# THE LEFSCHETZ NUMBER OF AN $n$-VALUED MULTIMAP 

## ROBERT F. BROWN

Department of Mathematics
University of California
Los Angeles, CA 90095-1555, U. S. A.
e-mail: rfb@math.ucla.edu


#### Abstract

An $n$-valued multimap is a continuous multivalued function $\phi: X \multimap Y$ such that $\phi(x)$ is an unordered subset of $n$ points of $Y$ for each $x \in X$. If $X$ and $Y$ are finite polyhedra, then $\phi$ induces a graded homomorphism of homology with rational coefficients. For $\phi: X \multimap X$ the Lefschetz number $L(\phi)$ of $\phi$ is defined to be the Lefschetz number of the induced homomorphism. If $L(\phi) \neq 0$, then every $n$-valued multimap homotopic to $\phi$ has a fixed point. If $X$ is the circle, then the Lefschetz number of $\phi$ is related to the Nielsen number $N(\phi)$ of Schirmer as in the single-valued case, that is, $N(\phi)=|L(\phi)|$.


## 1. Introduction

In 1957, O'Neill [4] introduced a very general concept of induced homology homomorphism for multivalued functions. If $X$ and $Y$ are compact metric spaces then, for any multivalued function $\phi: X \multimap Y$,

[^0]Keywords and phrases: $n$-valued multimap, simplicial approximation, Lefschetz fixed point theorem, Nielsen number.

Communicated by Peter Wong
Received January 19, 2007
there is a vector space of graded homomorphisms

$$
h=\left\{h_{k}: H_{k}(X) \rightarrow H_{k}(X)\right\}
$$

of homology groups with coefficients in a field, all of which he considered induced homomorphisms of $\phi$. When $\phi$ is a single-valued continuous function, this vector space consists of the scalar multiples of the usual induced homology homomorphism. If $X$ is a finite polyhedron, so the homology with rational coefficients $H_{*}(X)$ is finite-dimensional, and $h=\left\{h_{k}: H_{k}(X) \rightarrow H_{k}(X)\right\}$ is a graded homomorphism, then its Lefschetz number $\Lambda(h)$ is defined by

$$
\Lambda(h)=\sum_{k}(-1)^{k} \operatorname{trace}\left(h_{k}\right) .
$$

Note that if $r$ is a scalar, then the Lefschetz number of the graded homomorphism $r h$ has the property $\Lambda(r h)=r \Lambda(h)$. Thus, in O'Neill's theory, no specific value called a Lefschetz number can be assigned to a function.

A multimap $\phi: X \rightarrow Y$ is a multivalued function, that is, continuous, that is, both upper and lower semi-continuous [1]. O'Neill's version of the Lefschetz fixed point theorem [4, Theorem 9] states that if $\phi: X \mapsto X$ is a multimap of a finite polyhedron such that, for each $x \in X, \phi(x)$ is either homologically trivial or consists of exactly $n$ homologically trivial components and $\Lambda(h) \neq 0$ for some induced homology homomorphism $h$, then $\phi$ has a fixed point, that is, $x \in \phi(x)$ for some $x \in X$.

In 1984, Schirmer [5] initiated a study of multimaps $\phi: X \multimap Y$ of finite polyhedra that are $n$-valued, that is, $\phi(x)$ is an unordered set of exactly $n$ points of $Y$, for each $x \in X$. For this special case of the class of multivalued functions considered by O'Neill, she proved a simplicial approximation theorem [5, Theorem 4] and remarked that it "could, in fact, be used to define for an $n$-valued continuous multifunction $\phi$ an induced homomorphism $\phi_{*}: H_{*}(X) \rightarrow H_{*}(Y)$ in a way different from the one given by O'Neill, but we do not persue (sic) this topic".

