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MEROMORPHIC STARLIKE FUNCTIONS WITH NEGATIVE AND MISSING COEFFICIENTS

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Abstract

In this paper, we introduce a new class $\Sigma_p(n, \alpha)$ of meromorphic starlike functions with negative and missing coefficients in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$. We obtaine among others results, coefficient inequalities, distortion theorem, closure theorem and class preserving integral operator.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \quad (a_{-1} \neq 0,\ p \in N = \{1, 2, \ldots\}) \tag{1.1}$$

which are regular in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$.

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Define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$Df(z) = D^1 f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (p+k+2) a_{p+k} z^{p+k}$$

$$= \frac{(z^2 f(z))'}{z}, \tag{1.3}$$

$$D^2 f(z) = D(D^1 f(z)), \tag{1.4}$$

and for $n = 1, 2, 3, \ldots$

$$D^n f(z) = D(D^{n-1} f(z)) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (p+k+2)^n a_{p+k} z^{p+k}$$

$$= \frac{(z^2 D^{n-1} f(z))'}{z}. \tag{1.5}$$

Let $B_n(\alpha)$ denote the class consisting functions in A_p satisfying

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)} - 2\right\} < -\alpha \quad (z \in U^*,\ 0 \le \alpha < 1,\ n \in N_0 = N \cup \{0\}). \tag{1.6}$$

Let Λ_p be the subclass of A_p which consisting of functions of the form

$$f(z) = \frac{a_{-1}}{z} - \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \quad (a_{-1} > 0;\ a_{p+k} \ge 0,\ p \in N). \tag{1.7}$$

Further let

$$\Sigma_p(n, \alpha) = B_n(\alpha) \cap \Lambda_p. \tag{1.8}$$

We note that, in [3] Uralegaddi and Somanatha defined a class $B_n(\alpha)$ which consists functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k \quad (a_{-1} \ne 0)$$

which are analytic in U^* and obtained inclusion relation, the class preserving integral. Further, recently in [2] Darwish consider meromorphic p-valent starlike functions with negative coefficients and obtained coefficient inequalities, distortion theorem, closure theorems and integral operators. We in the present paper, have considered functions of the forms (1.7) and obtained basic properties which include, e.g., coefficient inequalities, distortion theorem, closure theorems and integral operators. Finally, the class preserving integral operators of the form

$$F_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)\,du \quad (0 \leq u < 1,\ 0 < c < \infty) \tag{1.9}$$

is considered. Techniques used are similar to those of Aouf and Hossen [1].

2. Coefficient Inequalities

Theorem 1. *Let the function* $f(z)$ *be defined by* (1.1). *If*

$$\sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) |a_{p+k}| \leq (1-\alpha)|a_{-1}|, \tag{2.1}$$

then $f(z) \in B_n(\alpha)$.

Proof. It suffices to show that

$$\left| \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1}f(z)}{D^n f(z)} - (3-2\alpha)} \right| < 1, \quad |z| < 1 \tag{2.2}$$

we have

$$\left| \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1}f(z)}{D^n f(z)} - (3-2\alpha)} \right|$$

$$= \left| \frac{\sum_{k=0}^{\infty} (p+k+2)^n (p+k+1) a_{p+k} z^{p+k}}{(2-2\alpha)a_{-1} - \sum_{k=0}^{\infty} (p+k+2)^n (p+k+2\alpha-1) a_{p+k} z^{p+k}} \right|$$

$$\leq \frac{\sum_{k=0}^{\infty} (p+k+2)^n (p+k+1) |a_{p+k}|}{2(1-\alpha)|a_{-1}| - \sum_{k=0}^{\infty} (p+k+2)^n (p+k+2\alpha-1) |a_{p+k}|}.$$

The last expression is bounded by 1 if

$$\sum_{k=0}^{\infty} (p+k+2)^n (p+k+1) |a_{p+k}|$$

$$\leq 2(1-\alpha)|a_{-1}| - \sum_{k=0}^{\infty} (p+k+2)^n (p+k+2\alpha-1) |a_{p+k}|$$

which reduces to

$$\sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) |a_{p+k}| \leq (1-\alpha)|a_{-1}| \tag{2.3}$$

but (2.3) is true by hypothesis. Hence the result follows.

