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MONITORING THE MEAN OF AUTOCORRELATED PROCESSES WITH GWMA CHARTS BASED ON RESIDUALS

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Abstract

When autocorrelation exists between observations, the performance of control charts may cause dramatic influence. Specifically, a high number of false alarms signals are generated. This work extends the exponentially weighted moving average (EWMA) control chart, called the generally weighted moving average (GWMA) control chart, to monitor the mean of autocorrelated processes based on the residuals from the forecast values of a first-order autoregressive (AR(1)) process with a random error. The GWMA control chart of residuals with time-varying control limits is more sensitive to shifts in the mean upon start-up. A simulation is conducted to evaluate the average run length (ARL) of the EWMA and GWMA control charts of residuals. The results of these simulations reveal that the GWMA control scheme is more sensitive than the EWMA control scheme to small shifts in an autocorrelated process mean. The composite Shewhart-GWMA control chart of residuals is presented to enhance the detection capacity of the GWMA control chart in detecting small shifts of an autocorrelated process mean.

1. Introduction

Numerous control charts are employed to monitor processes for the

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purpose of detecting special causes and then improve the processes. The design and evaluation of control charts typically assume that the quality characteristics of processes are independent. However, in many applications the dynamics of the process produce autocorrelation in observations which are closely spaced in time. The presence of significant autocorrelation can have a serious impact on the properties of traditional control charts (see, for example, Padgett et al. [7] and VanBrackle and Reynolds [16]). Specifically, a high false alarm rate is generated. Hence, considering the problem of monitoring an autocorrelated process is important.

The technique for detecting a process must be determined when an autocorrelation is present. Many authors have studied the characteristics of autocorrelated processes. Two general approaches to developing control charts can be adopted in cases of autocorrelation. First, standard control charts are used: parameters estimated according to models and the control limits adjusted for the autocorrelated observations (see, for example, Vasilopoulos and Stamboulis [18] and Yashchin [20]). Second, a proper time series model is applied to the process data and the residuals or the forecast errors are treated as control statistics (see, for example, Alwan and Roberts [1], Harris and Ross [4], Montgomery and Mastrangelo [6], Schmid [12], Superville and Adams [15], Vander Weil [17], Wardell et al. [19]). A fundamental conclusion is that autocorrelation markedly influences the properties of control charts. Lu and Reynolds [5] compared the performance of an EWMA control chart based on the original observations with that of an EWMA control chart based on residuals from the forecast values of a first-order autoregressive (AR(1)) process with a random error. They indicated that when the level of autocorrelation is strong, the EWMA control chart based on residuals form the AR(1) model requires less time to detect various shifts; but for low or moderate level of autocorrelation, the two EWMA charts require the same amount of time to detect various shifts.

Sheu and Griffith [13] and Sheu and Lin [14] developed and applied an expanded EWMA control chart, called the *generally weighted moving average* (GWMA), to enhance the detection ability of control charts. They demonstrated that the GWMA control chart performs substantially better than either the Shewhart or the EWMA control chart for monitoring small shifts in the process mean. However, they evaluated the properties of GWMA control charts based on independent observations instead of autocorrelation between observations. Hence, the aim of this work is to examine GWMA control charts of residuals to monitor the process mean of autocorrelated processes. The GWMA control chart performance is compared with that of the EWMA and Shewhart-GWMA control charts of residuals. A numerical simulation is performed to assess the average run length (ARL) properties of various mean shifts and adjusted parameters with different levels of autocorrelation.

2. General Model

Sheu and Lin [14] first proposed the GWMA control chart to enhance the detection ability of EWMA control charts. The various weights in the GWMA model were designed to drop from the present sample to past samples, such that the GWMA reflects crucial information on recent processes.

Suppose that events A and B are complementary and mutually exclusive. Let M count the number of samples until the first occurrence of event A since its previous occurrence. Let $\overline{P}_j = P(M > j)$. Hence, $\overline{P}_j$ satisfies $1 = \overline{P}_0 \geq P_1 \geq \cdots \geq 0$. Let $P(M = j) = \overline{P}_{j-1} - \overline{P}_j = \overline{P}_{j-1}\left(1 - \dfrac{\overline{P}_j}{\overline{P}_{j-1}}\right)$. Hence, event A occurs with probability $1 - \dfrac{\overline{P}_j}{\overline{P}_{j-1}}$ at the j-th sample whereas event B occurs with probability $\dfrac{\overline{P}_j}{\overline{P}_{j-1}}$. Due to

$$
\begin{aligned}
\sum_{j=1}^{\infty} P(M = j) &= P(M = 1) + P(M = 2) + \cdots + P(M = j) + P(M > j) \\
&= (\overline{P}_0 - \overline{P}_1) + (\overline{P}_1 - P_2) + \cdots + (\overline{P}_{j-1} - \overline{P}_j) + \vec{P}_j \\
&= \overline{P}_0 \\
&= 1
\end{aligned}
$$

$P(M = 1)$, $P(M = 2)$, ..., $P(M = j)$ can be regarded as the weights of the GWMA and be the weights of the current sample, the second updated sample , …, the remote sample, respectively.