We will show how Schirmer's simplicial approximation theorem for an $n$-valued multimap $\phi: X \multimap Y$ leads to a well-defined homomorphism of rational homology $\phi_{*}: H_{*}(X) \rightarrow H_{*}(Y)$. There is, therefore, associated to each $n$-valued multimap $\phi: X \mapsto X$ on a finite polyhedron a unique Lefschetz number defined by $L(\phi)=\Lambda\left(\phi_{*}\right)$. We will prove that the induced homomorphism, and therefore the Lefschetz number $L(\phi)$, is an invariant of the $n$-valued homotopy class of $\phi$ and that $L(\phi) \neq 0$ implies that $\phi$ has a fixed point. Moreover, we will demonstrate that, for $n$-valued multimaps of the circle, this Lefschetz number is related to the Nielsen number for $n$-valued multimaps developed by Schirmer in [6 and 7] in the same way that, for single-valued maps, the Lefschetz number is related to the classical Nielsen number. That is, letting $\phi: S^{1} \multimap S^{1}$ be an $n$-valued multimap of the circle. Then the Nielsen number $N(\phi)$ of Schirmer is equal to the absolute value of $L(\phi)$.

## 2. The Induced Homomorphism and the Lefschetz Number

Let $\phi: X \multimap Y$ be an $n$-valued multimap of finite polyhedra. For each $x \in X$, define $\gamma(x ; \phi)$ to be the minimum distance $d\left(y_{i}, y_{j}\right)$ among all pairs of points $y_{i}, y_{j} \in \phi(x)$. Let $\gamma(\phi)$ denote the infimum of $\gamma(x ; \phi)$ for $x \in X$. It is noted in [5] that the continuity of $\phi$, and hence of $\gamma(\cdot ; \phi)$, together with the compactness of $X$ implies that $\gamma(\phi)>0$.

The Splitting Lemma [5, Lemma 1] implies that if $S$ is a closed, simply-connected subset of $S$ of $X$, then the restriction of $\phi$ to $X$ splits, that is, there are single-valued maps $f_{1}, \ldots, f_{n}: S \rightarrow Y$ such that $\phi(x)=$ $\left\{f_{1}(x), \ldots, f_{n}(x)\right\}$ for all $x \in S$. Then, fixing simplicial structures $X=|K|$ and $Y=|L|$, an $n$-valued multimap $\phi^{\prime}: X \mapsto Y$ is said to be simplicial if, for each simplex $\sigma$ of $K$, the restriction of $\phi^{\prime}$ to the polyhedron whose geometric realization $\bar{\sigma}$ is the closure of $\sigma \in K$, splits into $n$ maps $f_{j}:|\bar{\sigma}| \rightarrow|L|$ each of which is simplicial. Suppose $\varepsilon>0$ is given. The Simplicial Approximation Theorem of [5, Theorem 4], states that if the
mesh of a subdivision $\left|L^{\prime}\right|$ of $|L|$ is less than both $\frac{1}{4} \gamma(\phi)$ and $\varepsilon$, then there is a subdivision $K^{\prime}$ of $K$ and a simplicial $n$-valued multimap $\phi^{\prime}:\left|K^{\prime}\right| \multimap\left|L^{\prime}\right|$ such that the Hausdorff distance $d\left(\phi(x), \phi^{\prime}(x)\right)<\varepsilon$ for all $x \in X$. Moreover, as noted on [5, p. 79], for each $\sigma \in K^{\prime}$, the image

$$
\phi^{\prime}(\bar{\sigma})=\bar{\tau}_{1} \cup \cdots \cup \bar{\tau}_{n}
$$

where the $\bar{\tau}_{j}$ are $n$ disjoint closed simplices of $|L|$.
Now let $\left\{C_{k}\left(K^{\prime}\right)\right\}$ and $\left\{C_{k}\left(L^{\prime}\right)\right\}$ denote the rational simplicial chain complexes and define $\phi_{k}: C_{k}\left(K^{\prime}\right) \rightarrow C_{k}\left(L^{\prime}\right)$ by setting