Theorem 2. *Let the function* $f(z)$ *be defined by* (1.7). *Then* $f(z) \in \Sigma_p(n, \alpha)$ *if and only if*

$$\sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) a_{p+k} \leq (1-\alpha) a_{-1}. \tag{2.4}$$

Proof. In view of Theorem 1, it is sufficient to prove the "only if" part. Let us assume that $f(z)$ defined by (1.7) is in $\Sigma_p(n, \alpha)$. Then

$$\mathrm{Re}\left\{ \frac{D^{n+1} f(z)}{D^n f(z)} - 2 \right\}$$

$$= \mathrm{Re}\left\{\frac{D^{n+1}f(z) - 2D^n f(z)}{D^n f(z)}\right\}$$

$$= \mathrm{Re}\left\{\frac{a_{-1} - \sum_{k=0}^{\infty}(p+k+2)^n (p+k) a_{p+k} z^{p+k}}{-a_{-1} - \sum_{k=0}^{\infty}(p+k+2)^n a_{p+k} z^{p+k}}\right\} < -\alpha, \quad |z| < 1. \qquad (2.5)$$

Choose values of z on the real axis so that $\frac{D^{n+1}f(z)}{D^n f(z)} - 2$ is real. Upon clearing the denominator in (2.5) and letting $z \to 1$ through real values, we obtain

$$a_{-1} - \sum_{k=0}^{\infty}(p+k+2)^n (p+k) a_{p+k} \geq \left(a_{-1} + \sum_{k=0}^{\infty}(p+k+2)^n a_{p+k}\right)\alpha.$$

Thus

$$\sum_{k=0}^{\infty}(p+k+2)^n (p+k+\alpha) a_{p+k} \leq (1-\alpha)a_{-1}.$$

Hence the result follows.

Corollary 1. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\Sigma_p(n, \alpha)$. *Then*

$$a_{p+k} \leq \frac{(1-\alpha)a_{-1}}{(p+k+2)^n (p+k+\alpha)}. \qquad (2.6)$$

The result is sharp for the function

$$f(z) = \frac{a_{-1}}{z} - \frac{(1-\alpha)a_{-1}}{(p+k+2)^n (p+k+\alpha)} z^{p+k} \qquad (k \geq 1). \qquad (2.7)$$

3. Distortion Theorem

Theorem 3. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\Sigma_p(n, \alpha)$. *Then for* $0 < |z| = r < 1$, *we have*

$$\frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} r \leq |f(z)| \leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} r, \quad (3.1)$$

where equality holds for the function

$$f(z) = \frac{a_{-1}}{z} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} z^p \quad (z = ir, r), \quad (3.2)$$

and

$$\frac{a_{-1}}{r^2} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} \leq |f'(z)| \leq \frac{a_{-1}}{r^2} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}, \quad (3.3)$$

where equality holds for the function $f(z)$ *given by* (3.2) *at* $z = \mp ir, \mp r$.

Proof. In view of Theorem 2, we have

$$\sum_{k=0}^{\infty} a_{p+k} \leq \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)}. \quad (3.4)$$

Thus, for $0 < |z| = r < 1$,

$$|f(z)| \leq \frac{a_{-1}}{r} + r\sum_{k=0}^{\infty} a_{p+k}$$

$$\leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} r \quad (3.5)$$

and

$$|f(z)| \geq \frac{a_{-1}}{r} - r\sum_{k=0}^{\infty} a_{p+k}$$

$$\geq \frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} r. \quad (3.6)$$

Thus (3.1) follows.

Since

$$(p+3)^n(p+\alpha+n)\sum_{k=0}^{\infty}(p+k)|a_{p+k}| \leq \sum_{k=0}^{\infty}(p+k+2)^n(p+k+\alpha)|a_{p+k}|$$

$$\leq (1-\alpha)|a_{-1}|,$$

where $\frac{(p+k+2)^n(p+k+\alpha)}{p+k}$ is an increasing function of k, from Theorem 2, it follows that

$$\sum_{k=0}^{\infty}(p+k)a_{p+k} \le \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}. \tag{3.7}$$

Hence

$$|f'(z)| \le \frac{a_{-1}}{r^2} + \sum_{k=0}^{\infty}(p+k)a_{p+k}\, r^{p+k-1}$$

$$\le \frac{a_{-1}}{r^2} + \sum_{k=0}^{\infty}(p+k)a_{p+k}$$

$$\le \frac{a_{-1}}{r^2} + \sum_{k=0}^{\infty}\frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} \tag{3.8}$$

and

$$|f'(z)| \ge \frac{a_{-1}}{r^2} - \sum_{k=0}^{\infty}(p+k)a_{p+k}\, r^{p+k-1}$$

$$\ge \frac{a_{-1}}{r^2} - \sum_{k=0}^{\infty}(p+k)a_{p+k}$$

$$\ge \frac{a_{-1}}{r^2} - \sum_{k=0}^{\infty}\frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)}. \tag{3.9}$$

Thus (3.3) follows. It can be easily seen that the function $f(z)$ defined by (3.2) is extremal for the theorem.