Let X_j represent the measurement at the j-th time period and assume that X_j, $j = 1, 2, 3, \ldots,$ are independent random variables with mean μ_0 and constant variance σ^2. Let Y_j denote the generally weighted moving average in the plotted statistics at time j. Usually, the initial value Y_0 is set equal to μ_0 for convenience. Then, Y_j can be configured as

$$\begin{aligned} Y_j &= \sum_{k=1}^{j} P(M = k) X_{j-k+1} + P(M > j)\mu_0 \\ &= \sum_{k-1}^{j} (\overline{P}_{k-1} - \overline{P}_k) X_{j-k+1} + \overline{P}_j \mu_0. \end{aligned} \tag{1}$$

The expected value of Eq. (1) is

$$\begin{aligned} E(Y_j) &= E\left(\sum_{k-1}^{j} (\overline{P}_{k-1} - \overline{P}_k) X_{j-k+1} + \overline{P}_j \mu_0 \right) \\ &= \mu_0. \end{aligned} \tag{2}$$

The variance of Eq. (1) is

$$\mathrm{Var}(Y_j) = Q_j \cdot \sigma^2, \tag{3}$$

where

$$Q_j = \sum_{k=1}^{j} (\overline{P}_{k-1} - \overline{P}_k)^2.$$

Thus, the GWMA control chart can be written as

$$\begin{aligned} UCL &= \mu_0 + L\sqrt{Q_j}\sigma \\ CL &= \mu_0 \\ LCL &= \mu_0 - L\sqrt{Q_j}\sigma, \end{aligned} \tag{4}$$

where L denotes the width of the control limits and is determined by the practitioner to achieve the desired in-control ARL for a GWMA control chart. When statistics Y_j falls outside the range of control limits, it indicates that the process is out of control and some actions should be taken adequately.

3. Modeling GWMA Control Chart of Residuals

A fundamental assumption that underlies conventional control charts is that process data are mutually independent. Unfortunately, data taken from several manufacturing processes, such as those in the chemical or pharmaceutical industries frequently violate the independence assumption. Despite the fact that quality characteristics associated with the discrete parts of the manufacturing processes will be autocorrelated when the sampling time is short. Accordingly, this work considers autocorrelated observations based on an AR(1) process with a random error, and utilizes GWMA control charts to monitor the mean of the autocorrelated process.

The AR(1) process with a random error is applied, as described by Lu and Reynolds [5]. X_t is assumed to be given by

$$X_t = \mu_t + \varepsilon_t, \quad t = 1, 2, 3, \ldots, \tag{5}$$

where the values of ε_t are independent and have a common normal distribution with a mean of 0 and a variance of σ_ε^2 where μ_t can be specialized as an AR(1) process with a process mean of ξ_0, thus that,

$$\mu_t = (1 - \phi)\xi_0 + \phi\mu_{t-1} + a_t, \quad t = 1, 2, 3, \ldots, \tag{6}$$

where the values of a_t are independent normal random variables with a mean of 0 and a variance of σ_a^2, and are independent of the ε_t values, and ϕ is the AR(1) parameter, which satisfies $|\phi| < 1$. The initial value μ_0 is assumed to be normally distributed with mean ξ_0 and variance $\sigma_a^2/(1 - \phi^2)$, implying that X_t will be a normal distribution with a mean of ξ_0 and a variance of

$$\sigma_X^2 = \sigma_\mu^2 + \sigma_\varepsilon^2 = \frac{\sigma_a^2}{1 - \phi^2} + \sigma_\varepsilon^2, \quad \text{for all } t \geq 1.$$

Conveniently define

$$\psi = \frac{\sigma_\mu^2}{\sigma_X^2} = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\varepsilon^2}$$

to be the proportion of the total process variance that is due to the AR(1) process. The covariance between X_t and X_{t+1} is $\phi\sigma_\mu^2$, and the autocorrelation coefficient between X_t and X_{t+1} is $p = \phi\psi$.

The AR(1) process with a random error is equivalent to an autoregressive moving average process of order 1 and 1 (ARMA(1, 1)) process (see, Box et al. [2]). The ARMA (1, 1) process can be represented as

$$X_t = (1 - \phi)\xi_0 + \phi X_{t-1} + b_t - \theta b_{t-1}, \tag{7}$$

where b_t is the random white noise of the ARMA (1, 1) process at time t and follows independent normal random variable with a mean of 0 and a variance of σ_b^2, where θ is the moving average parameter, and ϕ is the same autoregressive parameter as in Eq. (6). Equations for the parameters ϕ, θ and σ_b^2 in the ARMA(1, 1) model are available in terms of the parameters θ, σ_a^2 and σ_ε^2 in the AR(1) with a random error model and vice versa (see, Lu and Reynolds [5] and Reynolds et al. [8]). Most applications, for which parameters satisfy $0 \le \theta \le \phi < 1$ and $\sigma_b^2 > 0$, the ARMA (1, 1) model can be used to yield process observations.

In recent years, many scholars have sought to find effective monitoring techniques with significantly autocorrelated processes. A proper time series model can typically be fitted to the process data and the residuals or forecast errors regarded as nearly independent control statistics to monitor the mean of autocorrelated process.

Evaluating the performance of control charts based on the residuals depends on determining the distribution of the residuals. When the process is controlled, the minimum mean square error forecast is,

$$\hat{X}_t = (1 - \phi)\xi_0 + \phi X_{t-1} + b_t - \theta e_{t-1}, \tag{8}$$

where

$$e_t = X_t - \hat{X}_t$$

is the residual at time t and e_t is a sequence of independently and identically distributed random variables with mean 0 and variance σ_b^2 (see, Box et al. [2]). When the process is out of control, the shift in the process mean is supposed to be ξ_1. Then, the expected residual is,

$$E(e_t) = \frac{\theta^t(\phi - \theta) - \phi + 1}{1 - \theta}(\xi_1 - \xi_0), \quad t \geq 0. \tag{9}$$

The control statistic Y_j of a GWMA control chart based on the residuals can be expressed as:

$$Y_j = \sum_{k-1}^{j} (\overline{P}_{k-1} - \overline{P}_k)(e_{j-k+1}) + \overline{P}_j \xi_0. \tag{10}$$

