Far East J. Math. Sci. (FJMS) **26**(2) (2007), 345-358
This paper is available online at http://www.pphmj.com

ON THE THEORY OF LARGE DEVIATIONS AND MULTIPLE HYPOTHESES LAO TESTING FOR MANY INDEPENDENT OBJECTS

LEADER NAVAEI and MASOUD YARMOHAMMADI

Azad Eslamic University, Bonab, Iran
e-mail: ashkan_l1380@yahoo.com

Payame Noor University, Tehran, Iran
e-mail: masyar@pnu.ac.ir

Abstract

In this paper using the theory of large deviations, many hypotheses testing for a model consisting of three or more independent objects are studied (this problem has been proposed by Ahlswede and Haroutunian). We assume that M probability distributions are given and objects independent of each other follow to one among them. Through Sanov's theorem and its applications in hypotheses testing, we expand this procedure for the calculation of the matrix of all possible pairs of the error probability exponents in optimal testing.

1. Introduction

The approach to statistical problems based on considering probabilities of large deviations has been in use to statistical inference since the paper by Bahadur [3] where considering the problem of discrimination between two simple hypotheses he showed that, if the hypotheses are fixed, then the error probabilities decrease exponentially as the sample size tends to infinity; the corresponding optimal exponent

2000 Mathematics Subject Classification: 62P30.

Keywords and phrases: large deviation rate function, hypotheses testing, error probability, independent objects, logarithmically asymptotically optimal (LAO), reliabilities.

Received February 27, 2007

is specified by what is now known as Chernoff's function.

Recently Ahlswede and Haroutunian formulated an ensemble of new problems on multiple hypotheses testing for many objects on identification of hypotheses. The problem of hypotheses testing for the model consisting of two independent objects with two possible hypothetical distributions was investigated in [1, 2] (without usage of the theory of large deviations).

Our aim in the present paper is to obtain the solution of the problem proposed by Ahlswede and Haroutunian that generalizes those investigated in [11] via the theory of large deviations for testing of many hypotheses concerning one object. In fact, we study the model consisting of $K(\geq 3)$ objects which independently follow to one of the given $M(\geq 2)$ probability distributions. Recently, Tuncel [12] also published an interesting consideration of the problem of multiple hypotheses optimal testing, which differs from the approach provided in [1, 2], [11].

In the next section we recall main definitions, notations, basic concepts, theorems for the case of one object and theory of large deviation techniques and in Section 3 we formulate and prove the results on three independent objects testing.

2. Preliminaries

The large deviation principle (LDP) characterizes the limiting behavior, as $\delta \to 0$, of a family of probability measures $\{P_\delta\}$ on $(\mathcal{X}, \mathcal{B})$ in terms of a rate function. For any set $\mathcal{A}$, $\overline{\mathcal{A}}$ denotes the closure of $\mathcal{A}$, $\mathcal{A}^0$ the interior of $\mathcal{A}$ and $\mathcal{A}^c$ the complement of $\mathcal{A}$.

Definition 1. A *rate function* I is a lower semicontinuous mapping $I : \mathcal{X} \to [0, \infty)$.

Define the *level set* $M_I(\gamma) \overset{\Delta}{=} \{x : I(x) \leq \gamma\}, \quad \forall \gamma \geq 0$. It is a closed subset of $\mathcal{X}$.

A *good rate function* is a rate function for which all the level sets $M_I(\gamma)$ are compact subsets of $\mathcal{X}$.

The *effective domain* of I, denoted by $\mathcal{D}_I$, is the set of points in $\mathcal{X}$ of finite rate, that is, $\mathcal{D}_I \triangleq \{x : I(x) < \infty\}$.

Definition 2. $\{P_\gamma\}$ satisfies the LDP with a rate function I if, for all $\mathcal{A} \in \mathcal{B}$,

$$-\inf_{x \in \mathcal{A}^0} I(x) \leq \liminf_{\delta \to 0} \delta \log P_\delta(\mathcal{A}) \leq \limsup_{\delta \to 0} \delta \log P_\delta(\mathcal{A}) \leq -\inf_{x \in \overline{\mathcal{A}}} I(x). \quad (1)$$

Let $\mathcal{A} \in \mathcal{B}$ and P_δ satisfy the LDP and also,

$$\inf_{x \in \mathcal{A}^0} I(x) = \inf_{x \in \overline{\mathcal{A}}} I(x) \triangleq I_{\mathcal{A}}. \quad (2)$$

Then

$$\lim_{\delta \to 0} \log P_\delta(\mathcal{A}) = -I_{\mathcal{A}}. \quad (3)$$

The set $\mathcal{A}$ that satisfies (2) is called an I *continuity set*. In general, the LDP implies a precise limit in (3) only for I continuity sets.

