Far East J. Theo. Stat. **22**(1) (2007), 65-80

CHANDRASEKHAR-TYPE RECURSIONS FOR PERIODIC LINEAR SYSTEMS

ABDELHAKIM AKNOUCHE and FAYÇAL HAMDI

University of Sciences and Technology Houari Boumedienne
16111 BP. El Alia, Bab Ezzouar
Algiers, Algeria

Abstract

This paper extends the discrete-time Chandrasekhar recursions due to Morf et al. [22] to the case of periodic time-varying state-space models. Exploiting the S-periodicity of the model parameters, we show that the S-lagged increments of the Riccati variable satisfy certain recursions, from which we derive some algorithms for linear least squares estimation. The proposed methods may have potential computational advantages over the Kalman filter and, in particular, the periodic Riccati difference equation. Application of the proposed periodic Chandrasekhar recursions to the likelihood evaluation of periodic ARMA models is given.

1. Introduction

A considerable attention has been paid in the three recent decades to the Chandrasekhar type recursions since they represent attractive alternatives to the Kalman filter for time-invariant state-space models [8], [11], [13], [21], [22], [24] and [25]. At present, there exist several useful applications of the Chandrasekhar filter in improving

---

2000 Mathematics Subject Classification: Primary 62M10; Secondary 60F99.

Keywords and phrases: periodic state-space models, periodic Riccati difference equation, Chandrasekhar-type recursions, Kalman filter, PARMA likelihood evaluation.

Received December 3, 2006

computational aspects related to the building of linear time-invariant models. We mention non exhaustively the likelihood evaluation (see [16], [19], [23], for ARMA models and [26] for vector ARMA models), the calculation of the exact Fisher information matrix (see [20] for the ARMA case and [15] for general dynamic time-invariant models), and the development of fast variants of the recursive least squares algorithm [10], [24]. As is well known, the Chandrasekhar equations are restricted to the case of time-invariant state-space models because of their particular time invariance structure and it seems that there is no result tied to the class of all nonstationarity, except in very special cases [24]. A particular class of nonstationarity whose importance has no need to be proven is the one of periodic linear models. Important progress has been made recently in the building and analysis of periodic ARMA (PARMA) and periodic state-space characterizations. The objective was to develop extensions of similar methods for standard time-invariant models to their periodic counterparts, without transforming periodic systems to their corresponding multivariate time-invariant representations in order to simplify the computational burden. Despite the current abundance of computational methods for periodic state-space models (see [1], [17], [28] and the references therein) it seems that there is no result concerning extensions of the Chandrasekhar recursions to the periodic case. This paper proposes some algorithms for linear least squares estimation of periodic state-space models. Our methods extend the Chandrasekhar algorithms proposed by Morf et al. [22] to the periodic time-varying case and retain their desirable features. As a result, the periodic Chandrasekhar recursions are used through the innovation approach to efficiently evaluate the likelihood of periodic ARMA models.

The rest of this paper is organized as follows. Section 2 briefly recalls some preliminary definitions and facts about periodic state-space models and their corresponding Kalman filter. In Section 3 we develop some Chandrasekhar-type algorithms that substitute the Kalman filter for periodic state-space models. Application of the proposed periodic Chandrasekhar recursions to the PARMA likelihood evaluation is given in Section 4.

2. Preliminary Definitions and Notations

The basic model dealt with by this paper is the following periodic state-space form

$$\begin{cases} \mathbf{x}_{t+1} = F_t \mathbf{x}_t + G_t \varepsilon_t \\ \mathbf{y}_t = H_t' \mathbf{x}_t + \mathbf{e}_t \end{cases}, \quad t \in \mathbb{Z}, \tag{2.1}$$

where $\{\mathbf{x}_t\}, \{\mathbf{y}_t\}, \{\mathbf{e}_t\}$ and $\{\varepsilon_t\}$ are random processes of dimensions $r \times 1$, $m \times 1$, $m \times 1$, and $d \times 1$, respectively, with

