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SOME EXAMPLES OF HYPERBOLIC HYPERSURFACES IN THE COMPLEX PROJECTIVE SPACE

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Abstract

In 1970, Kobayashi [6] has asked whether generic enough degree hypersurfaces in $P^n(C)$ are hyperbolic. This problem is still open. Shirosaki [10] and Fujimoto [4] constructed a hyperbolic hypersurface in $P^3(C)$ of degrees 10 and 8, respectively. That is the best result at present in $P^3(C)$. In this paper we construct a family of hyperbolic hypersurfaces in $P^2(C)$ of degree $d \geq 3 \in Z^+$, and a family of hyperbolic hypersurfaces in $P^3(C)$ of degree $d \geq 7 \in Z^+$.

1. Introduction

In [6], Kobayashi asked whether a generic hypersurface in the complex projective space $P^n(C)$ of large degree is hyperbolic or not. This problem is still open, but some papers were devoted to giving various

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examples of hyperbolic hypersurfaces. In 1997, Brody and Green [1] gave an example of hyperbolic hypersurface in $P^3(C)$ of even degree ≥ 50. Afterwards, Nadel, Goul and Demailly obtained hyperbolic hypersurfaces in $P^3(C)$ of degree ≥ 21, of arbitrary degree ≥ 14 and of arbitrary degree ≥ 11 in their papers [8], [5] and [3], respectively. Shirosaki [10] and Fujimoto [4] constructed a hyperbolic hypersurface in $P^3(C)$ of degree 10 and 8, respectively. On the other hand, Masuda and Noguchi [7] proved that there exists a smooth hyperbolic hypersurface of every degree $d \geq d(n)$ for a positive integer $d(n)$ depending only on n and some concrete examples of hyperbolic hypersurfaces in $P^3(C)$ for $n \leq 5$.

The purpose of this paper is to give some new examples of hyperbolic hypersurfaces in the complex projective space. We construct a family of hyperbolic hypersurfaces in $P^3(C)$ of degree $d > 6$.

2. Lemmas

We use the terminology in [9]. Let $f_0, \ldots, f_n$ be entire functions on C such that $f_j \not\equiv 0$ for at least one j $(0 \leq j \leq n)$. Then $\widetilde{f} := (f_0, \ldots, f_n)$ becomes a representation of a holomorphic mapping f of C into $P^n(C)$. If $f(z) = (c_0 : \cdots : c_n)$ for all $z \in C - \widetilde{f}^{-1}(0)$, where $c_0, \ldots, c_n$ are constants at least one of which are not 0, then we say that f or $(f_0 : \cdots : f_n)$ is constant.

We shall need the following:

Lemma 1 [9, p. 291]. *Let f be a nonconstant meromorphic function on C and a_j $(1 \leq j \leq q)$ distinct points in $C \cup \{\infty\}$. If all the zeros of $f - a_j$ have the multiplicities at least m_j for each j, where m_j are arbitrarily fixed positive integers $(1 \leq j \leq q)$ and $f - \infty$ means $1/f$, then*

$$\sum_{j=1}^{q}\left(1 - \frac{1}{m_j}\right) \leq 2.$$

Remark. If $f - a_j$ has no zero, then we may consider $1 - 1/m_j$ as 1.

Let f be a holomorphic curve and H be a hyperplane of $P^n(C)$ which does not contain the image of f. We denote by $\deg_z f^*H$ the degree of the pull-backed divisor f^*H at $z \in C$. We say that f ramifies at least $d(> 0)$ over H if $\deg_z f^*H \geq d$ for all $z \in f^{-1}H$. In case $f^{-1}H = 0$, we set $d = \infty$.