Before paying attention to Sanov's theorem it is necessary to describe the concept of empirical distribution.

Let $\mathcal{X} = \{1, 2, \ldots, K\}$ be a finite set of size K. Then the set of all probability distributions (PDs) on $\mathcal{X}$ is denoted by $\mathcal{P}(\mathcal{X})$. For PDs, P and Q, $H(P)$ denotes entropy and $D(P \| Q)$ denotes the information divergence (or the Kullback-Leibler distance):

$$H(P) \triangleq -\sum_{x \in \mathcal{X}} P(x) \log P(x),\ D(P \| Q) \triangleq \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}.$$

The type of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathcal{X}^N$ is the empirical distribution given by $Q(x) \triangleq N^{-1} \cdot N(x \mid \mathbf{x})$ for all $x \in \mathcal{X}$, where $N(x \mid \mathbf{x})$ denotes the number of occurrences of x in $\mathbf{x}$.

The subset of $\mathcal{P}(\mathcal{X})$ consisting of all possible types of sequences $\mathbf{x} \in \mathcal{X}^N$ is denoted by $\mathcal{P}_N(\mathcal{X})$. For $Q \in \mathcal{P}_N(\mathcal{X})$ the set of sequences of type class Q will be denoted by $\mathcal{T}_Q^N(X)$.

Notice that by the definition of $Q(x)$ we can show that $I(x)$ is equal with Kullback-Leibler distance, for more details see [7].

Theorem 1 (Sanov's theorem) [6, 11]**.** *Let $\mathcal{A}$ be a set of distributions from $\mathcal{P}$ such that its closure is equal to the closure of its interior. Then for the empirical distribution $Q_{\mathbf{x}}$ of a vector $\mathbf{x}$ from a strictly positive distribution P on $\mathcal{X}$:*

$$\lim_{N\to\infty}\left(-\frac{1}{N}\log P^N(\mathbf{x} : Q_{\mathbf{x}} \in \mathcal{A})\right) = \inf_{Q_{\mathbf{x}}\in\mathcal{A}} D(Q_{\mathbf{x}} \,\|\, P).$$

Let $\mathcal{X} = \{1, 2, \ldots, K\}$ be the finite set such that M incompatible hypotheses $H_1, H_2, \ldots, H_M$ consist in that the random variable X taking values on $\mathcal{X}$ has one of M distributions $P_1, P_2, \ldots, P_M$. For decision making N independent experiences are carried out.

By means of non-randomized test $\varphi_N(\mathbf{x})$ on the basis of a sample $\mathbf{x}$ of length N we must accept one of the hypotheses. To this aim we can divide the sample space $\mathcal{X}^N$ into M disjoint subsets

$$\mathcal{A}_m^N \overset{\Delta}{=} \{\mathbf{x} : \varphi_N(\mathbf{x}) = m\}, \quad m = \overline{1, M}.$$

The probability of the erroneous acceptance of the hypothesis H_l provided that the hypothesis H_m is true, for $m \neq l$ is denoted as follows:

$$\alpha_{m|l}^N(\varphi_N) \overset{\Delta}{=} P_m^N(\mathcal{A}_l^N) = \sum_{\mathbf{x}\in\mathcal{A}_l^N} P_m^N(\mathbf{x}).$$

For $m = l$ we denote by $\alpha_{m|m}(\varphi_N)$ the probability to reject H_m when it is true and we have

$$\alpha_{m|m}^N(\varphi_N) \overset{\Delta}{=} \sum_{l\neq m} \alpha_{m|l}^N(\varphi_N). \tag{4}$$