The expected value of Eq. (10) will be

$$E(Y_j) = E\left(\sum_{k-1}^{j} (\overline{P}_{k-1} - \overline{P}_k)(e_{j-k+1}) + \overline{P}_j \xi_0\right)$$

$$= \xi_0. \tag{11}$$

The variance of Eq. (10) will be

$$\mathrm{Var}(Y_j) = \mathrm{Var}\left(\sum_{k-1}^{j} (\overline{P}_{k-1} - \overline{P}_k)(e_{j-k+1}) + \overline{P}_j \xi_0\right)$$

$$= Q_j \cdot \sigma_b^2, \tag{12}$$

where $Q_j = \sum_{k=1}^{j} (\overline{P}_{k-1} - \overline{P}_k)^2$. The time-varying control limits of the GWMA control chart of residuals will be

$$UCL = \xi_0 + L\sqrt{Q_j}\sigma_b$$

$$CL = \xi_0$$

$$LCL = \xi_0 - L\sqrt{Q_j}\sigma_b. \tag{13}$$

For computing convenience, we choose

$$\overline{P}_j = q^{j^\alpha}, \quad 0 \leq q < 1, \quad \alpha > 0, \quad j = 0, 1, 2, 3, \ldots,$$

where the design parameter q is constant, and the adjustment parameter α is determined by the practitioner. According to Eq. (10), we obtain

$$Y_j = \sum_{k-1}^{j} (q^{(k-1)^\alpha} - q^{k^\alpha})(e_{j-k+1}) + q^{j^\alpha} \xi_0,$$

$$0 \leq q < 1, \quad \alpha > 0, \quad j = 1, 2, 3, \ldots . \tag{14}$$

If $\alpha = 1$ and $\lambda = 1 - q$ is set, then Eq. (14) can be transformed into

$$Y_j = \sum_{k-1}^{j} \lambda(1-\lambda)^{j-1}(e_{j-k+1}) + (1-\lambda)^j \xi_0, \quad 0 < \lambda \leq 1, \quad j = 1, 2, 3, \ldots .$$

That is the EWMA control chart. When $\lambda = 1$, the EWMA control chart becomes the Shewhart control chart. Consequently, the EWMA and Shewhart control charts are special cases of the GWMA control chart of residuals.

4. Measuring the Performance of Control Charts

The control chart performance is generally assessed in terms of its ARL. The ARL is expected to be as large as possible when it is under control and as small as possible when a process is out of control. Accordingly, in measuring the statistical performance, ARLs of various shifts in process means for a single in-control ARL are compared. Greater detection ability corresponds to a shorter out-of-control ARL. Some work has already been done on computing the ARL of an EWMA control scheme (see, for example, Crowder [3], Roberts [9], Robinson and Ho [10], Saccucci and Lucas [11]). Many methods have been proposed, including integral equations, Markov chain, asymptotic approximation and simulation. Since the GWMA statistic is more complicated than that of EWMA, determining the exact ARL for given control limits is complex. This work determines appropriate parameters of GWMA control charts of residuals to estimate ARLs. The following five steps should be implemented in simulations:

Step 1. Specify parameters ϕ and Ψ, shift size ξ_1 and the charting parameter (q, α, L).

Step 2. Generate a set of simulation data in an AR(1) process with random error and calculate the residuals by fitting the ARMA (1, 1) model.

Step 3. Calculate the GWMA control statistic Y_j according to Eq. (10) at the target value $\xi_0 = 0$ and the corresponding time-varying upper control limit (UCL) and lower control limit (LCL) according to Eq. (13). Record the run length (RL) when Y_j exceeds the control limits and the trial halts; return to Step 2.

Step 4. Run n iterations; the average of n run lengths with the specific parameters is obtained. Modify the control limit constant (L) using a bisectional approach to achieve the desirable in-control ARL.

Step 5. Set the in-control parameters to monitor the process and calculate the out-of-control ARLs when the process shifts.

The simulation is performed using *R* Programming for Statistics. Each simulation runs 10,000 iterations. The approximate standard error of an ARL can be computed as $\frac{SDRL}{\sqrt{n}}$, where *SDRL* be the standard deviation of run length and n denotes the number of simulation iterations per run. The value of *SDRL* is approximately equal to that of ARL. For instance, when $n = 10{,}000$ and the ARL is roughly 370.4, the approximate standard error will be 3.70, an acceptable error level.

Table 1 presents the ARL results for the GWMA control charts of residuals with the time-varying control limits of interest for $\psi = 0.1$ and Table 2 shows the results for $\psi = 0.9$, with various design parameters $q(q = 0.5, q = 0.7, q = 0.9)$ and adjustment parameters $\alpha(\alpha = 0.5,\ \alpha = 0.7,$ $\alpha = 0.8, \alpha = 0.9, \alpha = 1.0)$. The values of ϕ, which quantifies the correlation between μ_{j-1} and μ_j, are set to 0.4 and 0.8. Usually, a positive correlation would be more likely in applications; consequently, the negative values of ϕ in the numerical results are neglected. Therefore, the correlation between X_{j-1} and X_j ranges from $\rho = \psi\phi = (0.1)(0.4) = 0.04$

to $(0.9)(0.8) = 0.72$, where $\psi = \sigma_\mu^2/\sigma_X^2$ is the proportion of the process variance due to the AR(1) process. The width of the control limits (L) is adjusted to maintain the in-control ($\xi_0 = 0$) ARL at around 370.4. Restated, type I errors are set to 0.0027 for different GWMA control schemes, whereas out-of-control ARLs are applied for comparison.