$$
\phi_{k}(\sigma)=\tau_{1}+\cdots+\tau_{n} .
$$

Since each $\bar{\tau}_{j}=f_{j}(\bar{\sigma})$ and the $f_{j}$ induce chain maps, $\left\{\phi_{k}\right\}$ is a chain map of these chain complexes. Moreover, since an $n$-valued homotopy of a split $n$-valued map is also split [3, Theorem 2.1], an $n$-valued homotopy of $\phi$ to $\psi: X \multimap Y$, when restricted to $\bar{\sigma}$, splits into $n$ homotopies of the $f_{j}$ which determines a splitting $g_{1}, \ldots, g_{n}$ of the restriction of $\psi$ to $\bar{\sigma}$. Those homotopies define chain homotopies $D_{1}, \ldots, D_{n}$ and thus, defining

$$
D(\sigma)=D_{1}(\sigma)+\cdots+D_{n}(\sigma)
$$

we obtain a chain homotopy between $\left\{\phi_{k}\right\}$ and $\left\{\psi_{k}\right\}$. The chain map $\left\{\phi_{k}: C_{k}\left(K^{\prime}\right) \rightarrow C_{k}\left(L^{\prime}\right)\right\}$ thus induces a homomorphism of rational homology

$$
\phi_{*}=\left\{\phi_{* k}: H_{k}(X) \rightarrow H_{k}(Y)\right\}
$$

such that if $\phi, \psi: X \multimap Y$ are homotopic $n$-valued multimaps, then $\phi_{*}=\psi_{*}$.

Now suppose $\phi: X \multimap X$ is an $n$-valued multimap of a finite polyhedron. The definition identified a specific homomorphism of rational homology $\phi_{*}$ as a finite set of nontrivial linear endomorphisms, so the definition

$$
L(\phi)=\Lambda\left(\phi_{*}\right)=\sum_{k}(-1)^{k} \operatorname{trace}\left(\phi_{* k}\right)
$$

determines a well-defined Lefschetz number for $\phi$.

Theorem 1 (Lefschetz fixed point theorem for $n$-valued multimaps). If $\phi: X \rightarrow X$ is an $n$-valued multimap of a finite polyhedron such that $L(\phi) \neq 0$, then $x \in \phi(x)$ for some $x \in X$.

Proof. We will prove that if $\phi$ has no fixed points, then $L(\phi)=0$. For each $x \in X$, let $\varepsilon(x ; \phi)$ be the minimum of the distance from $x$ to each point $y_{j} \in \phi(x)$. Let $\varepsilon(\phi)$ be the infimum of $\varepsilon(x ; \phi)$ for all $x \in X$. If $\phi$ has no fixed points, then the continuity of $\phi$, and thus of $\varepsilon(\cdot ; \phi)$ together with the compactness of $X$ imply that $\varepsilon(\phi)>0$. Let $X=|K|$ be a triangulation of $X$ of mesh smaller than both $\frac{1}{4} \gamma(\phi)$ and $\frac{1}{2} \varepsilon(\phi)$ and let $\phi^{\prime}:\left|K^{\prime}\right| \rightleftharpoons|K|$ be a simplicial approximation to $\phi$. Let $\sigma \in K^{\prime}$ be a $k$-simplex. Then, as above, we have the chain map $\phi_{k}(\sigma)=\tau_{1}+\cdots+\tau_{n}$, where $\bar{\tau}_{j}=f_{j}(\bar{\sigma})$ for a splitting $f_{1}, \ldots, f_{n}$ of the restriction of $\phi^{\prime}$ to $\bar{\sigma}$. As in the proof of Theorem II.A. 1 on page 26 of [2], it must be that $\bar{\sigma} \cap \bar{\tau}_{j}=\varnothing$ for all $j=1, \ldots, n$. Moreover, there is a chain map $h=\left\{h_{k}\right\}$ obtained from barycentric subdivision such that the diagonal entries of the matrix of $\phi_{k} h_{k}: C_{k}(K)$ $\rightarrow C_{k}(K)$ are all zero, so $\operatorname{trace}\left(\phi_{k} h_{k}\right)=0$. The Hopf Trace Theorem [2, Theorem I.D.3] then completes the proof that $L(\phi)=0$.