The matrix $\mathcal{A}(\varphi_N) \overset{\Delta}{=} \{\alpha_{m|l}^N(\varphi_N)\}$ is called the *power of the test*. We consider the rates of exponential decrease of the error probabilities and call them *reliabilities*

$$E_{m|l}(\varphi) \overset{\Delta}{=} \overline{\lim_{N\to\infty}} -\frac{1}{N}\log \alpha_{m|l}(\varphi_N). \tag{5}$$

From (4) and (5), we derive

$$E_{m|m} = \min_{l\neq m} E_{m|l}. \tag{6}$$

Definition 3. The test sequence $\varphi^* = (\varphi_1, \varphi_2, \dots)$ is called *LAO* if for given values of the elements $E_{1|1}, \dots, E_{M-1|M-1}$ it provides maximal values for all other elements of $E(\varphi^*)$.

Consider for given positive and finite numbers $E_{1|1}, \dots, E_{M-1|M-1}$ the following family of regions:

$$\mathcal{R}_l \overset{\Delta}{=} \{Q : D(Q\,\|\,P_l) \leq E_{l|l}\}, \quad l = \overline{1, M-1}, \tag{7a}$$

$$\mathcal{R}_M \overset{\Delta}{=} \{Q : D(Q\,\|\,P_l) > E_{l|l}\}, \quad l = \overline{1, M-1}, \tag{7b}$$

$$\mathcal{R}_l^N \overset{\Delta}{=} \mathcal{R}_l \cap \mathcal{P}_N(\mathcal{X}), \quad l = \overline{1, M} \tag{7c}$$

and the following numbers:

$$E^*_{l|l} = E^*_{l|l}(E_{l|l}) \overset{\Delta}{=} E_{l|l}, \quad l = \overline{1, M-1}, \tag{8a}$$

$$E^*_{m|l} = E^*_{m|l}(E_{l|l}) \overset{\Delta}{=} \inf_{Q\in\mathcal{R}_l}(D(Q\,\|\,P_m)),\ m = \overline{1, M},\ m \neq l,\ l = \overline{1, M-1}, \tag{8b}$$

$$E^*_{m|M} = E^*_{m|M}(E_{1|1}, \dots, E_{M-1|M-1}) \overset{\Delta}{=} \inf_{Q\in\mathcal{R}_l}(D(Q\,\|\,P_m)),\ m = \overline{1, M-1}, \tag{8c}$$

$$E^*_{M|M} = E^*_{M|M}(E_{1|1}, \dots, E_{M-1|M-1}) \overset{\Delta}{=} \min_{l=\overline{1, M-1}} E_{M|l}. \tag{8d}$$

With assumption $\mathcal{A} = \mathcal{R}_l$, $P = P_m$ in Sanov's theorem for conditions (7), (8) we have (see Figure 1)

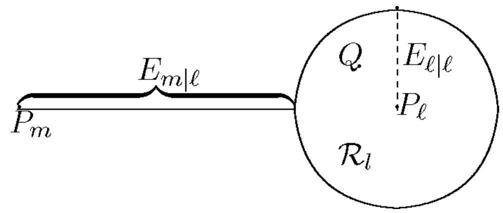

Figure 1. Interpretation of the construction of the test.

$$\lim_{N\to\infty} -\frac{1}{N} \log \alpha^N_{m|l}(\varphi^*_N) = \lim_{N\to\infty} -\frac{1}{N} \log P^N_m(\mathcal{R}_l) = \inf_{Q\in\mathcal{R}_l} D(Q \| P_m). \tag{9}$$

We use the notation $y_1^N \approx y_2^N$, when $g(y_1^N) = g(y_2^N) + \varepsilon_N$, where $\varepsilon_N \to 0$, for $N \to \infty$. Now, using (9) we write

$$E_{m|l}(\varphi^*) \approx \inf_{Q\in\mathcal{R}_l} D(Q \| P_m). \tag{10}$$

Therefore the value of $\alpha_{m|l}(\varphi^*_N)$ is equal to

$$\alpha_{m|l}(\varphi^*_N) \approx \exp(-N \inf_{Q\in\mathcal{R}_l} D(Q \| P_m)) \approx \exp(-NE_{m|l}(\varphi^*_N)). \tag{11}$$

In fact the error probability $\alpha_{m|l}(\varphi_N)$ still goes to zero with exponential rate $\inf_{Q\in\mathcal{R}_l} D(Q \| P_m)$ for P_m not in the set of $\mathcal{R}_l$.