Table 1. ARLs of GWMA charts with time-varying control limits when $\psi = 0.1$

			$\phi = 0.4$					$\phi = 0.8$		
			$q = 0.5$					$q = 0.5$		
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
ξ_1	$L = 2.992$	$L = 2.987$	$L = 2.986$	$L = 2.982$	$L = 2.980$	$L = 3.002$	$L = 2.996$	$L = 2.995$	$L = 2.993$	$L = 2.989$
0.00	370.39	370.03	370.16	370.60	370.31	370.62	370.81	370.43	370.56	370.81
0.25	190.92	196.64	202.45	205.82	209.30	232.78	239.41	243.01	247.26	249.90
0.50	69.55	72.41	75.38	77.91	80.86	100.73	105.21	108.41	111.49	113.82
0.75	31.33	31.78	32.72	33.63	34.55	46.80	47.56	49.26	51.11	52.80
1.00	16.74	16.23	16.48	16.76	17.20	24.89	24.63	25.23	25.97	26.69
1.25	10.26	9.61	9.57	9.60	9.74	14.52	13.89	14.15	14.37	14.74
1.50	6.88	6.38	6.28	6.23	6.27	9.40	8.72	8.70	8.78	8.86
2.00	3.82	3.55	3.49	3.45	3.41	4.74	4.42	4.35	4.32	4.31
3.00	1.76	1.71	1.69	1.68	1.67	2.01	1.93	1.90	1.88	1.87
			$q = 0.7$					$q = 0.7$		
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
ξ_1	$L = 2.978$	$L = 2.956$	$L = 2.948$	$L = 2.939$	$L = 2.934$	$L = 2.992$	$L = 2.973$	$L = 2.963$	$L = 2.957$	$L = 2.951$
0.00	370.72	370.27	370.67	370.33	370.91	370.70	370.01	370.43	370.29	370.17
0.25	131.11	140.10	147.57	154.81	162.45	169.33	182.22	190.01	199.28	207.10
0.50	44.43	44.17	46.42	49.08	52.75	62.19	63.71	66.93	71.48	76.78
0.75	21.38	19.87	20.25	21.00	22.10	29.85	28.63	29.55	31.28	33.37
1.00	12.58	11.25	11.20	11.30	11.57	17.01	15.56	15.61	16.11	16.77
1.25	8.18	7.30	7.13	7.11	7.17	10.89	9.76	9.61	9.72	9.96
1.50	5.85	5.26	5.13	5.04	5.04	7.44	6.66	6.51	6.48	6.54
2.00	3.47	3.18	3.10	3.05	3.01	4.17	3.80	3.69	3.65	3.63
3.00	1.70	1.65	1.63	1.61	1.60	1.92	1.83	1.81	1.79	1.77

	$q = 0.9$					$q = 0.9$				
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
ξ_1	$L = 2.896$	$L = 2.798$	$L = 2.762$	$L = 2.739$	$L = 2.729$	$L = 2.920$	$L = 2.829$	$L = 2.799$	$L = 2.777$	$L = 2.766$
0.00	370.48	370.55	370.62	370.38	370.41	370.53	370.22	370.10	370.08	370.51
0.25	81.78	78.66	82.27	89.01	97.49	106.50	104.48	110.76	120.46	131.75
0.50	30.43	26.94	26.83	27.67	29.25	40.51	36.19	36.70	38.41	41.29
0.75	16.05	13.82	13.54	13.59	13.85	21.24	18.32	18.11	18.38	18.97
1.00	10.01	8.57	8.31	8.26	8.33	12.98	11.13	10.91	10.91	11.09
1.25	6.84	5.91	5.73	5.66	5.68	8.74	7.55	7.36	7.29	7.33
1.50	5.05	4.38	4.24	4.18	4.17	6.25	5.43	5.28	5.24	5.25
2.00	3.12	2.79	2.70	2.65	2.64	3.70	3.27	3.19	3.14	3.14
3.00	1.62	1.52	1.49	1.48	1.47	1.81	1.69	1.65	1.64	1.63

Table 2. ARLs of GWMA charts with time-varying control limits when $\psi = 0.9$

	$\phi = 0.4$, $q = 0.5$					$\phi = 0.8$, $q = 0.5$				
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
ξ_1	$L = 2.994$	$L = 2.987$	$L = 2.986$	$L = 2.983$	$L = 2.981$	$L = 2.993$	$L = 2.988$	$L = 2.986$	$L = 2.984$	$L = 2.981$
0.00	370.76	370.21	370.98	370.97	370.41	370.06	370.21	370.81	370.89	370.81
0.25	254.16	260.30	263.79	267.43	270.28	333.25	334.62	335.46	339.38	339.11
0.50	121.77	126.89	130.20	134.30	137.61	244.93	250.42	254.59	256.63	261.07
0.75	60.37	62.12	64.52	66.84	69.43	163.29	170.00	173.78	177.02	179.66
1.00	33.66	34.11	35.08	36.15	37.51	105.46	110.95	114.22	116.72	120.50
1.25	20.34	19.77	20.20	20.65	21.37	67.91	70.26	72.95	75.12	77.19
1.50	13.23	12.49	12.56	12.74	13.03	43.13	43.73	45.10	46.37	47.86
2.00	6.43	5.90	5.87	5.80	5.84	16.05	15.86	16.13	16.51	16.88
3.00	2.15	2.01	1.97	1.94	1.92	1.89	1.80	1.77	1.75	1.75
	$q = 0.7$					$q = 0.7$				
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
ξ_1	$L = 2.980$	$L = 2.959$	$L = 2.949$	$L = 2.941$	$L = 2.932$	$L = 2.980$	$L = 2.960$	$L = 2.950$	$L = 2.941$	$L = 2.932$
0.00	370.54	370.74	370.34	370.95	370.88	370.60	370.71	370.55	370.37	370.24
0.25	191.52	206.42	214.52	224.02	230.52	303.70	309.90	315.20	318.28	318.45
0.50	76.22	79.21	83.90	90.60	95.78	180.61	195.16	203.26	212.31	217.66