$$\begin{cases} E(\varepsilon_t) = E(\mathbf{e}_t) = 0 \\ E(\varepsilon_t \varepsilon_{t+h}') = \delta_{h,0} Q_t \\ E(\mathbf{e}_t \mathbf{e}_{t+h}') = \delta_{h,0} R_t \end{cases}$$

and

$$\begin{cases} E(\varepsilon_t \mathbf{x}_{t-k}') = 0 \\ E(\mathbf{e}_t \mathbf{y}_{t-k}') = 0, \\ E(\mathbf{x}_t \mathbf{x}_t') = W_t \end{cases} \quad \begin{matrix} \forall t, h \in \mathbb{Z}, \\ \forall k \geq 0 \end{matrix}$$

(δ stands for the Kronecker function). The nonrandom matrix coefficients F_t, G_t, and H_t' and the covariance matrices Q_t, R_t, and W_t are periodic in time with period S. To simplify the exposition we suppose without loss of generality that

$$E(\mathbf{e}_t \varepsilon_l') = 0 \quad \forall t, \, l \in \mathbb{Z}.$$

Let $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{y}}_t$ be the linear least squares forecasts of $\mathbf{x}_t$ and $\mathbf{y}_t$, respectively, based on $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{t-1}$. Then as is well known, $\hat{\mathbf{x}}_t$ and $\hat{\mathbf{y}}_t$ may be uniquely obtained from the Kalman filter [14] which is given by the following recursions:

$$\Omega_t = H_t' \Sigma_t H_t + R_t, \tag{2.2a}$$

$$K_t = F_t \Sigma_t H_t, \tag{2.2b}$$

$$\hat{\mathbf{y}}_t = H_t' \hat{\mathbf{x}}_t, \tag{2.2c}$$

$$\hat{\mathbf{x}}_{t+1} = F_t \hat{\mathbf{x}}_t + K_t \Omega_t^{-1} \hat{\mathbf{e}}_t, \tag{2.2d}$$

$$\Sigma_{t+1} = F_t \Sigma_t F_t' - K_t \Omega_t^{-1} K_t' + G_t Q_t G_t', \tag{2.2e}$$

with starting values

$$\hat{\mathbf{x}}_1 = E(\mathbf{x}_1) = 0, \tag{2.2f}$$

$$\Sigma_1 = E(\mathbf{x}_1 \mathbf{x}_1') = W_1, \tag{2.2g}$$

where $\hat{\mathbf{e}}_t = \mathbf{y}_t - \hat{\mathbf{y}}_t$ is the $\mathbf{y}_t$-residuals with covariance matrix Ω_t, $\Sigma_t = E[(\mathbf{x}_t - \hat{\mathbf{x}}_t)(\mathbf{x}_t - \hat{\mathbf{x}}_t)']$ is interpreted as the covariance matrix of the one-step state prediction errors, and $K_t = E(\mathbf{x}_{t+1} \hat{\mathbf{e}}_t')$ is known as the *Kalman gain*. The notation $A \geq 0$ means that the matrix A is nonnegative definite.

Recursion (2.2e) based on the starting equation (2.2f) is usually called the *periodic Riccati difference equation* (PRDE). If Σ_{t+Sk} converges as $k \to \infty$ for all $t \in \{1, \ldots, S\}$, then the S-periodic limiting solution $P_t = \lim_{k\to\infty} \Sigma_{t+Sk}$ will satisfy the following *discrete-time matrix periodic Riccati equation* (*DPRE*)

$$\begin{aligned} P_{t+1} = {} & F_t P_t F_t' - F_t P_t H_t (H_t' P_t H_t + R_t)^{-1} H_t' P_t F_t' \\ & + G_t Q_t G_t', \ t \in \{1, \ldots, S\}, \end{aligned} \tag{2.3}$$

which has been extensively studied (see for example [3] for some theoretical aspects and [9] for a numerical resolution). As is well-known, the resolution of (2.2e) requires $O(r^3)$ operations per iteration. This can be significantly reduced taking into account the periodic invariance property of the model parameters. Furthermore, the solution Σ_t must be nonnegative definite, a property that is not easy to preserve in a numerical resolution of (2.2e). The following section proposes some recursions that avoid these drawbacks and have further advantages over the Kalman filter (2.2).