Theorem 2. *For a fixed family of distributions* $P_1, \ldots, P_m$ *on a finite set* $\mathcal{X}$ *the following two statements hold. If positive finite numbers* $E_{1|1}, \ldots, E_{M-1|M-1}$ *satisfy conditions*:

$$E_{1|1} < \min_{l=\overline{2, M}} D(P_l \| P_1),$$

$$\vdots$$

$$E_{M|M} < \min\left[\min_{l=\overline{1, m-1}} E^*_{m|l}(E_{l|l}), \min_{l=\overline{m+1, M}} D(P_l \| P_m)\right], \quad m = \overline{2, M-1}, \tag{12}$$

then:

(a) *There exists a LAO sequence of tests* φ^*_N, *the reliability matrix* $E^* = \{E^*_{m|l}(\varphi^*)\}$ *of which is defined in* (8), *and all the elements* $E^*_{m|m}$ *of it are positive.*

(b) *Even if one of conditions in* (12) *is violated, then the reliability matrix of an arbitrary test necessarily has an element equal to zero* (*the corresponding error probability does not tend exponentially to zero*).

Proof. See [10].

Remark 1. From definitions (8) and (12), it follows that

$$E^*_{m|m} = E^*_{m|M}, \quad m = \overline{1, M-1}, \text{ and } E^*_{m|m} = E^*_{m|l}, \quad l \neq m, M. \tag{13}$$

Remark 2. If one preliminary given element $E_{m|m}$, $m = \overline{1, M}$, of the reliability matrix of an object is equal to zero, then the corresponding element of the matrix determined as functions of $E_{m|m}$, is defined as in the case of Stain's lemma [6]:

$$E'_{l|m} = D(P_m \| P_l), \quad l = \overline{1, M}, \quad l \neq m,$$

and the remaining elements of the matrix are defined by $E_{l|l} > 0$, $l \neq m$, $l = \overline{1, M-1}$, as follows from Theorem 2.

3. LAO Testing of Hypotheses for Three Independent Objects

Let x_1, x_2 and x_3 be independent RV taking values in some finite set $\mathcal{X}$ with one of M PDs, which are characteristics of the corresponding independent objects, the random vector (X_1, X_2, X_3) assume values $(x^1, x^2, x^3) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$.

Let $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = ((x_1^1, x_1^2, x_1^3), \ldots, (x_n^1, x_n^2, x_n^3), \ldots, (x_N^1, x_N^2, x_N^3))$, $x^i \in \mathcal{X}$, $i = 1, 2, 3$, $n = \overline{1, N}$, be a sequence of results of N independent observations of the vector (X_1, X_2, X_3). We must define unknown PDs of the objects on the base of observed data. The selection for each object is denoted by Φ_N. The objects independence test Φ_N may be considered to be the tests φ_N^1, φ_N^2 and φ_N^3 for the respective separate objects. We denote the whole compound test sequence by Φ.

Assume $\alpha_{m_1,m_2,m_3|l_1,l_2,l_3}(\Phi_N)$ to be the probability of the erroneous acceptance by the test Φ_N of the hypotheses $(H_{l_1}, H_{l_2}, H_{l_3})$ provided that $(H_{m_1}, H_{m_2}, H_{m_3})$ is true, where $(m_1, m_2, m_3) \neq (l_1, l_2, l_3)$, m_i, $l_i = \overline{1, M}$, $i = 1, 2, 3$. The probability to reject a true hypothesis $(H_{m_1}, H_{m_2}, H_{m_3})$ in analogy with (4) is the following:

$$\alpha^N_{m_1, m_2, m_3 | m_1, m_2, m_3}(\Phi_N) \stackrel{\Delta}{=} \sum_{(l_1, l_2, l_3) \neq (m_1, m_2, m_3)} \alpha^N_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi_N). \quad (14)$$

We also study corresponding limits $E_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi_N)$ of error probability exponents of the sequence of tests Φ, called *reliabilities*:

$$E_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi)$$

$$\stackrel{\Delta}{=} \overline{\lim_{N \to \infty}} -\frac{1}{N} \log \alpha_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi_N), \quad m_i, l_i = \overline{1, M}, \quad i = 1, 2, 3. \quad (15)$$

We denote by $E(\varphi^i)$ the reliability matrices of the sequences of tests φ^i, $i = 1, 2, 3$, for each of the objects.

