Far East J. Math. Sci. (FJMS) **26**(2) (2007), 419-431
This paper is available online at http://www.pphmj.com

COMPOSITION OPERATORS BETWEEN VECTOR-VALUED WEIGHTED BERGMAN SPACES

MAOFA WANG

School of Mathematics and Statistics
Wuhan University
P. R. China

Abstract

In this paper, we study the boundedness of composition operators between different version vector-valued weighted Bergman spaces. Some surprising sufficient and necessary conditions for such composition operators to be bounded are obtained, which are an extension of the characterizations for these composition operators to be Hilbert-Schmidt class on the corresponding scalar-valued weighted Bergman spaces.

1. Introduction

Throughout this paper, D denotes the open unit disk $\{z \in \mathrm{C} : |z| < 1\}$ in the finite complex plane C. Let X be any complex Banach space and $\alpha > -1$ the vector-valued weighted Bergman space $A_\alpha^2(X)$ consists of all vector-valued analytic functions $f : \mathrm{D} \to X$ such that

$$\| f \|_{A_\alpha^2(X)} = \left(\int_{\mathrm{D}} \| f(z) \|_X^2 \, dA_\alpha(z) \right)^{\frac{1}{2}} < \infty,$$

2000 Mathematics Subject Classification: 46E15, 47B38.

Keywords and phrases: vector-valued analytic function, Bergman space, composition operator, boundedness, Hilbert-Schmidt class.

Foundation item: National Natural Science Foundation of China.

Received March 16, 2007

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and dA is the normalized 2-dimensional Lebesgue area measure on the unit disk D. These classes of vector-valued Bergman spaces have been quite intensively studied, see, e.g., [3] for a recent survey of vector-valued Bergman spaces. The following weak version of these spaces were considered by, e.g., Blasco [1] and Bonet et al. [4]: the weak version Bergman spaces $wA_\alpha^2(X)$ consists of all vector-valued analytic functions $f : \mathrm{D} \to X$ for which

$$\| f \|_{wA_\alpha^2(X)} = \sup_{\| x^* \| \le 1} \| x^* \circ f \|_{A_\alpha^2(\mathrm{C})} < \infty.$$

(In fact, such weak spaces $wE(X)$ can be introduced under more general conditions on any Banach spaces E consisting of analytic functions $\mathrm{D} \to \mathrm{C}$, see, e.g., [4]).

Let φ be an analytic self-map of D into itself. It is known from Littlewood's subordination theorem that the analytic composition operator

$$C_\varphi : f \to f \circ \varphi$$

always defines a bounded linear operator on $A_\alpha^2(X)$ for any Banach spaces X and $\alpha > -1$ see, e.g., [15]. The operator C_φ is also bounded on the weak spaces $wA_\alpha^2(X)$, see, e.g., [4, 15]. For the case that $X = \mathrm{C}$, composition operators acting on spaces of scalar-valued analytic functions in the disk D have been well understood. In particular, Smith [18] gave some characterizations in terms of the generalized Nevanlinna counting functions of the inducing map φ for C_φ to be bounded between the different Bergman spaces. Now it is a natural question whether it is possible to characterize the class of analytic maps $\varphi : \mathrm{D} \to \mathrm{D}$ for which the operator C_φ is bounded between these strong and weak Bergman spaces. This problem is motivated by the fact that the spaces $A_\beta^2(X)$ and $wA_\alpha^2(X)$ are different significantly for any infinite-dimensional Banach space X and $\alpha \ge \beta > -1$. In fact, $A_\beta^2(X) \subseteq wA_\alpha^2(X)$, $A_\beta^2(X) \ne wA_\alpha^2(X)$ and the norm $\|\cdot\|_{A^2_\beta(X)}$ is not equivalent to $\|\cdot\|_{wA^2_\alpha(X)}$ on $A^2_\beta(X)$, see [11, 13, 14] for more details and concrete examples. Thus the properties of C_φ from $wA^2_\alpha(X)$ to $A^2_\beta(X)$ can reflect the differences between these weak and strong spaces. Note that $wA^2_\alpha(\mathbb{C}) = A^2_\alpha(\mathbb{C})$ for any $\alpha > -1$ (for simplification, denoted by A^2_α the space $A^2_\alpha(\mathbb{C})$), so that it goes to the classic case.