3. Periodic Chandrasekhar-type Equations

For the time-invariant coefficient case, Morf et al. [22] proposed recursions that substitute the Kalman filter (2.2) with a simpler computational complexity. The new algorithms have been called *Chandrasekhar-type recursions* because they are analog of certain differential equations encountered in continuous-time models [13]. The recursions proposed in this section and which are aimed to generalize Morf et al.'s algorithms [22] to the periodic case will be called *analogously periodic Chandrasekhar-type equations*. This, of course, will not mean that there is an analog of our recursions in the periodic continuous-time case.

A. Periodic Chandrasekhar factorization

The derivation of our recursions is similar to its classical counterpart and is based on the factorization result given below (see Proposition 3.1).

Let $\Delta^S \Sigma_t = \Sigma_{t+S} - \Sigma_t$ denote the S-lagged increment of the Riccati variable, for given $\Sigma_1, \Sigma_2, \ldots, \Sigma_S \geq 0$. Then, one can prove the following result.

Proposition 3.1. *The S-lagged increment $\Delta^S \Sigma_t$ satisfies the following difference equations*

$$\Delta^S \Sigma_{t+1} = (F_t - K_{t+S}\Omega_{t+S}^{-1}H_t')[\Delta^S \Sigma_t + \Delta^S \Sigma_t H_t \Omega_t^{-1} H_t' \Delta^S \Sigma_t] (F_t - K_{t+S}\Omega_{t+S}^{-1}H_t')' \tag{3.1}$$

$$= (F_t - K_t\Omega_t^{-1}H_t')[\Delta^S \Sigma_t + \Delta^S \Sigma_t H_t \Omega_{t+S}^{-1} H_t' \Delta^S \Sigma_t] (F_t - K_t\Omega_t^{-1}H_t')'. \tag{3.2}$$

Proof. (i) From (2.2e) we have

$$\Delta^S \Sigma_{t+1} = F_t \Delta^S \Sigma_t F_t' - \widetilde{K}_{t+S}\Omega_{t+S}\widetilde{K}_{t+S}' + \widetilde{K}_t \Omega_t \widetilde{K}_t', \tag{3.3}$$

where $\widetilde{K}_t = K_t \Omega_t^{-1}$ and from (2.2a) the covariance matrix Ω_t satisfies the following recursion:

$$\Omega_{t+S} = \Omega_t + H_t' \Delta^S \Sigma_t H_t$$

$$\overset{def}{=} \Omega_t + \Delta^S \Omega_t. \tag{3.4}$$

On the other hand, $\widetilde{K}_t$ may be written in the following backward recursive form:

$$\begin{aligned}
\widetilde{K}_t &= (F_t \Sigma_{t+S} H_t - F_t \Delta^S \Sigma_t H_t)\Omega_t^{-1} \\
&= (\widetilde{K}_{t+S}\Omega_{t+S} - F_t \Delta^S \Sigma_t H_t)\Omega_t^{-1} \\
&= [\widetilde{K}_{t+S}(H_t' \Delta^S \Sigma_t H_t + H_t' \Sigma_t H_t + R_t) - F_t \Delta^S \Sigma_t H_t]\Omega_t^{-1} \\
&= \widetilde{K}_{t+S} - (F_t - \widetilde{K}_{t+S} H_t')\Delta^S \Sigma_t H_t \Omega_t^{-1} \\
&\overset{def}{=} \widetilde{K}_{t+S} - \Delta^S \widetilde{K}_t.
\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) into (3.3) we obtain