4. Closure Theorems

Let the function $f_j(z)$ be defined for $j \in \{1, 2, 3, \ldots, m\}$, by

$$f_j(z) = \frac{a_{-1,j}}{z} - \sum_{k=0}^{\infty} a_{p+k,j} z^{p+k}, \quad (a_{-1,j} > 0,\ a_{p+k,j} \ge 0) \tag{4.1}$$

for $z \in U^*$.

Now, we shall prove the following result for the closure of function in the class $\Sigma_p(n, \alpha)$.

Theorem 4. *Let the functions* $f_j(z)$ *defined by* (4.1) *be in the class* $\Sigma_p(n, \alpha)$ *for every* $j \in \{1, 2, 3, \ldots, m\}$. *Then the function* $F(z)$ *defined by*

$$F(z) = \frac{b_{-1}}{z} - \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \quad (b_{-1} > 0;\ b_{p+k} \geq 0,\ p \in N). \tag{4.2}$$

is a member of the class $\Sigma_p(n, \alpha)$, *where*

$$b_{-1} = \frac{1}{m} \sum_{j=1}^{m} a_{-1, j} \text{ and } b_{p+k} = \frac{1}{m} \sum_{j=1}^{m} a_{p+k, j} \quad (k = 1, 2, \ldots). \tag{4.3}$$

Proof. Since $f_j(z) \in \Sigma_p(n, \alpha)$, it follows from Theorem 2 that

$$\sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) a_{p+k, j} \leq (1 - \alpha) a_{-1, j} \tag{4.4}$$

for every $j \in \{1, 2, 3, \ldots, m\}$. Hence

$$\sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) b_{p+k}$$

$$= \sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) \left(\frac{1}{m} \sum_{j=1}^{m} a_{p+k, j} \right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{k=0}^{\infty} (p + k + 2)^n (p + k + \alpha) a_{p+k, j} \right)$$

$$\leq (1 - \alpha) \left(\frac{1}{m} \sum_{j=1}^{m} a_{-1, j} \right) = (1 - \alpha) b_{-1},$$

which (in view of Theorem 2) implies that $F(z) \in \Sigma_p(n, \alpha)$.

Theorem 5. *The class* $\Sigma_p(n, \alpha)$ *is closed under convex linear combination.*

Proof. Let the function $f_j(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $\Sigma_p(n, \alpha)$, it is sufficient to prove that the function

$$H(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \le \lambda \le 1) \tag{4.5}$$

is also in the class $\Sigma_p(n, \alpha)$. Since, for $0 \le \lambda \le 1$,

$$H(z) = \frac{\lambda a_{-1,1} + (1-\lambda) a_{-1,2}}{z} - \sum_{k=0}^{\infty} \{\lambda a_{p+k,1} + (1-\lambda) a_{p+k,2}\} z^{p+k}, \tag{4.6}$$

we observe that

$$\begin{aligned} &\sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) \{\lambda a_{p+k,1} + (1-\lambda) a_{p+k,2}\} \\ &= \lambda \sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) a_{p+k,1} \\ &\quad + (1-\lambda) \sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) a_{p+k,2} \\ &\le (1-\alpha) \{\lambda a_{-1,1} + (1-\lambda) a_{-1,2}\} \end{aligned} \tag{4.7}$$

with the aid of Theorem 2. Hence $H(z) \in \Sigma_p(n, \alpha)$. This completes the proof of Theorem 5.

Theorem 6. *Let*

$$f_0(z) = \frac{a_{-1}}{z} \tag{4.8}$$

and

$$f_{p+k}(z) = \frac{a_{-1}}{z} - \frac{(1-\alpha) a_{-1}}{(p+k+2)^n (p+k+\alpha)} z^{p+k} \qquad (k \ge 1). \tag{4.9}$$

Then $f(z) \in \Sigma_p(n, \alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{4.10}$$

where

$$\lambda_{p+k} \geq 0 \ (k \geq 0) \ \textit{and} \ \sum_{k=0}^{\infty} \lambda_k = 1.$$

Proof. Let

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z),$$

where

$$\lambda_{p+k} \geq 0 \ (k \geq 0) \ \text{and} \ \sum_{k=0}^{\infty} \lambda_{p+k} = 1.$$

Then

$$f(z) = \sum_{k=0}^{\infty} \lambda_{p+k} f_{p+k}(z)$$

$$= \lambda_0 f_0(z) + \sum_{k=1}^{\infty} \lambda_{p+k} f_{p+k}(z)$$

$$= \left(1 - \sum_{k=0}^{\infty} \lambda_{p+k}\right) \frac{a_{-1}}{z}$$

$$+ \sum_{k=0}^{\infty} \lambda_{p+k} \left\{ \frac{a_{-1}}{z} - \frac{(1-\alpha) a_{-1}}{(p+k+2)^n (p+k+\alpha)} z^{p+k} \right\}$$

$$= \frac{a_{-1}}{z} - \sum_{k=0}^{\infty} \frac{(1-\alpha) a_{-1} \lambda_{p+k}}{(p+k+2)^n (p+k+\alpha)} z^{p+k}. \tag{4.11}$$

