}$, then the labeling is known as *odd sequential labeling* and the corresponding graph is called *odd sequential graph*.

Another labeling has been suggested by Bermond [2], and named as *graceful labeling*. A graph $G = (V, E)$ is numbered if each vertex v is assigned to nonnegative integer $f(v)$ and each edge uv is attributed the absolute value of the difference of numbers of its end points, that is, $|f(u) - f(v)|$. The numbering is called *graceful* if furthermore, we have the following three conditions: (a) All the vertices are labeled with distinct integers (i.e., f is an one-to-one). (b) The largest value of vertex labels is equal to the number of edges, i.e., $f(v) \in \{0, 1, 2, ..., q\}$ for all $v \in V(G)$. (c) The edges of G are distinctly labeled with the integers from 1 to q.

If a graph G admits such a numbering, then G is said to be *graceful*.

The notion of prime labeling of graph was defined in [4]. A graph G with n-vertices is said to have a *prime labeling* if its vertices are labeled with distinct integers $1, 2, ..., n$ such that for each edge uv the labels assigned to u and v are relatively prime. A graph which admits prime labeling is known as a *prime graph*.

2. On Ladder Graphs

Definition 2.1. The Ladder graph L_n is defined by the Cartesian product of the path P_n with K_2 and it is denoted by $L_n = P_n \times K_2$. In [1], it is shown that, the ladder graph L_n is graceful and in [6], found that prime labeling of L_n and $L_n \odot K_1$. Now, we have the following.

Theorem 2.1. *The ladder graph L_n is odd sequential for all $n \geq 2$.*

Proof. Consider the ladder graph L_n, $n \geq 2$. Let $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ be the vertex set of L_n. Note that, L_n is a graph with $2n$ vertices and $3n - 2$ edges.

The result is obvious for the case $n = 2$. Since $P_2 \times K_2$ is nothing but C_4 which is obviously odd sequential. Hence consider $n \geq 3$.

Define a function $f : V(L_n) \to \{0, 1, 2, ..., 3n - 2 = q\}$ such that

$$f(u_{2i-1}) = 6i - 6; \qquad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}$$

$$1 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd,}$$

$$f(v_{2i-1}) = 6i - 5; \qquad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}$$

$$1 \leq i \leq \frac{n+1}{2} \text{ if } n \text{ is odd,}$$

$$f(u_{2i}) = 6i - 3; \qquad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}$$

$$1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd,}$$

and

$$f(v_{2i}) = 6i - 2; \qquad 1 \leq i \leq \frac{n}{2} \text{ if } n \text{ is even}$$

$$1 \leq i \leq \frac{n-1}{2} \text{ if } n \text{ is odd.}$$

(a) Clearly f is injective.

(b) Also, $\max_{v \in V} f(v) = 3n - 2,$ the number of edges.

Thus, $f(v) \in \{0, 1, ..., 3n - 1\}$.

(c) It is obvious that the labels of the edges of L_n are all integers of the interval $[1, 3n - 2]$.

That is, f is an odd sequential numbering. Hence, L_n is odd sequential.

The numbering for L_{10} is exhibited in Fig. 2.1

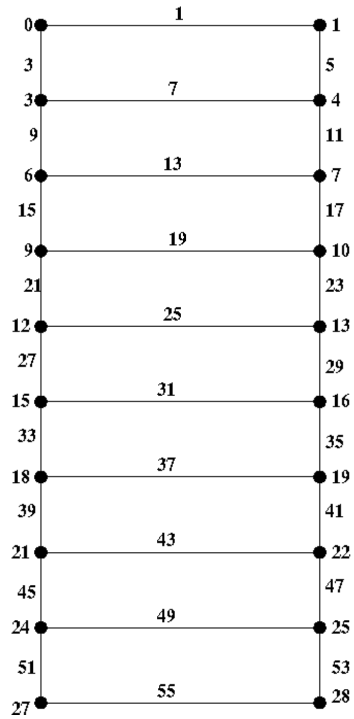

Figure 2.1

Theorem 2.2. *The ladder graph* $L_n + K_1$, $n \geq 3$ *is prime if* $2n + 1$ *is prime.*