0.75	38.38	37.90	39.46	41.95	44.53	106.74	113.54	121.01	127.57	133.89
1.00	33.42	20.84	21.20	22.16	23.37	64.69	66.76	70.46	74.77	79.74
1.25	14.45	13.01	12.96	13.25	13.62	41.14	40.87	42.60	44.89	47.73
1.50	9.94	8.79	8.62	8.63	8.76	25.98	25.04	25.80	26.97	28.60
2.00	5.33	4.73	4.58	4.50	4.47	10.27	9.03	8.97	9.19	9.49
3.00	1.99	1.83	1.79	1.75	1.74	1.59	1.44	1.41	1.39	1.38
	$q = 0.9$					$q = 0.9$				
	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$
ξ_1	$L = 2.898$	$L = 2.798$	$L = 2.763$	$L = 2.739$	$L = 2.726$	$L = 2.901$	$L = 2.800$	$L = 2.766$	$L = 2.741$	$L = 2.726$
0.00	370.71	370.19	370.35	370.02	370.14	370.25	370.68	370.72	370.03	370.53
0.25	122.69	123.58	130.66	140.87	153.11	230.90	236.18	244.64	256.99	269.08
0.50	49.36	44.63	45.78	48.23	52.41	114.02	113.11	119.44	128.79	139.90
0.75	26.55	23.02	22.80	23.28	24.47	65.18	59.56	61.84	66.55	72.76
1.00	16.45	13.97	13.59	13.67	13.92	40.24	34.88	35.21	36.92	40.07
1.25	11.12	9.30	9.00	8.91	8.99	26.13	21.61	21.37	21.88	23.07
1.50	7.88	6.58	6.34	6.26	6.26	17.04	12.51	13.01	12.93	13.43
2.00	4.47	3.75	3.68	3.47	3.44	6.95	5.21	4.88	4.75	4.74
3.00	1.80	1.61	1.56	1.53	1.51	1.38	1.23	1.19	1.18	1.17

Tables 1 and 2 concern the effectiveness of the GWMA and EWMA control charts of residuals with time-varying control limits in monitoring the autocorrelated process mean. When $\alpha = 1.0$, the GWMA control scheme reduces to the EWMA control scheme. For the same level of autocorrelation, the adjustment parameter α of the GWMA control chart of residuals is more sensitive to small shifts in the autocorrelated process mean than to those of the EWMA control chart of residuals with time-varying control limits. The boldface figures in Tables 1 and 2, especially when q is small, clarify the results. For instance, when $\psi = 0.1$, $\phi = 0.8$, $\alpha = 0.8$ and $q = 0.5$ within a shift of 1.50, the out-of-control ARL is smaller than the ARL of the EWMA control chart of residuals with time-varying control limits. However, when q is large, no enhancement in detection ability is evident. For example, when $\rho = 0.08$, $\alpha = 0.8$ and q is increased to 0.9, the GWMA control chart is superior to the EWMA control chart of residuals with time-varying control limits within a shift of 1.0.

Numerical results also demonstrated that the detection ability of GWMA control charts of residuals with time-varying control limits increases with the level of autocorrelation. In particular, at the high level of autocorrelation $\rho = 0.72$ and $q = 0.5$, the GWMA control chart is more sensitive than the EWMA control chart of residuals with time-varying control limits at a shift of up to 2σ. Since $\rho = \psi\phi$, parameters ψ and ϕ affect the autocorrelation coefficient. Furthermore, ψ represents the proportion of the process variance that is due to the AR(1) process and has little impact on the ability to detect small shifts in an autocorrelated process mean. However, ϕ denotes the strength of correlations among preceding data. Therefore, the influence of ϕ on detection ability exceeds that of ψ.