Since

$$\sum_{k=0}^{\infty} (p+k+2)^n (p+k+\alpha) \cdot \frac{(1-\alpha) a_{-1} \lambda_{p+k}}{(p+k+2)^n (p+k+\alpha)}$$

$$= (1-\alpha) a_{-1} \sum_{k=0}^{\infty} \lambda_{p+k} = (1-\alpha) a_{-1} (1 - \lambda_0)$$

$$\leq (1-\alpha) a_{-1}, \tag{4.12}$$

by Theorem 2, $f(z) \in \Sigma_p(n, \alpha)$. Conversely, we suppose that $f(z)$ defined by (1.7) is in the class $\Sigma_p(n, \alpha)$. Then by using (2.6), we get

$$a_{p+k} \leq \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)} \quad (k \geq 1). \tag{4.13}$$

Setting

$$\lambda_{p+k} = \frac{(p+k+2)^n(p+k+\alpha)}{(1-\alpha)a_{-1}} a_{p+k} \quad (k \geq 1) \tag{4.14}$$

and

$$\lambda_0 = 1 - \sum_{k=0}^{\infty} \lambda_{p+k}, \tag{4.15}$$

we have (4.10). This completes the proof of Theorem 6.

5. Integral Operators

In this section we consider integral transforms of functions in the class $\Sigma_p(n, \alpha)$.

Theorem 7. *Let the function* $f(z)$ *defined by* (1.7) *be in the class* $\Sigma_p(n, \alpha)$. *Then the integral transforms*

$$F_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)du \quad (0 < u \leq 1,\ 0 < c < \infty), \tag{5.1}$$

are in $\Sigma_p(n, \alpha)$, *where*

$$\delta(\alpha, c, p) = \frac{(p+c+3)(p+\alpha+1)-(p+1)(1-\alpha)(c+1)}{(p+c+3)(p+\alpha+1)+(c+1)(1-\alpha)}. \tag{5.2}$$

The result is sharp for the function

$$f(z) = \frac{a_{-1}}{z} - \frac{(1-\alpha)a_{-1}}{(p+3)^n(p+\alpha+1)} z^p. \tag{5.3}$$

Proof. Let

$$F_{c+1}(z) = (c+1)\int_0^1 u^{c+1} f(uz)du$$

$$= \frac{a_{-1}}{z} - \sum_{k=0}^{\infty} \frac{c+1}{p+k+c+2} a_{p+k} z^{p+k} \tag{5.4}$$

in view of Theorem 2, it is sufficient to show that

$$\sum_{k=0}^{\infty} \frac{(p+k+2)^n (p+k+\delta)}{(1-\delta)a_{-1}} \left(\frac{c+1}{p+k+c+2} \right) a_{p+k} \leq 1. \tag{5.5}$$

Since $f(z) \in \Sigma_p(n, \alpha)$, we have

$$\sum_{k=0}^{\infty} \frac{(p+k+2)^n (p+k+\alpha)}{(1-\alpha)a_{-1}} a_{p+k} \leq 1. \tag{5.6}$$

Thus (5.4) will be satisfied if

$$\frac{(p+k+\delta)(c+1)}{(1-\delta)(p+k+c+2)} \leq \frac{p+k+\delta}{1-\alpha} \quad \text{for each } k,$$

or

$$\delta \leq \frac{(p+k+c+2)(p+k+\alpha) - (p+k)(1-\alpha)(c+1)}{(p+k+c+2)(p+k+\alpha) + (c+1)(1-\alpha)}. \tag{5.7}$$

For each α, p and, c fixed let

$$F(k) = \frac{(p+k+c+2)(p+k+\alpha) - (p+k)(1-\alpha)(c+1)}{(p+k+c+2)(p+k+\alpha) + (c+1)(1-\alpha)}.$$

Then

$$\begin{aligned} &F(k+1) - F(k) \\ &= (c+1)(1-\alpha)(p+k+1)(p+k+2)/[(p+k+c+2)(p+k+\alpha) \\ &\qquad + (c+1)(1-\alpha)][(p+k+c+3)(p+k+\alpha+1) + (c+1)(1-\alpha)] \\ &> 0 \end{aligned}$$

for each k. Hence, $F(k)$ is an increasing function of k.

Since

$$F(1) = \frac{(p+c+3)(p+\alpha+1) - (p+1)(1-\alpha)(c+1)}{(p+c+3)(p+\alpha+1) + (c+1)(1-\alpha)}.$$

The result follows.

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