$$\begin{aligned}
\Delta^S \Sigma_{t+1} = F_t \Delta^S \Sigma_t F_t' - \widetilde{K}_{t+S}(\Omega_t + \Delta^S \Omega_t)\widetilde{K}_{t+S}' \\
+ (\widetilde{K}_{t+S} - \Delta^S \widetilde{K}_t)\Omega_t(\widetilde{K}_{t+S} - \Delta^S \widetilde{K}_t)'.
\end{aligned} \tag{3.6}$$

Expansion of terms in the right hand-side of (3.6) with some straightforward manipulations gives (3.1).

(ii) A similar argument can be used to derive (3.2). It suffices to show that

$$\begin{aligned}
\widetilde{K}_{t+S} &= \widetilde{K}_t + (F_t - K_t H_t')\Delta^S \Sigma_t H_t \Omega_{t+S}^{-1} \\
&= \widetilde{K}_t + \Delta^S \widetilde{K}_t,
\end{aligned}$$

and replacing this latter relation with (3.4) into (3.3) we obtain (3.2).

Proposition 3.1 shows that $\Delta^S \Sigma_t$ may be factorized as follows:

$$\Delta^S \Sigma_t = Y_t M_t Y_t', \tag{3.7}$$

where M_t is a square symmetric matrix, non necessarily nonnegative definite, of dimension rank$(\Delta^S \Sigma_1)$, which is at least equal to rank$(\Delta^S \Sigma_t)$. Indeed, from (3.1) we have

$$\text{rank}(\Delta^S \Sigma_{t+1}) \leq \text{rank}(\Delta^S \Sigma_t) \leq \cdots \leq \text{rank}(\Delta^S \Sigma_1) \leq r.$$

This can be exploited to derive some recursions with the best computational complexity than the filter (2.2).

B. Algorithms

Thanks to the factorization result given by Proposition 3.1, the matrices Y_t and M_t can be obtained recursively. The following algorithm shows that the periodic Riccati difference equation (2.2e) may be replaced by a set of recursions on Ω_t, K_t, Y_t and M_t with a reduction in computational efforts, especially when the state dimension r is much larger than m, the dimension of $\mathbf{y}_t$.

Algorithm 3.1. The Kalman filter (2.2) can be replaced by (2.2c), (2.2d) and the following recursions:

$$\Omega_{t+S} = \Omega_t + H_t' Y_t M_t Y_t' H_t, \tag{3.8a}$$

$$K_{t+S} = (K_t + F_t Y_t M_t Y_t' H_t), \tag{3.8b}$$

$$Y_{t+1} = (F_t - K_{t+S}\Omega_{t+S}^{-1} H_t') Y_t, \tag{3.8c}$$

$$M_{t+1} = M_t + M_t Y_t' H_t \Omega_t^{-1} H_t' Y_t M_t, \tag{3.8d}$$

with initialization given by

$$\Omega_s = H_s' \Sigma_s H_s, \quad s = 1, \dots, S, \tag{3.8e}$$

$$K_s = F_s \Sigma_s H_s, \quad s = 1, \dots, S, \tag{3.8f}$$

where $\sum_s$, $s = 1, \ldots, S$ are found from (2.2e) and (2.2g), while Y_1 and M_1 are obtained by factorizing nonuniquely

$$\begin{aligned}\Delta^S \Sigma_1 &= \Sigma_{S+1} - \Sigma_1 \\ &= F_S \Sigma_S F_S' - K_S \Omega_S^{-1} K_S' + G_S Q_S G_S' - \Sigma_1, \end{aligned} \tag{3.8g}$$

as

$$Y_1 M_1 Y_1'.$$

Derivation (3.8a) is just (3.4) when using (3.7), while (3.8b) follows from (3.7) and the relation