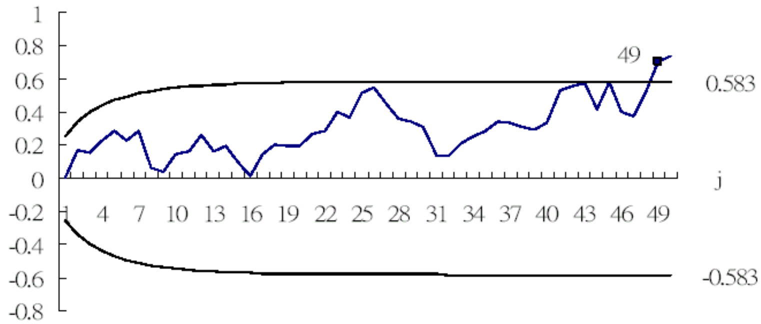
5. Example

Table 3 presents a numerical example of the GWMA control chart of residuals, using simulated data to illustrate the GWMA control scheme. 50 simulation data are listed in Table 3. The first ten samples are assumed to be in control at a target value $\xi_0 = 0$ and a shift in the mean of 0.5σ is assumed to occur during the last 40 samples. Original observations X_j obtained from the AR(1) model in Eqs. (5) and (6) $\phi = 0.4$, $\sigma = 1.0$, $\sigma_\varepsilon^2 = 0.1$ and $\psi = 0.9$ are evaluated, indicating that 90% of the variability in this process is associated with the variation in μ_t and the correlation between adjacent observations is given by $\rho = \psi\phi = 0.36$. By the equivalence of the relationships, the corresponding parameters in the ARMA $(1, 1)$ model in Eq. (7) are $\phi = 0.4$, $\theta = 0.046$ and $\sigma_b^2 = 0.870$. After the ARMA $(1, 1)$ model is fitted, the residuals can be calculated and then these independent residuals adopted to construct the corresponding EWMA control statistics (Z_j) and GWMA control statistics (Y_j). Within this table, the threshold for in-control is set to $ARL \cong 370$ and the parameters $q = 0.9$, $\alpha = 1.0$ and $L = 2.726$ for the EWMA control chart of residuals are compared to $q = 0.9$, $\alpha = 0.5$ and $L = 2.898$ for the GWMA control chart of residuals. The EWMA control chart of residuals has an out-of-control signal at residual 49. The GWMA control chart of residuals obtains an out-of-control signal at residual 41. A comparison to Table 2 reveals that only an average of 49.36 samples is required to enable the GWMA control chart of residuals to identify an out-of-control signal, while 52.41 samples are needed to enable the EWMA control chart of residuals. That is, the GWMA control scheme detects small shifts more rapidly than the EWMA control scheme. Figure 1(a) plots the EWMA control statistics and Figure 1(b) plots the GWMA control statistics, together with the original data from Table 3.

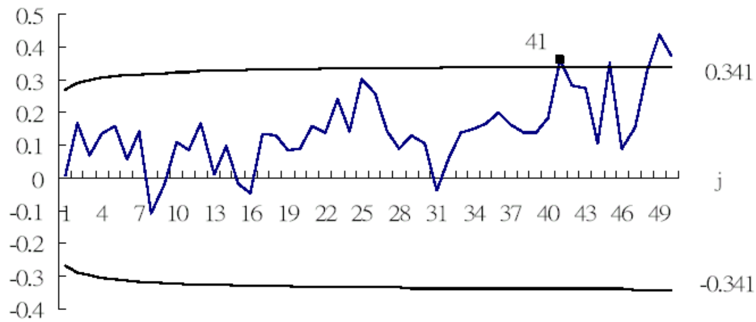
Table 3. Examples of EWMA and GWMA control charts of residuals using data from AR(1) process initially in control

			EWMA			GWMA	
j	X_j	Z_j	LCL	UCL	Y_j	LCL	UCL
1	0.365	0.005	–0.254	0.254	0.005	–0.270	0.270
2	0.369	0.171	–0.342	0.342	0.168	–0.289	0.289
3	1.591	0.157	–0.399	0.399	0.068	–0.299	0.299
4	0.743	0.227	–0.440	0.440	0.135	–0.306	0.306
5	1.482	0.288	–0.470	0.470	0.157	–0.310	0.310
6	1.624	0.227	–0.494	0.494	0.058	–0.314	0.314
7	0.582	0.285	–0.512	0.512	0.142	–0.317	0.317
8	1.501	0.062	–0.526	0.526	–0.108	–0.319	0.319
9	–1.115	0.039	–0.537	0.537	–0.018	–0.321	0.321
10	0.008	0.143	–0.546	0.546	0.110	–0.323	0.323
11	1.182	0.165	–0.553	0.553	0.085	–0.324	0.324
12	0.252	0.263	–0.559	0.559	0.169	–0.326	0.326
13	1.123	0.159	–0.564	0.564	0.012	–0.327	0.327
14	–0.604	0.197	–0.567	0.567	0.098	–0.328	0.328
15	0.384	0.097	–0.570	0.570	–0.016	–0.329	0.329
16	–0.955	0.015	–0.573	0.573	–0.047	–0.330	0.330
17	–1.204	0.148	–0.575	0.575	0.134	–0.331	0.331
18	0.600	0.204	–0.576	0.576	0.129	–0.331	0.331
19	0.298	0.194	–0.578	0.578	0.086	–0.332	0.332

20	–0.174	0.198	–0.579	0.579	0.091	–0.333	0.333
21	–0.105	0.268	–0.579	0.579	0.157	–0.333	0.333
22	0.529	0.285	–0.580	0.580	0.139	–0.334	0.334
23	0.249	0.401	–0.581	0.581	0.241	–0.334	0.334
24	1.268	0.366	–0.581	0.581	0.144	–0.335	0.335
25	0.168	0.513	–0.581	0.581	0.302	–0.335	0.335
26	1.825	0.544	–0.582	0.582	0.257	–0.336	0.336
27	1.195	0.453	–0.582	0.582	0.141	–0.336	0.336
28	0.053	0.358	–0.582	0.582	0.088	–0.336	0.336
29	–0.366	0.344	–0.582	0.582	0.129	–0.337	0.337
30	0.067	0.306	–0.582	0.582	0.106	–0.337	0.337
31	–0.236	0.134	–0.582	0.582	–0.040	–0.337	0.337
32	–1.731	0.141	–0.583	0.583	0.060	–0.338	0.338
33	–0.594	0.208	–0.583	0.583	0.138	–0.338	0.338
34	0.047	0.249	–0.583	0.583	0.151	–0.338	0.338
35	0.058	0.289	–0.583	0.583	0.169	–0.338	0.338
36	0.204	0.341	–0.583	0.583	0.198	–0.339	0.339
37	0.470	0.333	–0.583	0.583	0.162	–0.339	0.339
38	0.071	0.312	–0.583	0.583	0.140	–0.339	0.339
39	–0.105	0.297	–0.583	0.583	0.137	–0.339	0.339
40	–0.121	0.337	–0.583	0.583	0.185	–0.339	0.339
41	0.356	0.532	–0.583	0.583	0.362	–0.340	0.340
42	2.058	0.553	–0.583	0.583	0.282	–0.340	0.340
43	1.062	0.570	–0.583	0.583	0.276	–0.340	0.340
44	1.067	0.415	–0.583	0.583	0.107	–0.340	0.340
45	–0.644	0.582	–0.583	0.583	0.351	–0.340	0.340
46	1.999	0.398	–0.583	0.583	0.091	–0.340	0.340
47	–0.873	0.372	–0.583	0.583	0.154	–0.341	0.341
48	–0.017	0.523	–0.583	0.583	0.329	–0.341	0.341
49	1.660	0.705	–0.583	0.583	0.440	–0.341	0.341
50	2.547	0.738	–0.583	0.583	0.373	–0.341	0.341



(a) The EWMA control chart of residuals.