## 3. The Circle

In the case $n=1$, the definition of the Lefschetz number and the proof of the Lefschetz theorem reduce to the classical ones. However, there might be ways to extend Lefschetz theory to $n$-valued multimaps with $n>1$ other than that described in the previous section. As evidence that the generalization we have presented is the appropriate one, we will demonstrate that a key relationship between the classical Lefschetz and Nielsen fixed point theories holds also for this Lefschetz number and the Nielsen number for $n$-valued multimaps that was developed by Schirmer in [6, 7].

Theorem 2. Let $\phi: S^{1} \rightarrow S^{1}$ be an n-valued multimap of the circle with Nielsen number $N(\phi)$ and Lefschetz number $L(\phi)$. Then $N(\phi)=|L(\phi)|$.

Proof. Let $p: \mathbb{R} \rightarrow S^{1}$ be the map defined by $p(t)=e^{i 2 \pi t}$. For an integer $d$, define an $n$-valued multimap $\phi_{n, d}: S^{1} \multimap S^{1}$ by

$$
\phi_{n, d}(p(t))=\left\{p\left(\frac{d}{n} t\right), p\left(\frac{d}{n} t+\frac{1}{n}\right), \ldots, p\left(\frac{d}{n} t+\frac{n-1}{n}\right)\right\} .
$$

By Theorem 3.1 of [3], the map $\phi$ is homotopic to $\phi_{n, d}$ for some $d$. Since the Lefschetz number is homotopy invariant and, by Theorem 6.5 of [6], the Nielsen number is also homotopy invariant, it is sufficient to prove that $N\left(\phi_{n, d}\right)=\left|L\left(\phi_{n, d}\right)\right|$ for all positive integers $n$ and integers $d$.

For $r \geq 2$, let $S_{r}^{1}$ denote the triangulation of $S^{1}$ with vertex set $\left\{v_{j}^{r}=p\left(\frac{j}{r}\right)\right\}$ for $j=0,1, \ldots, r-1, r$. Note that $v_{r}^{r}=v_{0}^{r}$. The group $C_{1}\left(S_{r}^{1}\right)$ of simplicial 1-chains with rational coefficients is generated by the 1-simplices $\left\{\left[v_{j}^{r}, v_{j+1}^{r}\right]\right\}$. The homology class $\bar{z}$ of the cycle

$$
z_{r}=\left[v_{0}^{r}, v_{1}^{r}\right]+\left[v_{1}^{r}, v_{2}^{r}\right]+\cdots+\left[v_{r-1}^{r}, v_{r}^{r}\right]
$$

generates $H_{1}\left(S^{1}\right)$. The multimap $\phi_{n, d}: S_{n|d|}^{1}{ }^{\circ} S_{n^{2}}^{1}$ is simplicial because, for $0 \leq j \leq n|d|-1$, we have

$$
\phi_{n, d}\left(v_{j}^{n|d|}\right)=\left\{v_{\sigma(j)}^{n^{2}}, v_{\sigma(j)+n}^{n^{2}}, v_{\sigma(j)+2 n}^{n^{2}}, \ldots, v_{\sigma(j)+(n-1) n}^{n^{2}}\right\}
$$

where $\sigma(j)=j$ if $d \geq 0$ and $\sigma(j)=-j$ if $d<0$. The sets $\phi_{n, d}\left(v_{j}^{n|d|}\right)$ and $\phi_{n, d}\left(v_{j^{\prime}}^{n|d|}\right)$ are identical if and only if $j$ and $j^{\prime}$ are congruent $\bmod n$. Otherwise they are disjoint. The set of vertices of $S_{n|d|}^{1}$ is thus partitioned, in terms of the images under $\phi_{n, d}$, into $n$ subsets, each consisting of $|d|$ vertices.