Using (14) and (15), it follows that

$$E_{m_1, m_2, m_3 | m_1, m_2, m_3}(\Phi) = \min_{(l_1, l_2, l_3) \neq (m_1, m_2, m_3)} E_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi). \quad (16)$$

In this section we use the following lemma.

Lemma 1. *If elements* $E_{m|l}(\varphi^i)$, $m, l = \overline{1, M}$, $i = 1, 2, 3$, *are strictly positive*, *then the following equalities hold for* $\Phi = (\varphi^1, \varphi^2, \varphi^3)$:

$$E_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi) = \sum_{i=1}^{3} E_{m_i | l_i}(\varphi^i), \text{ if } m_i \neq l_i, \ i = 1, 2, 3, \quad (17a)$$

$$E_{m_1, m_2, m_3 | l_1, l_2, l_3}(\Phi)$$

$$= \sum_{i \neq k} E_{m_i | l_i}(\varphi^i), \text{ if } m_k = l_k, \ m_i \neq l_i, \ i \neq k, \ i, k = 1, 2, 3, \quad (17b)$$

$$E_{m_1, m_2, m_3 \mid l_1, l_2, l_3}(\Phi)$$

$$= E_{m_i \mid l_i}(\varphi^i), \text{ if } m_k = l_k, \; m_i \neq l_i, \; i \neq k, \; k, i = 1, 2, 3. \tag{17c}$$

The relation (17a) *also holds, if the reliabilities* $E_{m \mid l}(\varphi^i) = 0$, *for several* m, l *and several* i.

Proof. From the independence of the objects we can write

$$\alpha^N_{m_1, m_2, m_3 \mid l_1, l_2, l_3}(\Phi_N) = \prod_{i=1}^{3} \alpha_{m_i \mid l_i}(\varphi^i), \text{ if } m_i \neq l_i, \; i \neq k, \tag{18a}$$

$$\alpha^N_{m_1, m_2, m_3 \mid l_1, l_2, l_3}(\Phi_N) = [1 - \alpha_{m_k \mid l_k}](\varphi^k_N) \prod_{i \neq k} \alpha_{m_i \mid l_i}(\varphi^i_N),$$

$$\text{if } m_k = l_k, \; i \neq k, \; m_i \neq l_i, \; i, k = 1, 2, 3, \tag{18b}$$

$$\alpha^N_{m_1, m_2, m_3 \mid l_1, l_2, l_3}(\Phi_N) = \alpha_{m_i \mid l_i}(\varphi^i_N) \prod_{i \neq k} [1 - \alpha_{m_k \mid l_k}(\varphi^k_N)],$$

$$\text{if } m_k = l_k, \; i \neq k, \; m_i \neq l_i, \; k, i = 1, 2, 3. \tag{18c}$$

In view of the definitions (14) and (15), from the equalities (18) we obtain relations (17).

Definition 4. The test sequence $\Phi^* = (\varphi_1, \varphi_2, \ldots)$ is called *LAO* for the model with three objects if for given positive values of certain $3(M-1)$ elements of the reliability matrix $E(\Phi^*)$ the procedure provides maximal values for other elements in it.

Our aim is to find LAO test from the set of compound tests $\Phi = (\varphi^1, \varphi^2, \varphi^3)$ when strictly positive elements $E_{m,m,m|M,m,m}$, $E_{m,m,m|m,M,m}$, and $E_{m,m,m|m,m,M}$, $m = \overline{1, M-1}$, of the reliability matrix are given.