(b) The GWMA control chart of residuals.

Figure 1. The EWMA and GWMA control chart of residuals.

6. Composite Shewhart-GWMA Control Charts of Residuals

Previously, the GWMA control chart of residuals has outperformed the EWMA control scheme in detecting changes in autocorrelated process means. The Shewhart control chart is known to detect effectively relatively large process mean shifts. Accordingly, this section presents the composite Shewhart-GWMA control chart of residuals to monitor the process mean of autocorrelated processes. A Shewhart chart is combined with a GWMA chart. Both can be conveniently plotted on a single graph with control limits for the Shewhart chart of $\xi_0 \pm 3\sigma_b$ and those for GWMA of $\xi_0 \pm L\sqrt{Q_j}\sigma_b$. Figure 2 plots the composite Shewhart-GWMA control chart of residuals derived from the original data in Table 3. A false alarm occurs when the statistic exceeds either control limit.

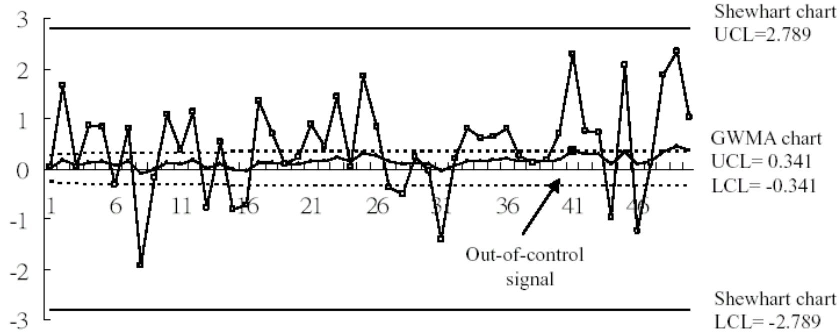


Figure 2. Composite Shewhart-GWMA control chart of residuals.

Simulations are performed herein to estimate the various ARLs, including GWMA charts, Shewhart charts and composite Shewhart-GWMA charts of residuals. Tables 4 and 5 are for $\psi = 0.1$, and $\psi = 0.9$, respectively, with various design parameters (q, α, L). For a fair comparison, the design parameters of the composite Shewhart-GWMA chart are adjusted to match the in-control ARL of the GWMA control charts of residuals. Regardless of the level of autocorrelation, the composite Shewhart-GWMA control chart outperforms the Shewhart chart in detecting an autocorrelated process shift. For small shifts, the composite Shewhart-GWMA control chart outperforms the GWMA control chart, but neither is detectably better when the shifts are moderate or large. For instance, when $\rho = 0.36$, $q = 0.9$, $\alpha = 1$, $L = 2.726$ and shift $= 0.25$, the ARL of the composite Shewhart-GWMA control chart, 113.20, is smaller than that of the GWMA control chart, 153.11. Although the composite Shewhart-GWMA control chart of residuals is more sensitive to small mean shifts, its rate of false alarms is higher. However, when the small shifts in the autocorrelated process mean and the high cost of quality-failure are considered, the composite Shewhart-GWMA control chart may be recommended.

Table 4. Comparison of the ARLs of composite Shewhart-GWMA control charts of residuals at $\psi = 0.1$

Types	Design parameters			Mean shifts						
				0.00	0.25	0.50	1.00	1.50	2.00	3.00
							$\phi = 0.4$			
GWMA chart	$q = 0.9$	$\alpha = 1.0$	$L = 2.729$	370.41	97.49	29.25	8.33	4.17	2.64	1.47
Shewhart chart	$q = 0.0$	$\alpha = 1.0$	$L = 3.000$	371.26	293.09	171.40	50.36	17.27	7.34	2.14
Composite Shewhart-GWMA	$q = 0.9$	$\alpha = 1.0$	$L = 2.729$	199.55	79.16	27.56	8.15	4.13	2.62	1.47
		$\alpha = 0.9$	$L = 2.735$	199.60	74.21	26.24	8.08	4.13	2.63	1.47
		$\alpha = 0.7$	$L = 2.784$	199.54	68.00	25.59	8.39	4.32	2.76	1.51
		$\alpha = 0.5$	$L = 2.836$	199.91	69.21	28.00	9.46	4.82	3.01	1.58
	$q = 0.5$	$\alpha = 1.0$	$L = 2.886$	199.70	130.79	59.34	14.58	5.68	3.19	1.60
		$\alpha = 0.9$	$L = 2.879$	199.52	128.44	57.15	14.16	5.65	3.20	1.60
		$\alpha = 0.7$	$L = 2.860$	199.42	122.59	52.75	13.45	5.64	3.24	1.61
		$\alpha = 0.5$	$L = 2.837$	199.61	117.32	49.73	13.61	5.91	3.38	1.62
							$\phi = 0.8$			
GWMA chart	$q = 0.9$	$\alpha = 1.0$	$L = 2.766$	370.51	131.75	41.29	11.09	5.25	3.14	1.63
Shewhart chart	$q = 0.0$	$\alpha = 1.0$	$L = 3.000$	370.87	313.22	211.20	76.42	27.90	11.16	2.80