$$K_{t+S} = (F_t \Sigma_t H_t + F_t \Delta^S \Sigma_t H_t).$$

On the other hand, (3.8c) and (3.8d) follow from (3.1) which we rewrite using (3.7) as

$$\begin{aligned}\Delta^S \Sigma_{t+1} &= (F_t - K_{t+S}\Omega_{t+S}^{-1} H_t') Y_t (M_t + M_t Y_t' H_t \Omega_t^{-1} H_t' Y_t M_t) \\ &\quad Y_t' (F_t - K_{t+S}\Omega_{t+S}^{-1} H_t')' \\ &= Y_{t+1} M_{t+1} Y_{t+1}'.\end{aligned}$$

Note that the PRDE (2.2e) must be executed for $s = 1, \ldots, S$ to start the recursions (3.8). However, for $t > S$ the recursive calculation of $\sum_t$ is not dealt with by the above algorithm but can be deduced from it through the following equation:

$$\Sigma_{kS+s} = \Sigma_s + \sum_{j=0}^{k-1} Y_{jS+s} M_{jS+s} Y_{jS+s}', \quad s = 1, \ldots, S.$$

Similarly to the time-invariant case [22], other forms of Algorithm 3.1 can be derived from Proposition 3.1. The following variant is particularly well adapted when $M_1 < 0$, in which case we have $M_t \leq 0$ for any t. This case is encountered whenever the periodic state-space model (2.1) is periodically stationary as we can see in the following subsection.

Algorithm 3.2. The following set of recursions in which (3.8a), (3.8b) and (3.8e)-(3.8g) are unchanged while (3.8c) and (3.8d) are replaced by

$$Y_{t+1} = (F_t - K_t \Omega_t^{-1} H_t') Y_t, \tag{3.9a}$$

$$M_{t+1} = M_t - M_t Y_t' H_t \Omega_{t+S}^{-1} H_t' Y_t M_t, \tag{3.9b}$$

provides the same results as Algorithm 3.1.

Derivation. The derivation is similar to that of Algorithm 3.1, but is based on the factorization (3.2) rather than (3.1).

It is still possible to derive other forms similarly to the standard time-invariant case. The homogenous periodic Riccati difference equation (3.8d) can be linearized using the matrix inversion lemma [22] through which, we obtain a recursion on M_t^{-1} rather than on M_t as follows:

$$M_{t+1}^{-1} = M_t^{-1} - Y_t' H_t \Omega_{t+S}^{-1} H_t' Y_t. \tag{3.10}$$

The periodic Chandrasekhar recursions given above will be preferred to the Kalman filter (2.2) whenever the dimension of Y_t and/or M_t are significantly less than that of $\sum_t$. These dimensions are conditioned on the good choice of the factorization $\Delta^S \sum_1 = Y_1 M_1 Y_1'$ in the initialization step which will be studied in the following subsection.

C. The initialization problem

As is well known, the most important step in the development of a Chandrasekhar algorithm is the initialization step because it modulates the computational complexity and hence the lack of numerical advantage over the Kalman filter. In our periodic case, this step depends on the relation between the period *S*, the output dimension *m*, and the state dimension *r*. First of all, suppose the process $\{\mathrm{x}_t\}$ given by (2.1) is periodically stationary, that is, the monodromy matrix $\prod_{i=0}^{S} F_{S-i}$ has its eingenvalues less than unity in modulus. Let us consider two cases.