Remark 3. Notice that the elements $E_{m,m,m|M,m,m}$, $E_{m,m,m|m,M,m}$, $E_{m,m,m|m,m,M}$, $m = \overline{1, M-1}$, of the test for three objects can be positive on the three subsets of tests $\Phi = (\varphi^1, \varphi^2, \varphi^3)$:

$$\mathcal{A} \stackrel{\Delta}{=} \{\Phi = (\varphi^1, \varphi^2, \varphi^3) : E_{m|m}(\varphi^i) > 0, \quad i = 1, 2, 3, \quad m = \overline{1, M-1}\},$$

$$\mathcal{B} \stackrel{\Delta}{=} \{\Phi = (\varphi^1, \varphi^2, \varphi^3) : \exists m' \in [1, M-1] : E_{m'|m'}(\varphi^i) = 0, \text{ for two } i, \text{ but}$$
$$E_{m'|m'}(\varphi^j) > 0, \ i \neq j, \text{ and for other } m < M,$$
$$E_{m|m}(\varphi^i) > 0, \ i, j = 1, 2, 3\},$$

$$\mathcal{C} \stackrel{\Delta}{=} \{\Phi = (\varphi^1, \varphi^2, \varphi^3) : \exists m' \in [1, M-1] : E_{m'|m'}(\varphi^i) = 0, \text{ and for other } m < M,$$
$$E_{m|m}(\varphi^i) > 0, \ i = 1, 2, 3\}.$$

Consider for given positive elements $E_{m,m,m|M,m,m}$, $E_{m,m,m|m,M,m}$, and $E_{m,m,m|m,m,M}$, $m = \overline{1, M}$, the family of regions:

$$\mathcal{R}_m^{(i)} \stackrel{\Delta}{=} \{Q : D(Q \| P_m) \leq E_{m,m,m|m_1,m_2,m_3}, \ m_i = M, \ m_j = m, \ i \neq j\},$$
$$m = \overline{1, M-1}, \ i = 1, 2, 3,$$

$$\mathcal{R}_M^{(i)} \stackrel{\Delta}{=} \{Q : D(Q \| P_m) > E_{m,m,m|m_1,m_2,m_3}, \ m_i = M, \ m_j = m, \ i \neq j\},$$
$$m = \overline{1, M-1}, \ i = 1, 2, 3$$

and the following numbers:

$$E^*_{m,m,m|M,m,m} \stackrel{\Delta}{=} E_{m,m,m|M,m,m},$$

$$E^*_{m,m,m|m,M,m} \stackrel{\Delta}{=} E_{m,m,m|m,M,m}, \quad l = \overline{1, M-1}$$

$$E^*_{m,m,m|m,m,M} \stackrel{\Delta}{=} E_{m,m,m|m,m,M} \tag{19a}$$

$$E^*_{m_1,m_2,m_3|l_1,l_2,l_3}$$

$$\stackrel{\Delta}{=} \inf_{Q:Q\in\mathcal{R}^i_{l_i}} D(Q \| P_{m_i}), \ m_i \neq l_i, \quad m_k = l_k, \quad i \neq k, \quad i, k = 1, 2, 3 \tag{19b}$$

$$E^{*}_{m_1, m_2, m_3 \mid l_1, l_2, l_3} \triangleq \sum_{i \neq k,\, Q : Q \in \mathcal{R}^{i}_{l_i}} \inf D(Q \,\|\, P_{m_i}), \quad m_i \neq l_i, \quad m_k = l_k, \quad i, k = 1, 2, 3 \tag{19c}$$

$$E^{*}_{m_1, m_2, m_3 \mid l_1, l_2, l_3} \triangleq E^{*}_{m_1, m_2, m_3 \mid l_1, m_2, m_3} + E^{*}_{m_1, m_2, m_3 \mid m_1, l_2, m_3} + E^{*}_{m_1, m_2, m_3 \mid m_1, m_2, l_3}, \quad m_i \neq l_i, \quad i = 1, 2, 3. \tag{19d}$$

Theorem 3. *If all distributions* P_m, $m = \overline{1, M}$, *are different, that is,* $D(P_l \,\|\, P_m) > 0$, $l \neq m$, $m = \overline{1, M}$, *then the following three statements are valid*:

(a) *When given elements* $E_{m, m, m \mid M, m, m}$, $E_{m, m, m \mid m, M, m}$ *and* $E_{m, m, m \mid m, m, M}$, $m = \overline{1, M-1}$, *satisfy the following conditions*:

$$\max(E_{1,1,1 \mid M,1,1}, E_{1,1,1 \mid 1, M, 1}, E_{1,1,1 \mid 1,1,M}) < \min_{l = \overline{2, M}} D(P_l \,\|\, P_1), \tag{20}$$