Proof. Consider the ladder graph L_n, $n \geq 3$. Let $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertex set of L_n and w be the unique vertex of K_1. Note that, $L_n + K_1$ has $2n + 1$ vertices and $5n - 2$ edges. Define a function

$$f : V(L_n + K_1) \to \{1, 2, \ldots, 2n + 1\}$$

such that $f(u_i) = i$ and $f(v_i) = 2n + 1 - i$ for $i = 1, 2, \ldots, n$ and $f(w) = 2n + 1$. From the definition, it is clear that f is injective. Next, we have to show that each f values are relatively prime. Since $f(u_i) = i$ and $f(v_i) = 2n + 1 - i$. It is obvious that $(f(u_i), f(v_i)) = 1$, $(f(u_i), f(w)) = 1$, $(f(v_i), f(w)) = 1$, $(f(u_i), f(u_{i+1})) = 1$ and $(f(v_i), f(v_{i+1})) = 1$ for $i = 1, 2, 3, \ldots, n - 1$. It remains to show that $(f(u_i), f(v_i)) = 1$ for $i = 1, 2, 3, \ldots, n$.

That is, to show that $(i, 2n+1-i) = 1$.

Suppose

$$\begin{aligned} d &= (i, 2n+1-i) \\ &\Rightarrow d \mid i \text{ and } d \mid 2n+1-i \\ &\Rightarrow d \mid i + 2n + 1 - i \\ &\Rightarrow d \mid 1, \text{ since } 2n+1 \text{ is a prime.} \end{aligned}$$

Thus, f is a prime labeling. Hence, the graph $L_n + K_1$ is prime.

3. On Grid Graphs

Definition 3.1. The *planar grid* is the graph obtained by the cartesian product of two paths P_m and P_n and is denoted by $P_m \times P_n$.

In [5], it is shown that $P_m \times P_n$ has an α-valuation and $P_m \times P_n$ has a sequential labeling if m and n are even, $n > 2$. Now we have the following.

Theorem 3.1. *The planar grid* $P_m \times P_n$ *is odd sequential for* m, $n \geq 2$.

Proof. Let $G = P_m \times P_n$. Let u_{ij} be the (i, j)th vertex of G, for $i = 1, 2, ..., m$; $j = 1, 2, ..., n$. Note that, G is a graph with $2mn - m - n$ edges. Define a function $f : V(G) \to \{0, 1, 2, ..., q = 2mn - m - n\}$ such that $f(u_{ij}) = (i-1) + (2m-1)(j-1)$ for $i = 1, 2, ..., m$; $j = 1, 2, ..., n$.

Also,

$$\max_{v \in V} f(v) = \max\{\max_{1 \leq i \leq m, 1 \leq j \leq n} (i-1) + (2m-1)(j-1)\} = 2mn - m - n,$$

the number of edges of $P_m \times P_n$.

Thus, $f(v) \in \{0, 1, 2, ..., q\}$. Clearly f is injective and it is easily verified that the above defined f is an odd sequential numbering. Thus, the graph G is odd sequential. Hence the theorem.

The numbering for $P_6 \times P_5$ is exhibited in Fig. 3.1

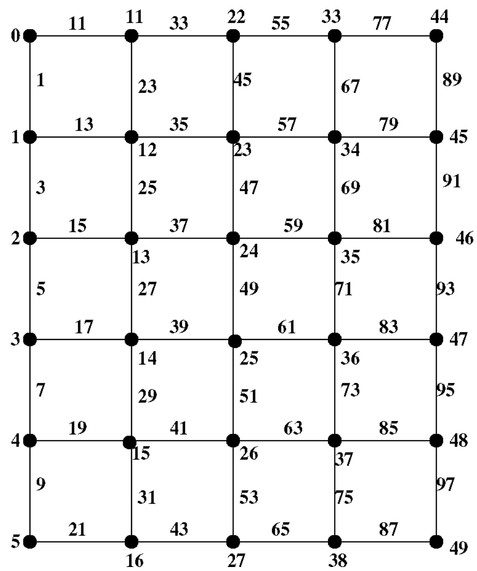

Figure 3.1

4. On Cycle-Related Graphs

In [7], proved that $C_n \odot P_3$ is sequential for all odd n and $K_2 \odot C_n$ is sequential for all odd n. Now, we have the following.

Theorem 4.1. *Odd cycles are not odd sequential.*

Theorem 4.2. C_n *is odd sequential if* $n = 0(\text{mod } 4)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the set of vertices of the cycle C_n and it is of length n.