Lemma 2 [2]**.** *Assume that f is linearly non-degenerate and ramifies at least d over H_j, $1 \leq j \leq q$, where the hyperplanes H_j are in general position in $P^n(C)$. Then*

$$\sum_{j=1}^{q}\left(1 - \frac{n}{d_j}\right) \leq (n+1).$$

Lemma 3. *Every compact Riemann surface of genus greater than one is hyperbolic.*

For a projective algebraic curve V in $P^2(C)$ we can take the normalization $\mu : \widetilde{V} \to V$, namely, a compact Riemann surface $\widetilde{V}$ and a holomorphic map μ onto V which is injective outside the inverse image of the singular locus of V. By definition, the genus of V is the genus of $\widetilde{V}$. Since each holomorphic map f of C into V can be lifted to a holomorphic map $\widetilde{f}$ of C into $\widetilde{V}$ with $f = \mu \cdot \widetilde{f}$, by Lemma 3, every holomorphic map of C into V is a constant if the genus of V is greater than one. Therefore, Lemma 3 remains valid for an algebraic curve in $P^2(C)$ possibly with singularities.

Next, we explain Plücker's formula on the genus of an algebraic curve in $P^2(C)$.

Let $Q(u_0, u_1, u_2)$ be an irreducible homogeneous polynomial of degree $d(\geq 1)$ and consider the algebraic curve

$$V := \{(u_0 : u_1 : u_2) \in P^2(C);\ Q(u_0 : u_1 : u_2) = 0\},$$

where $(u_0 : u_1 : u_2)$ denotes homogeneous coordinates in $P^2(C)$. A point $p := (p_0 : p_1 : p_2) \in V$ is a singular point of V if and only if

$$Q_{u_0}(p_0, p_1, p_2) = Q_{u_1}(p_0, p_1, p_2) = Q_{u_2}(p_0, p_1, p_2) = 0.$$

For a singular point $p \in V$, we say that p is a node of V if the complex Hessian

$$\varphi := \begin{vmatrix} \dfrac{\partial^2 Q}{\partial v^2} & \dfrac{\partial^2 Q}{\partial v \partial w} \\ \dfrac{\partial^2 Q}{\partial w \partial v} & \dfrac{\partial^2 Q}{\partial w^2} \end{vmatrix}$$

does not vanish at p, where v, w denote a system of inhomogeneous coordinates in a neighborhood of p.

Plücker's genus formula is stated as follows:

Lemma 4. *Let V be an algebraic curve of degree d which is given as the above and assume that V has no singularities than k nodes. Then the genus g of V is given by*

$$g = \frac{(d-1)(d-2)}{2} - k.$$

3. A Family of Hyperbolic Hypersurface of Degree $d(> 6)$ in $P^3(C)$

Theorem. *Define the hypersurface X in $P^3(C)$ by*

$$A^d + B^{\alpha_1} C^{\beta_1} + D^{\alpha_2} E^{\beta_2} = 0,$$

where $d(> 6)$, α_i, β_i are integers, and $\alpha_i + \beta_i = d$, $\alpha_i \geq 3$, $\beta_i \geq 3$, $i = 1, 2$.

$$A = \sum_{i=1}^{4} a_i x_i,\ B = \sum_{i=1}^{3} b_i x_i,\ C = \sum_{i=1}^{3} c_i x_i,\ D = \sum_{i=1}^{3} d_i x_i,\ E = \sum_{i=1}^{3} e_i x_i.$$

If the coefficients satisfy the following conditions:

(1) $\operatorname{rank}(b, c, d) = \operatorname{rank}(b, c, e) = \operatorname{rank}(e, c, d) = \operatorname{rank}(b, e, d) = 3$, *where* $b = (b_1, b_2, b_3)^T$;

(2) $\operatorname{rank}(b - mc, d - ne) = 2$, *where* $m = \det(b, e, d)/\det(e, c, d)$, $n = \det(b, c, d)/\det(b, c, e)$.

Then there exists no nonconstant holomorphic mapping f of C into $P^3(C)$ *such that* $f(C) \subset X$, *i.e., X is hyperbolic.*

Proof. Assume that a holomorphic mapping f of C into $P^3(C)$ with reduced representation (f_1, f_2, f_3, f_4) satisfies $f(C) \subset X$, we denote

$$A_f = \sum_{i=1}^{4} a_i f_i,\ B_f = \sum_{i=1}^{3} b_i f_i,\ C_f = \sum_{i=1}^{3} c_i f_i,\ D_f = \sum_{i=1}^{3} d_i f_i,\ E_f = \sum_{i=1}^{3} e_i f_i,$$

then

$$A_f^d + B_f^{\alpha_1} C_f^{\beta_1} + D_f^{\alpha_2} E_f^{\beta_2} = 0. \tag{1}$$

Now we proof $(f_1 : f_2 : f_3 : f_4) =$ constant.