Our main results show that for $\alpha, \beta > -1$ and any infinite-dimensional Banach space X, the operator $C_\varphi : wA^2_\alpha(X) \to A^2_\beta(X)$ is bounded if and only if

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2}} dA(z) < \infty. \tag{1}$$

It is noticing that this result was given for $\alpha = \beta = 0$ by Laitila et al. [14]. The appearance of the condition as above is perhaps surprising. In fact, we shall see that φ satisfies the condition as above if and only if $C_\varphi : A^2_\alpha \to A^2_\beta$ is a Hilbert-Schmidt operator (that is, $\sum_{n=1}^\infty \|C_\varphi e_n\|^2_{A^2_\beta} < \infty$ for all orthonormal bases $\{e_n\}$ of A^2_α). Although our main results maybe hold for more other spaces, but they must not be a general phenomenon as we know in [14] that $C_\varphi : wBMOA(l^2) \to BMOA(l^2)$ is bounded if and only if $C_\varphi : \mathcal{B} \to BMOA(\mathbb{C})$ is bounded, which is not related to the Hilbert-Schmidt conditions, where $\mathcal{B}$ is the Bloch space, $wBMOA(X)$ and $BMOA(X)$ the weak and strong spaces of X-valued analytic functions of bounded mean oscillation, see these definitions in [12, 14].

2. Composition Operators between Bergman Spaces

In this section we first give the following Hilbert-Schmidt conditions of composition operators, which does not seem to have been made explicit in the literature. Our work uses a characterization of measures μ on $\mathbb{D}$

which satisfies $A_\alpha^2 \subset L^2(\mathrm{D}, d\mu)$. This characterization is given using what are called *Carleson measure criteria*. For $\xi \in \partial\mathrm{D}$ and $0 < \delta < 2$, let $S(\xi, \delta) = \{z \in \mathrm{D} : |z - \xi| < \delta\}$, we say a finite positive Borel measure μ on D is η-*Carleson measure* if $\sup_{\xi\in\partial\mathrm{D},\, 0<\delta<2} \mu(S(\xi, \delta))/\delta^\eta < \infty$.

The following Carleson measure criteria for the inclusion $A_\alpha^2 \subset L^2(\mathrm{D}, d\mu)$ is known.

Lemma 1 [16]**.** *Let* μ *be a finite positive Borel measure* μ *on* D *and* $\alpha > -1$. *Then* $A_\alpha^2 \subset L^2(\mathrm{D}, d\mu)$ *if and only if* μ *is an* $\alpha + 2$*-Carleson measure.*

Some complicated calculation can give the following elementary estimation that will be applied below.

Lemma 2 [14]**.** *Let* $\alpha > -1$ *and* $1/2 \le t < 1$. *Then there is a constant* $c = c(\alpha) > 0$ *such that*

$$\sum_{k=1}^{\infty} k^\alpha t^k \ge \frac{c}{(1-t)^{\alpha+1}}.$$

Theorem 1. *Let* φ *be an analytic self-map of* D *and* $\alpha, \beta > -1$. *Then* $C_\varphi : A_\alpha^2 \to A_\beta^2$ *is Hilbert-Schmidt if and only if*

$$\int_{\mathrm{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2}}\, dA(z) < \infty. \tag{2}$$