Composite Shewhart-GWMA	$q = 0.9$	$\alpha = 1.0$	$L = 2.766$	191.61	98.61	37.43	10.76	5.15	3.10	1.62
		$\alpha = 0.9$	$L = 2.774$	191.57	92.85	35.21	10.63	5.15	3.11	1.63
		$\alpha = 0.7$	$L = 2.809$	191.60	84.08	33.33	10.80	5.31	3.22	1.67
		$\alpha = 0.5$	$L = 2.848$	191.37	83.34	35.84	12.07	5.89	3.51	1.75
	$q = 0.5$	$\alpha = 1.0$	$L = 2.884$	191.41	142.68	75.98	20.91	7.66	3.91	1.77
		$\alpha = 0.9$	$L = 2.880$	191.67	141.73	74.51	20.35	7.53	3.90	1.78
		$\alpha = 0.7$	$L = 2.862$	191.60	136.97	69.48	19.16	7.38	3.93	1.79
		$\alpha = 0.5$	$L = 2.838$	191.90	132.22	65.63	18.68	7.62	4.08	1.83

Table 5. Comparison of the ARLs of composite Shewhart-GWMA control charts of residuals at $\psi = 0.9$

Types	Design Parameters			Mean shifts						
				0.00	0.25	0.50	1.00	1.50	2.00	3.00
							$\phi = 0.4$			
GWMA chart	$q = 0.9$	$\alpha = 1.0$	$L = 2.726$	370.14	153.11	52.41	13.92	6.26	3.44	1.51
Shewhart chart	$q = 0.0$	$\alpha = 1.0$	$L = 3.000$	371.81	327.55	234.13	98.62	39.87	17.18	3.59
Composite Shewhart-GWMA	$q = 0.9$	$\alpha = 1.0$	$L = 2.726$	198.26	113.20	46.77	13.56	6.17	3.41	1.51
		$\alpha = 0.9$	$L = 2.732$	198.53	106.32	43.56	13.27	6.15	3.43	1.52
		$\alpha = 0.7$	$L = 2.775$	198.09	97.39	40.79	13.47	6.40	3.67	1.59

		$\alpha = 0.5$	$L = 2.831$	198.11	95.94	43.62	15.22	7.36	4.21	1.72
	$q = 0.5$	$\alpha = 1.0$	$L = 2.884$	198.88	157.33	92.23	29.66	11.05	5.14	1.79
		$\alpha = 0.9$	$L = 2.875$	198.70	155.63	89.74	28.34	10.68	5.07	1.79
		$\alpha = 0.7$	$L = 2.856$	198.34	150.39	84.12	26.17	10.30	5.07	1.82
		$\alpha = 0.5$	$L = 2.832$	198.68	145.21	78.94	25.09	10.52	5.31	1.87
							$\phi = 0.8$			
GWMA chart	$q = 0.9$	$\alpha = 1.0$	$L = 2.729$	370.53	269.08	139.90	40.07	13.43	4.74	1.17
Shewhart chart	$q = 0.0$	$\alpha = 1.0$	$L = 3.000$	371.72	359.32	321.75	215.52	119.13	52.46	4.62
Composite Shewhart-GWMA	$q = 0.9$	$\alpha = 1.0$	$L = 2.729$	197.95	165.44	104.84	36.15	12.95	4.66	1.17
		$\alpha = 0.9$	$L = 2.732$	197.98	159.14	98.11	33.60	12.39	4.64	1.17
		$\alpha = 0.7$	$L = 2.776$	197.54	150.83	89.65	32.03	12.78	5.00	1.21
		$\alpha = 0.5$	$L = 2.831$	197.44	149.17	89.10	35.16	15.15	6.24	1.31
	$q = 0.5$	$\alpha = 1.0$	$L = 2.880$	197.56	183.80	151.24	79.69	34.35	12.57	1.52
		$\alpha = 0.9$	$L = 2.874$	197.42	183.47	150.01	77.58	33.35	12.18	1.52
		$\alpha = 0.7$	$L = 2.856$	197.48	182.56	144.80	72.49	30.99	11.08	1.50
		$\alpha = 0.5$	$L = 2.833$	197.29	180.19	140.07	68.12	29.19	10.89	1.52

7. Conclusions

When quality characteristics no longer meet a standard independent assumption, a control chart for autocorrelated observations must be considered to monitor the process to avoid an increase in the frequency of false alarms. Many authors have recently noted this characteristic. Lu and Reynolds [5] demonstrated that when the observations are drawn from an AR(1) process with a random error, the EWMA control chart of residuals needs less time to identify a special cause than the Shewhart control chart of residuals. This study presented a statistical approach for extending the EWMA scheme to the GWMA scheme by adding an adjustment parameter. Of course, any chart based on residuals will fit a model that can be employed to calculate the residuals. This investigation considers observations taken from the AR(1) process with a random error and residuals fitted by the ARMA(1, 1) model. Numerical results have shown that the GWMA control chart of residuals is superior to the EWMA control chart of residuals in detecting small shifts in the autocorrelated process mean-specifically those under 1σ. The GWMA control chart of residuals requires less time to detect small process mean shifts as the level of autocorrelation declines. However, the GWMA and EWMA control charts of residuals exhibit similar detection abilities at large shift.

The Shewhart and EWMA control schemes are merely special cases of the GWMA control chart of residuals in relation to autocorrelated problems. The composite Shewhart-GWMA control chart has the advantage of simultaneously detecting small and large process shifts. Numerical analyses have established that the composite Shewhart-GWMA control chart can detect small shifts more quickly than the GWMA control chart. If detecting small shifts in the autocorrelated process mean is more important than avoiding spending time to identify false alarms, then the use of the composite Shewhart-GWMA control chart may be favored. Besides, another concern is the increase in variation over time, rather than only a mean shift.

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