1. Assume $A_f \not\equiv 0$, $B_f \not\equiv 0$, $C_f \not\equiv 0$, $D_f \not\equiv 0$, $E_f \not\equiv 0$;

Consider a holomorphic curve g in $P^1(C)$ defined by

$$g : z \in C \mapsto (A_f^d, B_f^{\alpha_1} C_f^{\beta_1}) \in P^1(C).$$

Take the following hypersurfaces in general position:

$$H_1 = \{z_1 = 0\},\ H_2 = \{z_2 = 0\},\ H_3 = \{z_1 + z_2 = 0\}.$$

By the conditions of Theorem we see that g ramifies at least 6, 3, 3 over H_1, H_2, H_3 respectively. It follows from Lemma 2 that

$$\left(1 - \frac{1}{6}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{3}\right) \le 2.$$

But it is impossible, a contradiction. So g is linearly degenerate, on the other hand, $A_f \not\equiv 0$, $B_f \not\equiv 0$, $C_f \not\equiv 0$, $D_f \not\equiv 0$, $E_f \not\equiv 0$. Hence there exists $c \neq 0, -1$ such that

$$A_f^d = cB_f^{\alpha_1}C_f^{\beta_1}. \tag{2}$$

Combination with (1) we get

$$(c+1)B_f^{\alpha_1}C_f^{\beta_1} + D_f^{\alpha_2}E_f^{\beta_2} = 0. \tag{3}$$

We denote

$$Q(x_1, x_2, x_3) = (c+1)B^{\alpha_1}C^{\beta_1} + D^{\alpha_2}E^{\beta_2} \tag{4}$$

obviously (2) is a hypersurface of degree $d(> 6)$ in $P^2(C)$, we can compute its genus by Lemma 4. According to

$$Q_{x_1}(x_1, x_2, x_3) = Q_{x_2}(x_1, x_2, x_3) = Q_{x_3}(x_1, x_2, x_3) = 0$$

we get

$$(c+1)\alpha_1 b_1 B^{\alpha_1 - 1}C^{\beta_1} + \beta_1 c_1 B^{\alpha_1}C^{\beta_1 - 1} + \alpha_2 d_1 D^{\alpha_2 - 1}E^{\beta_2} + \beta_2 e_1 D^{\alpha_2}E^{\beta_2 - 1} = 0 \tag{5}$$

$$(c+1)\alpha_1 b_2 B^{\alpha_1 - 1}C^{\beta_1} + \beta_1 c_2 B^{\alpha_1}C^{\beta_1 - 1} + \alpha_2 d_2 D^{\alpha_2 - 1}E^{\beta_2} + \beta_2 e_2 D^{\alpha_2}E^{\beta_2 - 1} = 0 \tag{6}$$

$$(c+1)\alpha_1 b_3 B^{\alpha_1 - 1}C^{\beta_1} + \beta_1 c_3 B^{\alpha_1}C^{\beta_1 - 1} + \alpha_2 d_3 D^{\alpha_2 - 1}E^{\beta_2} + \beta_2 e_3 D^{\alpha_2}E^{\beta_2 - 1} = 0. \tag{7}$$

Since $\det(b, c, d) \neq 0$, we get

$$B^{\alpha_1 - 1}C^{\beta_1} = \beta_2 D^{\alpha_2}E^{\beta_2 - 1} \det(e, c, d)/(c+1)\alpha_1 \det(b, c, d) \tag{8}$$

$$B^{\alpha_1}C^{\beta_1 - 1} = \beta_2 D^{\alpha_2}E^{\beta_2 - 1} \det(b, e, d)/(c+1)\beta_1 \det(b, c, d) \tag{9}$$

$$D^{\alpha_2 - 1}E^{\beta_2} = \beta_2 D^{\alpha_2}E^{\beta_2 - 1} \det(b, c, e)/(c+1)\alpha_2 \det(b, c, d). \tag{10}$$