Proof. First we notice that the condition (2) implies $C_\varphi : A_\alpha^2 \to A_\beta^2$ is bounded. In fact,

$$\|C_\varphi f\|_{A_\beta^2}^2 = \int_{\mathrm{D}} |f \circ \varphi(z)|^2 dA_\beta(z) = \int_{\mathrm{D}} |f(w)|^2 dA_\beta \circ \varphi^{-1}(w),$$

where the last equality used a change of variables formula from measure theory, see, e.g., [8]. So $C_\varphi : A_\alpha^2 \to A_\beta^2$ is bounded if and only if $dA_\beta \circ \varphi^{-1}$ is $\alpha + 2$-Carleson measure by Lemma 1, that is,

$$\sup_{\xi \in \partial D, 0<\delta<2} \frac{\int_{S(\xi, \delta)} dA_\beta \circ \varphi^{-1}(w)}{\delta^{\alpha+2}} < \infty,$$

where $S(\xi, \delta) = \{z \in D : |z - \xi| < \delta\}$ is the Carleson box. And

$$\sup_{\xi \in \partial D, 0<\delta<2} \frac{\int_{S(\xi, \delta)} dA_\beta \circ \varphi^{-1}(w)}{\delta^{\alpha+2}} = \sup_{\xi \in \partial D, 0<\delta<2} \int_D \frac{1_{S(\xi, \delta)}(\varphi(z))}{\delta^{\alpha+2}} dA_\beta(z) < \infty,$$

which is obvious from the condition (2), where $1_{S(\xi, \delta)}$ is the characterization function of $S(\xi, \delta)$.

Since $\left\{\frac{z^n}{\beta(n)}\right\}_{n=0}^{+\infty}$ is an orthonormal basis for A_α^2, so we have $C_\varphi : A_\alpha^2 \to A_\beta^2$ is Hilbert-Schmidt if and only if

$$+\infty > \sum_{n=0}^{+\infty} \left\| C_\varphi \frac{z^n}{\beta(n)} \right\|_{A_\beta^2}^2 = \sum_{n=0}^{+\infty} \int_D \frac{|\varphi^n(z)|^2}{\beta(n)^2} (\beta + 1)(1 - |z|^2)^\beta dA(z)$$

$$= \int_D \sum_{n=0}^{+\infty} \frac{1}{\beta(n)^2} |\varphi(z)|^{2n} (\beta + 1)(1 - |z|^2)^\beta dA(z)$$

$$\sim \int_D \left(\sum_{n=0}^{+\infty} n^{\alpha+1} |\varphi(z)|^{2n} \right) (1 - |z|^2)^\beta dA(z)$$

$$\sim \int_D \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+2}} dA(z),$$

where

$$\beta(n)^2 = \| z^n \|_{A_\alpha^2}^2 = \int_D |z^n|^2 (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$$

$$= (\alpha + 1) \int_0^1 \int_0^{2\pi} r^{2n} (1 - r^2)^\alpha r dr \frac{d\theta}{\pi}$$

$$= (\alpha + 1) \int_0^1 r^{2n} (1 - r^2)^\alpha 2r dr$$

$$= (\alpha + 1)\int_0^1 x^n (1 - x)^\alpha dx$$

$$= (\alpha + 1)B(n + 1, \alpha + 1) \sim n^{-(\alpha+1)},$$

where $B(p, q)$ is the Beta function and the last estimate comes from Stirling's formula. Here and below the relation $A \sim B$ means $c_1 A \leq B \leq c_2 A$ for some inconsequential constants $c_1, c_2 > 0$. □

Now we estimate the norm of the composition operator $C_\varphi : A_\alpha^2(X) \to A_\beta^2(X)$ for arbitrary infinite-dimensional complex Banach space X and $\alpha, \beta > -1$. We shall use the following Dvoretzky's well-known theorem [7]:

Lemma 3 [7]. *If X is an infinite-dimensional Banach space, then for any $n \in N$ and $\varepsilon > 0$ there is a linear embedding operator $T_n : l_2^n \to X$ so that*

$$(1 + \varepsilon)^{-1}\left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \leq \left\|\sum_{j=1}^{n} a_j T_n e_j\right\| \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \tag{3}$$

for any scalars $a_1, a_2, \ldots, a_n$. Here $\{e_1, \ldots, e_n\}$ is some fixed orthonormal basis of l_2^n.