$$0 < E_{m, m, m \mid M, m, m} < \min\left[\min_{l = \overline{1, m-1}} E^{*}_{m, m, m \mid l, m, m}, \min_{l = \overline{m+1, M}} D(P_l \,\|\, P_m)\right], \quad m = \overline{2, M-1}, \tag{21}$$

$$0 < E_{m, m, m \mid m, M, m} < \min\left[\min_{l = \overline{1, m-1}} E^{*}_{m, m, m \mid m, l, m}, \min_{l = \overline{m+1, M}} D(P_l \,\|\, P_m)\right], \quad m = \overline{2, M-1}, \tag{22}$$

$$0 < E_{m, m, m \mid m, m, M} < \min\left[\min_{l = \overline{1, m-1}} E^{*}_{m, m, m \mid m, m, l}, \min_{l = \overline{m+1, M}} D(P_l \,\|\, P_m)\right], \quad m = \overline{2, M-1}, \tag{23}$$

then there exists a LAO test sequence $\Phi^{*} \in \mathcal{A}$, *the reliability matrix of which* $E(\Phi^{*}) = \{E_{m_1, m_2, m_3 \mid l_1, l_2, l_3}(\Phi^{*})\}$ *is defined in* (19) *and all elements of it are positive.*

(b) *Even if one of conditions* (20)-(23) *is violated, there exists at least one element of the matrix* $E(\Phi^{*})$ *equal to* 0.

(c) *For given positive numbers* $E_{m,m,m|m,M,m}$ *and* $E_{m,m,m|M,m,m}$, $E_{m,m,m|m,m,M}$, $m = \overline{1, M-1}$, *the reliability matrix* $E(\Phi)$ *of the tests* $\Phi \in \mathcal{B}$ *and* $\Phi \in \mathcal{C}$ *necessarily contains elements equal to zero.*

Proof. (a) Equalities (20)-(23) imply that inequalities (12) hold simultaneously for the three objects. Using equality (13) we can rewrite (12) for three objects as follows:

$$\max(E_{1|M}(\varphi^i)) < \min_{l=\overline{2, M}} D(P_l \,\|\, P_1), \quad i = 1, 2, 3, \tag{24}$$

$$0 < E_{m|M}(\varphi^i)$$

$$< \min\left[\min_{l=\overline{1, m-1}} E^*_{m|i}(\varphi^1), \min_{l=\overline{m+1, M}} D(P_l \,\|\, P_m)\right], m = \overline{2, M-1}, i = 1, 2, 3. \tag{25}$$

We shall prove, for example, inequality (25), as consequences of the inequality (22). Consider the test $\Phi \in \mathcal{A}$ such that $E_{m,m,m|m,M,m}(\Phi) = E_{m,m,m|m,M,m}$ and $E_{m,m,m|m,l,m}(\Phi) = E^*_{m,m,m|m,l,m}$, $l = \overline{1, m-1}$, $m = \overline{1, M-1}$. The corresponding error probabilities $\alpha_{m,m,m|m,M,m}(\Phi_N)$ and $\alpha_{m,m,m|m,l,m}(\Phi_N)$ are given as products defined by (18b). Because $\Phi \in \mathcal{A}$,

$$E_{m|l}(\varphi^i) \triangleq \overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1 - \alpha_{m|l}(\varphi^i_N)) = 0, \quad m = \overline{2, M-1}, \quad i = 1, 2, 3. \tag{26}$$

Due to (15), (18c), (22), and (26) we obtain

$$E^*_{m,m,m|m,M,m}(\Phi) = E^*_{m|M}(\varphi^2), \quad m = \overline{2, M-1}, \tag{27}$$

$$E^*_{m,m,m|m,l,m}(\Phi) = E^*_{m|l}(\varphi^2), \quad m = \overline{2, M-1}. \tag{28}$$

Therefore (25) is a consequence of (19).

It follows from (13), (24) and (25) that conditions (12) of Theorem 2 hold for three objects.

According to Theorem 2, there exist LAO sequences of tests $\varphi^{*,1}$ and $\varphi^{*,2}$ for the first and the second objects such that the elements of the matrices $E(\varphi^{*,1})$, $E(\varphi^{*,2})$ and $E(\varphi^{*,3})$ are determined through (8). We consider the sequence of tests Φ^*, which is composed of the three of the sequences of tests $\varphi^{*,1}$, $\varphi^{*,2}$ and $\varphi^{*,3}$. We shall show that Φ^* is LAO and other elements of the matrix $E(\Phi^*)$ are determined according to (24), (25).