Solve this equations we get

$$\begin{cases} D = 0 \\ B = 0 \end{cases} \text{or} \begin{cases} D = 0 \\ C = 0 \end{cases} \text{or} \begin{cases} E = 0 \\ C = 0 \end{cases} \text{or} \begin{cases} E = 0 \\ B = 0 \end{cases} \text{or} \begin{cases} B = mC \\ D = nE. \end{cases}$$

Combine with this results and the conditions (1) and (2) of Theorem, obviously

$$Q(x_1, x_2, x_3) = (c+1)B^{\alpha_1}C^{\beta_1} + D^{\alpha_2}E^{\beta_2} = 0$$

at most five singularities. By Lemma 4 its genus:

$$g = \frac{(d-1)(d-2)}{2} - k \geq \frac{(6-1)(6-2)}{2} - 5 = 5 > 1.$$

By Lemma 3 it is hyperbolic. So $(f_1 : f_2 : f_3) = \text{const.}$ by the equation (2) again we get $(f_1 : f_2 : f_3 : f_4) = \text{const.}$

2. (i) If $A_f \equiv 0$, then

$$B_f^{\alpha_1}C_f^{\beta_1} + D_f^{\alpha_2}E_f^{\beta_2} = 0.$$

It is easy to get this equation when we take $c = 0$ in the equation (3), from the above proof we know $(f_1 : f_2 : f_3) = \text{const.}$, according to $A_f \equiv 0$ again, $(f_1 : f_2 : f_3 : f_4) = \text{const.}$

(ii) If there is one equal to zero among B_f, C_f, E_f, D_f, then we can always get the following equation:

$$(M_f + a_4 f_4)^d + P_f^{\alpha} Q_f^{\beta} = 0, \tag{11}$$

where α, β are positive integers and $\alpha + \beta = d$; $\alpha \geq 3$, $\beta \geq 3$. M_f, P_f, Q_f is three linear combinations of the arbitrary two of $\{f_1, f_2, f_3\}$, and $P_f \not\equiv 0$, $Q_f \not\equiv 0$ (because of the condition (2) of Theorem).

The corresponding homogeneous polynomial of the equation (11) is:

$$Q(x_1, x_2, x_4) = (M + a_4 x_4)^d + P^{\alpha} Q^{\beta}, \tag{12}$$

where $M = m_1 x_1 + m_2 x_2$; $P = p_1 x_1 + p_2 x_2$; $Q = q_1 x_1 + q_2 x_2$. We can compute its singularities by Lemma 4:

$$Q_{x_1} = dm_1(M + a_4 x_4)^{d-1} + \alpha p_1 P^{\alpha-1} Q^{\beta} + \beta q_1 P^{\alpha} Q^{\beta-1} = 0 \tag{13}$$

$$Q_{x_2} = dm_2(M + a_4 x_4)^{d-1} + \alpha p_2 P^{\alpha-1} Q^{\beta} + \beta q_2 P^{\alpha} Q^{\beta-1} = 0 \tag{14}$$

$$Q_{x_4} = da_4(M + a_4x_4)^{d-1} = 0 \tag{15}$$

$$\Rightarrow \begin{cases} P = 0 \\ M + a_4x_4 = 0 \end{cases} \text{or} \begin{cases} Q = 0 \\ M + a_4x_4 = 0 \end{cases} \text{or} \begin{cases} \alpha p_1 Q + \beta q_1 P = 0 \\ \alpha p_2 Q + \beta q_2 P = 0 \\ M + a_4x_4 = 0. \end{cases}$$

By the conditions (1) and (2) of the Theorem, obviously $Q(x_1, x_2, x_4) = 0$ has at most three singularities, so its genus > 1, thus it is hyperbolic, so $(f_1 : f_2 : f_3 : f_4) = \text{const.}$ The proof is completed.

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