Theorem 2. *Let X be any infinite-dimensional complex Banach space, $\alpha, \beta > -1$ and φ be an analytic self-map of* D. *Then $C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X)$ is bounded if and only if*

$$\int_{\mathrm{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+2}} dA(z) < \infty.$$

Moreover

$$\|C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X)\| \sim \left(\int_{\mathrm{D}} \frac{(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+2}} dA(z)\right)^{1/2}. \tag{4}$$

Proof. Any analytic function $f : \mathrm{D} \to \mathrm{C}$ satisfies $|f(z)|^2 \leq \frac{1}{(1-|z|^2)^{\alpha+2}} \| f \|_{A_\alpha^2}^2$ for any $z \in \mathrm{D}$ (see, e.g., [9]). Thus

$$\| f(z) \|_X^2 = \sup_{\|x^*\| \leq 1} |(x^* \circ f)(z)|^2 \leq \frac{1}{(1-|z|^2)^{\alpha+2}} \| f \|_{wA_\alpha^2(X)}^2,$$

for $f \in wA_\alpha^2(X)$. Consequently,

$$\| C_\varphi f \|_{A_\beta^2(X)}^2 = \int_{\mathrm{D}} \| f(\varphi(z)) \|_X^2 \, dA_\beta(z)$$

$$\leq \| f \|_{wA_\alpha^2(X)}^2 \int_{\mathrm{D}} \frac{(\beta+1)(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2}} dA(z).$$

So that

$$\| C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X) \|^2 \leq (\beta+1) \left(\int_{\mathrm{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2}} dA(z) \right).$$

For the converse estimation, we let $x \in X$ with $\| x \| = 1$ and consider the constant function $g(z) = x$ on D. Clearly $\| g \|_{wA_\alpha^2(X)} = 1$, so that $\| C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X) \| \geq \| g \circ \varphi \|_{A_\beta^2(X)} = \| x \| = 1$. So that

$$\int_{\left\{ z \in \mathrm{D} : |\varphi(z)|^2 < \frac{1}{2} \right\}} \frac{(\beta+1)(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2}} dA(z)$$

$$\leq 2^{\alpha+2} \left| \left\{ z \in \mathrm{D} : |\varphi(z)|^2 < \frac{1}{2} \right\} \right|$$

$$\leq 2^{\alpha+2} \leq 2^{\alpha+2} \| C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X) \|^2,$$

where $\left| \left\{ z \in \mathrm{D} : |\varphi(z)|^2 < \frac{1}{2} \right\} \right|$ is the normalized A_β-area measure of $\left\{ z \in \mathrm{D} : |\varphi(z)|^2 < \frac{1}{2} \right\}$. Consequently it will be enough to show that there is a uniform constant $K > 0$ such that

$$\int_{\left\{z\in D:|\varphi(z)|^2\geq\frac{1}{2}\right\}}\frac{(\beta+1)(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha+2}}dA(z)\leq K\|C_{\varphi}:wA_{\alpha}^2(X)\to A_{\beta}^2(X)\|^2. \quad (5)$$

Let $n \in N$ and $\varepsilon > 0$. Use Dvoretzky's theorem to fix a linear embedding $T_n : l_2^n \to X$ so that $\| T_n \| = 1$ and $\| T_n^{-1} \| \leq 1 + \varepsilon$ as in Lemma 3. Let $x_k^{(n)} = T_n e_k$ for $k = 1, ..., n$, where $\{e_1, ..., e_n\}$ is some fixed orthonormal basis of l_2^n. Now consider the sequence (f_n) of analytic polynomials $D \to X$ defined by

$$f_n(z) = \sum_{k=1}^{n} k^{\frac{\alpha+1}{2}} z^k x_k^{(n)} = T_n\left(\sum_{k=1}^{n} k^{\frac{\alpha+1}{2}} z^k e_k\right), \quad z \in D.$$