If $d=0$, then $\phi_{n, d}$ is a constant multimap so it induces the trivial homomorphism $\phi_{n, d^{*}}: H_{1}\left(S^{1}\right) \rightarrow H_{1}\left(S^{1}\right)$. Represent the conjugacy class
$\bmod n$ of an integer $j$ by $\langle j\rangle$, where $0 \leq\langle j\rangle \leq n-1$. If $d>0$, then the chain map $C_{1}\left(\phi_{n, d}\right): C_{1}\left(S_{n d}^{1}\right) \rightarrow C_{1}\left(S_{n^{2}}^{1}\right)$ is given by

$$
C_{1}\left(\phi_{n, d}\right)\left[v_{j}^{n d}, v_{j+1}^{n d}\right]=\sum_{k=0}^{n-1}\left[v_{\langle j\rangle+k n}^{n^{2}}, v_{\langle j\rangle+k n+1}^{n^{2}}\right]
$$

and therefore

$$
C_{1}\left(\phi_{n, d}\right)\left(z_{n d}\right)=\sum_{j=0}^{n d-1 n-1} \sum_{k=0}\left[v_{\langle j\rangle+k n}^{n^{2}}, v_{\langle j\rangle+k n+1}^{n^{2}}\right]=d z_{n^{2}}
$$

which implies that $\phi_{n, d *}(\bar{z})=d \bar{z}$.
If $d<0$, then

$$
\begin{aligned}
C_{1}\left(\phi_{n, d}\right)\left[v_{j}^{n|d|}, v_{j+1}^{n|d|}\right] & =\sum_{k=0}^{n-1}\left[v_{\langle-j\rangle+k n}^{n^{2}}, v_{\langle-j\rangle+k n-1}^{n^{2}}\right] \\
& =\sum_{k=0}^{n-1}-\left[v_{\langle-j\rangle+k n-1}^{n^{2}}, v_{\langle-j\rangle+k n}^{n^{2}}\right] .
\end{aligned}
$$

Each generator of $C_{1}\left(S_{n^{2}}^{1}\right)$ appears in the chain $C_{1}\left(\phi_{n, d}\right)\left(z_{n|d|}\right)$ exactly $|d|$ times, so

$$
C_{1}\left(\phi_{n, d}\right)\left(z_{n|d|}\right)=-|d| z_{n^{2}}=d z_{n^{2}}
$$

which implies that $\phi_{n, d *}(\bar{z})=d \bar{z}$ in this case also.
Letting $1 \in H_{0}\left(S^{1}\right)$ be a generator, it is clear from the definition that $\phi_{*}(1)=n 1$ for any $n$-valued multimap $\phi: S^{1} \multimap S^{1}$. Thus $L\left(\phi_{n, d}\right)=n-d$ for all positive integers $n$ and integers $d$. By Theorem 5.1 of [3], $N\left(\phi_{n, d}\right)$ $=|n-d|$ so $N\left(\phi_{n, d}\right)=\left|L\left(\phi_{n, d}\right)\right|$ and the proof is complete.

## Acknowledgement

Author thanks the referee for a thoughtful analysis of this paper.

## References

[1] C. Berge, Topological Spaces, Oliver \& Boyd, 1963.
[2] R. Brown, The Lefschetz Fixed Point Theorem, Scott Foresman, 1971.
[3] R. Brown, Fixed points of $n$-valued multimaps of the circle, Bull. Pol. Acad. Sci. Math. 54 (2006), 153-162.
[4] B. O'Neill, Induced homology homomorphisms for set-valued maps, Pacific J. Math. 7 (1957), 1179-1184.
[5] H. Schirmer, Fix-finite approximations of $n$-valued multifunctions, Fund. Math. 121 (1984), 73-80.
[6] H. Schirmer, An index and Nielsen number for $n$-valued multifunctions, Fund. Math. 124 (1984), 207-219.
[7] H. Schirmer, A minimum theorem for $n$-valued multifunctions, Fund. Math. 126 (1985), 83-92.


[^0]:    2000 Mathematics Subject Classification: 55M20, 55N25.