Define a function $f : V(C_n) \to \{0, 1, 2, \ldots, n\}$ such that $f(v_{2i-1}) = 2i - 1$, $i = 1, 2, \ldots, \frac{n}{2}$, and

$$f(v_{2i}) = \begin{cases} 2(i-1); & i = 1, 2, \ldots, \frac{n}{4} \\ 2i; & i = \frac{n}{4} + 1, \frac{n}{4} + 2, \ldots, \frac{n}{4} + \frac{n}{4} = \frac{n}{2}. \end{cases}$$

Clearly, we can see that f is injective. It is obvious that f is an odd sequential numbering. Hence, the graph C_n is odd sequential.

The odd sequential labeling for C_{16} is exhibited in Fig. 4.1

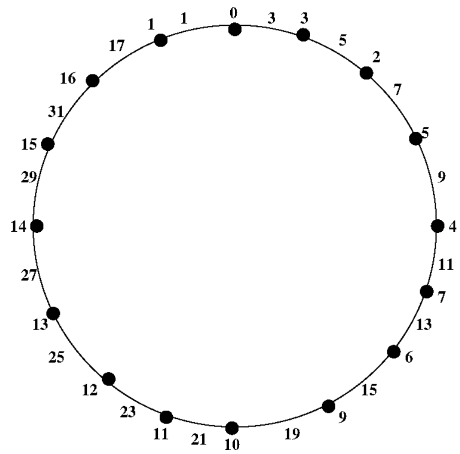

Figure 4.1

5. On Graceful Graphs

Definition 5.1. An α-*valuation* of a graph G is a graceful labeling f of G such that for each edge uv of G either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$ for some integer k, [4].

Definition 5.2. By $H_{n,n}$, we mean that the graph with vertex set $V(H_{n,n}) = \{x_1, x_2, ..., x_n; y_1, y_2, ..., y_n\}$ and the edge set $E(H_{n,n}) = \{x_i y_i \mid 1 \leq i \leq n;\ n - i + 1 \leq j \leq n\}$.

The graph $H_{4,4}$ is as shown in Figure 5.1

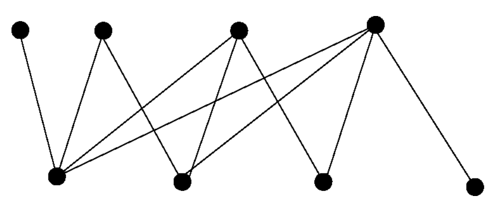

Figure 5.1

Theorem 5.1. *The graph* $H_{n,n}$ *is graceful for all* $n \geq 3$. *Furthermore it has an* α-*valuation.*

Proof. Let (X, Y) be the partition of $H_{n,n}$, where $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$. Note that, $H_{n,n}$ has $2n$ vertices and $\frac{n(n+1)}{2}$ edges.

Define a function $f : V(H_{n,n}) \to \left\{0, 1, 2, ..., \frac{n(n+1)}{2}\right\}$ such that

$$f(x_i) = \frac{n(n+1)}{2} - n + i \text{ for } i = 1, 2, ..., n$$

and

$$f(y_i) = \left(\frac{n(n+1)}{2} - n\right) - \left(\frac{i(i+1)}{2} - i\right)$$

for $i = 1, 2, ..., n$. Clearly, the definition of f tells that f is injective. Also, it is obvious that f is a graceful labeling. Hence, the graph $H_{n,n}$ is graceful.

Let $K = \max f(y_i)$ for all $i = 1, 2, ..., n$. Then $f(x_i) > K$ for all $i = 1, 2, ..., n$.

Therefore $f(y_i) \leq K < f(x_i)$ for all $i = 1, 2, ..., n$. Therefore, f is an α-valuation.

The graceful numbering for $H_{4,4}$ is exhibited in Figure 5.2.

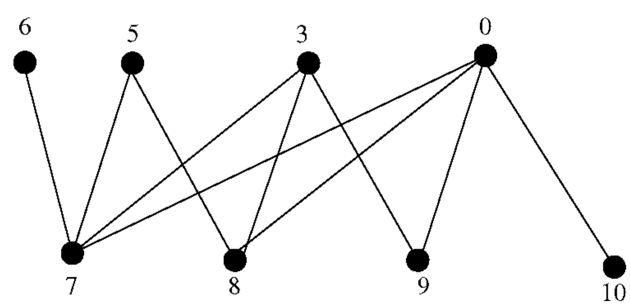

Figure 5.2

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