Since

$$\left\|\sum_{k=1}^{n} k^{\frac{\alpha+1}{2}} a_k z^k\right\|_{A_\alpha^2} \sim \int_D \left|\sum_{k=1}^{n} k^{\frac{\alpha+1}{2}} a_k z^k\right|^2 \left(\log\frac{1}{|z|^2}\right)^{\alpha} \frac{dA(z)}{\Gamma(\alpha+1)}$$

$$= \left(\sum_{k=1}^{n}\left(\frac{k}{1+k}\right)^{\alpha+1}|a_k|^2\right)^{1/2} \leq \left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2},$$

for all $n \in N$ and complex polynomials $\sum_{k=1}^{n} a_k z^k$, here $\Gamma(s)$ stands for the usual Gamma function. Hence there is a constant $c_1 > 0$ so that for $x^* \in B_{X^*}$, we get from the preceding estimation that

$$\| x^* \circ f_n \|_{A_\alpha^2} = \left\|\sum_{k=1}^{n} k^{\frac{\alpha+1}{2}} x^*(x_k^{(n)}) z^k\right\|_{A_\alpha^2} \leq c_1\left(\sum_{k=1}^{n}|x^*(x_k^{(n)})|^2\right)^{1/2} \leq c_1.$$

Thus

$$\sup_n \| f_n \|_{wA_\alpha^2(X)} \leq c_1$$

and

$$\| C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X) \| \geq (1/c_1)\sup_n \| f_n \circ \varphi \|_{A_\beta^2(X)}$$

for any n. In particular, by monotone convergence theorem we get that

$$\| C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X) \|^2$$

$$\geq \frac{1}{c_1^2} \limsup_n \int_{\mathrm{D}} \left\| \sum_{k=1}^n k^{\frac{\alpha+1}{2}} \varphi(z)^k x_k^{(n)} \right\|_X^2 dA_\beta(z)$$

$$\geq \frac{1}{c_1^2(1+\varepsilon)^2} \limsup_n \int_{\mathrm{D}} \left\| \sum_{k=1}^n k^{\frac{\alpha+1}{2}} \varphi(z)^k e_k \right\|_{l_2^n}^2 dA_\beta(z)$$

$$= \frac{1}{c_1^2(1+\varepsilon)^2} \limsup_n \int_{\mathrm{D}} \left(\sum_{k=1}^n k^{\alpha+1} |\varphi(z)|^{2k} \right) dA_\beta(z)$$

$$\geq \frac{1}{c_1^2(1+\varepsilon)^2} \int_{\mathrm{D}} \left(\sum_{k=1}^\infty k^{\alpha+1} |\varphi(z)|^{2k} \right) dA_\beta(z).$$

Recall next that

$$\sum_{k=1}^\infty k^{\alpha+1} |\varphi(z)|^{2k} \geq \frac{c_2}{(1 - |\varphi(z)|^2)^{\alpha+2}}$$

holds for all $z \in D$ satisfying $|\varphi(z)|^2 \geq 1/2$, by applying Lemma 2. By combining these estimations we get that

$$\| C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X) \|^2$$

$$\geq \frac{1}{c_1^2(1+\varepsilon)^2} \int_{\left\{z \in \mathrm{D}: |\varphi(z)|^2 \geq \frac{1}{2}\right\}} \left(\sum_{k=1}^\infty k^{\alpha+1} |\varphi(z)|^{2k} \right) dA_\beta(z)$$

$$\geq \frac{c_2(\beta+1)}{c_1^2(1+\varepsilon)^2} \int_{\left\{z \in \mathrm{D}: |\varphi(z)|^2 \geq \frac{1}{2}\right\}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+2}} dA(z).$$

This proves the claim with $K = c_1^2(1+\varepsilon)^2 c_2^{-1}(\beta+1)^{-1}$, which completes the proof. □