From (24), (25), (13) and (12), it follows that the requirements of lemma are fulfilled. Using lemma we can deduce that the reliability matrix $E(\Phi^*)$ can be obtained from matrices $E(\varphi^{*,1})$ and $E(\varphi^{*,2})$ as in (17).

When conditions (20)-(23) hold, we obtain (19) according to (17), (8), (13), (27) and (28), that the elements $E_{m_1,m_2|l_1,l_2}(\Phi^*)$, $m_i \neq l_i$, $m_{3-i} = l_{3-i}$, $i = 1, 2$ of the matrix $E(\Phi^*)$ are determined by relation (19b). The equality in (19b) is a particular case of (16). From (19), it follows that all elements of $E(\Phi^*)$ are positive.

Now, we show that the compound test Φ^* for two objects is LAO, that is, it is optimal.

Suppose that for given $E_{m,m,m|M,m,m}$, $E_{m,m,m|m,M,m}$ and $E_{m,m,m|m,m,M}$, $m = \overline{1, M-1}$, there exists a test $\Phi' \in \mathcal{A}$ with matrix $E(\Phi')$, such that it has at least one element exceeding the respective element of the matrix $E(\Phi^*)$.

This contradicts the fact that LAO tests have been used for the objects X_1, X_2 and X_3.

(b) When one of the inequalities (20)-(23) is violated, then from (19b) we see that some of the elements in the matrix $E(\Phi^*)$ must be equal to zero.

(c) When $\Phi \in \mathcal{B}$, then from (13) and (17a) it follows that $E_{m',m',m'|M,M,M} = 0$. Consider $\Phi = (\varphi^1, \varphi^2, \varphi^3) \in \mathcal{C}$ and $E_{m'|m'}(\varphi^2) > 0$, then

$$E_{m',m',m'|M,m',M} = 0 + \overline{\lim_{N\to\infty}} -\frac{1}{N}\log(1-\alpha_{m'|m'}(\varphi^2)) + 0 = 0.$$

Acknowledgement

We are grateful to Professor E. A. Haroutunian for his very helpful comments which substantially improved the presentation of the paper.

References

[1] R. F. Ahlswede and E. A. Haroutunian, Testing of hypotheses and identification, Electronic Notes on Discrete Mathematics 21 (2005), 185-189.

[2] R. F. Ahlswede and E. A. Haroutunian, On statistical hypotheses optimal testing and identification, Mathematical Problems of Computer Science 24 (2005), 16-33.

[3] R. R. Bahadur, On the asymptotic efficiency of tests and estimates, Sankhya 22 (1960), 229-252.

[4] I. Csiszár and J. Körner, Information Theory: Coding Theorem for Discrete Memoryless Systems, Academic Press, New York, 1981.

[5] I. Csiszár and G. Longo, On the error exponent for source coding and for testing simple statistical hypotheses, Studia Sci. Math. Hungar. 6 (1971), 181-191.

[6] I. Csiszár and P. Shields, Information Theory and Statistics, Fundamentals and Trends in Communications and Information Theory, Publishers Inc., 2004.

[7] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications, Jones and Bartlett Publishers, London, 1993.

[8] E. A. Haroutunian, Logarithmically asymptotically optimal testing of multiple statistical hypotheses, Problems Control Inform. Theory 19 (1990), 413-421.

[9] W. Hoeffding, Asymptotically optimal tests for multinomial distributions, Ann. Math. Statist. 36 (1965), 369-401.

[10] G. Longo and A. Sgarro, The error exponent for the testing of simple statistical hypotheses: a combinatorial approach, J. Combin. Inform. System Sci. 5(1) (1980), 58-67.

[11] L. Navaei, On many hypotheses logarithmically asymptotically optimal testing via the theory of large deviations, Far East J. Math. Sci. (FJMS) 25(2) (2007), 335-344.

[12] E. Tuncel, On error exponents in hypotheses testing, IEEE Trans. Inform. Theory 51(8) (2005), 2945-2950.

[13] G. Tusnady, On asymptotically optimal tests, Ann. Statist. 5(2) (1977), 385-393.