Remark 1. From Theorems 1 and 2 we know that $C_\varphi : wA_\alpha^2(X)$

$\to A_\beta^2(X)$ is bounded for any infinite-dimensional complex Banach space X if and only if $C_\varphi : A_\alpha^2 \to A_\beta^2$ is a Hilbert-Schmidt operator. Moreover, if φ maps D into a compact subset of D, then $C_\varphi : A_\alpha^2 \to A_\beta^2$ is a Hilbert-Schmidt operator. So it is easy to give some bounded $C_\varphi : wA_\alpha^2(X) \to A_\beta^2(X)$. On the other hand, there exists compact $C_\varphi : A_\alpha^2 \to A_\alpha^2$ which is not Hilbert-Schmidt see [6], that is, in contrast to the scalar-valued case, there are some unbounded composition operators $C_\varphi : wA_\alpha^2(X) \to A_\alpha^2(X)$.

Remark 2. Because the measure dA_α converges to $\frac{d\theta}{2\pi}$ in the weak-star topology as α goes to -1, that is, for any bounded analytic function in D,

$$\lim_{\alpha\to -1}\int_{\mathrm{D}} |f(z)|^2 dA_\alpha(z) = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

So if we let $\alpha = \beta = -1$, then we have $C_\varphi : wH^2(X) \to H^2(X)$ is bounded for any infinite-dimensional Banach spaces X if and only if $\int_0^{2\pi} \frac{1}{1-|\varphi(e^{i\theta})|^2}\frac{d\theta}{2\pi} < \infty$. Here the Hardy space $H^2(X)$ is the set of analytic functions $f : \mathrm{D} \to X$ satisfying

$$\| f \|_{H^2(X)}^2 = \sup_{0<r<1}\int_0^{2\pi} \| f(re^{i\theta})\|_X^2 \frac{d\theta}{2\pi} < \infty$$

and $wH^2(X)$ is the set of analytic functions $f : \mathrm{D} \to X$ with

$$\| f \|_{wH^2(X)} = \sup_{\| x^* \|\le 1}\| x^* \circ f \|_{H^2(\mathrm{C})} < \infty.$$

3. Composition Operators between Other Spaces

Let $(E, \|\cdot\|_E)$ be a Banach space consisting of analytic functions $f : \mathrm{D} \to \mathrm{C}$ such that

(i) E contains the constant functions,

(ii) the unit ball B_E of E is τ_k-compact, where τ_k is the topology of uniform convergence on compact subsets of D.

In this case one may define corresponding weak vector-valued spaces $wE(X)$ for any complex Banach space X, where the analytic function $f : \mathrm{D} \to X$ belongs to $wE(X)$ if

$$\| f \|_{wE(X)} = \sup_{x^* \in B_{X^*}} \| x^* \circ f \|_E < \infty.$$

The space $wE(X)$ is a Banach space which is isometrically isomorphic to $L(V_*, X)$, where V_* is a certain predual of E, see [4].

Composition operators on such weak spaces of analytic functions were studied in [4, 11, 13]. It is easy to check that C_φ is bounded $wE(X) \to wE(X)$ if and only if $C_\varphi : E \to E$ is bounded. This fact suggests the general problem of characterizing the bounded operator $C_\varphi : wE(X) \to E(X)$, provided $E(X)$ has some fixed definition.

Remark 3. Careful examination of the proof of our main results can give a more general result (see Theorem 3 below), whose proof is similar to those of Theorems 1 and 2, we omit its details to the reader.

Theorem 3. *Let* H_1, H_2 *be two weighted Hardy spaces defined in the disk* D, *and* $\varphi : \mathrm{D} \to \mathrm{D}$ *is analytic. Then* $C_\varphi : wH_1(X) \to H_2(X)$ *is bounded if and only if* $C_\varphi : H_1 \to H_2$ *is Hilbert-Schmidt, where* $wH_1(X)$, $H_2(X)$ *are the corresponding weak and strong vector-valued spaces of* H_1 *and* H_2 *for any infinite-dimensional Banach space* X *for some fixed definitions.*

Remark 4. But Theorem 3 is not a special case for more general phenomenon. In fact, a counter-example about the characterization of analytic self-maps φ of D for which $C_\varphi : wBMOA(l_2) \to BMOA(l_2)$ to be bounded are given in [14], where the resulting condition turns out to be unrelated to the Hilbert-Schmidt conditions. Recall that $BMOA(X)$

consists of the analytic functions $f : \mathrm{D} \to X$ for which

$$\| f \|_{BMOA(X)} = \| f(0) \|_X + \sup_{a \in \mathrm{D}} \| f \circ \sigma_a - f(a) \|_{H^2(X)} < \infty.$$

Here $\sigma_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ is a Möbius transformation related to $a \in \mathrm{D}$. It is known that the weak space $wBMOA(X)$ differs from $BMOA(X)$ for any infinite-dimensional X, see [11, 12].

Example [14]**.** $C_\varphi : wBMOA(l_2) \to BMOA(l_2)$ is bounded if and only if

$$\sup_{a \in \mathrm{D}} \int_{\mathrm{D}} \frac{| \varphi'(z) |^2 (1 - | \sigma_a(z) |^2)}{(1 - | \varphi(z) |^2)^2} dA(z) < \infty \tag{6}$$

if and only if $C_\varphi : \mathcal{B} \to BMOA$ is bounded, where $\mathcal{B}$ is the Bloch space.

Acknowledgement

The author wishes to thank Prof. Hans-Olav Tylli and Dr. Jussi Laitila for their inspiring thoughts and valuable discussions.

References

[1] O. Blasco, Boundary values of vector-valued harmonic functions considered as operators, Studia Math. 86 (1987), 19-33.

[2] O. Blasco, Remarks on vector-valued BMOA and vector-valued multipliers, Positivity 4 (2000), 339-356.

[3] O. Blasco, Introduction to vector-valued Bergman spaces, University of Joensuu, Department of Mathematics, Report Series 8, 2005, pp. 9-30.

[4] J. Bonet, P. Domanski and M. Lindström, Weakly compact composition operators on analytic vector-valued function spaces, Ann. Acad. Sci. Fenn. Math. 26 (2001), 233-248.

[5] D. M. Boyd, Composition operators on the Bergman spaces, Colloq. Math. 34 (1975), 127-136.

[6] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.

[7] J. Diestel, H. Jarchow and A. Tonge, Absolutely Summing Operators, Cambridge University Press, Cambridge, 1995.

[8] P. R. Halmos, Measure Theory, Springer-Verlag, New York, 1974.

[9] H. Hedenmalm, B. Korenblum and K. Zhu, Theory of Bergman Spaces, Springer-Verlag, New York, 2000.

[10] Y. Katznelson, An Introduction to Harmonic Analysis, Dover, New York, 1976.

[11] J. Laitila, Weakly compact composition operators on vector-valued BMOA, J. Math. Anal. Appl. 308 (2005), 730-745.

[12] J. Laitila, Composition operators on vector-valued BMOA and related function spaces, Ph.D. Thesis, University of Helsinki, 2006.

[13] J. Laitila and H.-O. Tylli, Composition operators on vector-valued harmonic functions and Cauchy transforms, Indiana Univ. Math. J. 55 (2006), 719-746.

[14] J. Laitila, H.-O. Tylli and M. Wang, Composition operators from weak to strong spaces of vector-valued analytic functions (to appear).

[15] P. D. Liu, E. Saksman and H.-O. Tylli, Small composition operators on analytic vector-valued function spaces, Pacific J. Math. 184 (1998), 295-309.

[16] D. Luecking, Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, Amer. J. Math. 107 (1985), 85-111.

[17] J. H. Shapiro, The essential norm of a composition operator, Annals of Math. 125 (1987), 375- 404.

[18] W. Smith, Composition operators between Bergman and Hardy spaces, Trans. Amer. Math. Soc. 248(6) (1996), 2331-2348.

[19] M. Tjani, Compact composition operators on some Möbius invariant Banach spaces, Ph.D. Thesis, Michigan State University, 1996.

[20] D. Vukotic, On the coefficient multipliers of Bergman spaces, J. London Math. Soc. 50 (1994), 341-348.