(i) Case where $Sm < r$

As pointed out in (3.8g) the start up values Y_1 and M_1 are determined by factorizing $\Delta^S \Sigma_1$ as $Y_1 M_1 Y_1'$. Iterating (3.8g) S times and invoking the fact that under the periodic stationarity assumption, Σ_1 satisfies the following discrete-time periodic Lyapunov equation (DPLE) (e.g., [27])

$$\Sigma_1 = \left(\prod_{j=0}^{S-1} F_{S-j}\right) \Sigma_1 \left(\prod_{j=0}^{S-1} F_{S-j}\right)'$$

$$+ \sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j}\right) G_{S-k} Q_{S-k} G'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j}\right)',$$

we obtain

$$\Delta^S \Sigma_1 = -\sum_{k=0}^{S-1} \left(\prod_{j=0}^{k-1} F_{S-j}\right) K_{S-k} \Omega_{S-k}^{-1} K'_{S-k} \left(\prod_{j=0}^{k-1} F_{S-j}\right)' \tag{3.11a}$$

$$= -L \begin{pmatrix} \Omega_S^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Omega_1^{-1} \end{pmatrix} L', \tag{3.11b}$$

where L is given by

$$L = \left[K_S,\ F_S K_{S-1},\ F_S F_{S-1} K_{S-2},\ \ldots,\ \prod_{j=0}^{S-1} F_{S-j} K_1 \right]. \tag{3.12}$$

Clearly, when S is fairly less than r, the nonhomogeneous PRDE (2.2e) may be replaced by the homogenous PRDE (3.9b) which is of lower dimension. For example, for $m = 1$, the complexity of solving (3.8d) or (3.9b) when using (3.11) as an initialization step is of order $O(S^3)$ which is computationally simple to solve compared to the PRDE (2.2e). It is still possible to improve the computation of (3.11) by alleviating the formation of the sums of products in (3.12) by using the periodic Schur decomposition [4], [9].

(ii) Case where $Sm \geq r$

From the fact that

$$\Sigma_1 = F_S W_0 F_S' + G_S Q_S G_S',$$

we have

$$\Delta^S \Sigma_1 = F_S \Sigma_S F_S' - K_S \Omega_S^{-1} K_S' - F_S W_0 F_S'$$

$$= F_S [\Sigma_S - W_0 - \Sigma_S H_S \Omega_S^{-1} H_S' \Sigma_S'] F_S'.$$

This identifies Y_1 and M_1 as

$$Y_1 = F_S,$$

$$M_1 = \Sigma_S - W_0 - \Sigma_S H_S \Omega_S^{-1} H_S' \Sigma_S'. \tag{3.13}$$

With such an initialization, the PRDE (3.9b) has the same dimension as that of the PRDE (2.2e), and it seems that there is no reduction in the computational cost compared to the Kalman filter. However, the difference from (2.2e) is that, unlike the Σ_t, the M_t are not required to be nonnegative-definite (see [22] for the particular time-invariant case). This helps alleviate the computational complexity of (3.9b).

4. Application to the PARMA Likelihood Evaluation

This section develops an algorithm for the exact likelihood evaluation for Gaussian PARMA models by means of the innovation approach (e.g. [18]) which generally gives the likelihood with a number of operations proportional to the sample size. The innovation approach has been previously used in the literature for the PARMA likelihood, where the sample innovations are evaluated either by the Kalman filter [12], [6], or the innovation algorithm [18]. Here, the sample innovations will be efficiently obtained through the Chandrasekhar recursion given in Section 3, whenever the underlying PARMA model is written in a state-space form.

Consider the time-invariant orders PARMA model of orders (p, q) and period S

$$y_{s+nS} - \sum_{j=1}^{p} \phi_{s,j} y_{s+nS-j} = \varepsilon_t - \sum_{j=1}^{q} \theta_{s,j} \varepsilon_{s+nS-j}, \quad n \in \mathbb{Z}, 1 \leq s \leq S, \tag{4.1}$$

where $\{\varepsilon_t, t \in \mathbb{Z}\}$ is a periodic white noise with variance σ_t^2. Let $\sigma = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_S^2)'$ be the vector of variance parameter and $\beta' = (\phi_1', \theta_1', \phi_2', \theta_2', \ldots, \phi_S', \theta_S')$ denote the $(p+q)S \times 1$ vector of autoregressive and moving average parameters where $\phi_s = (\phi_{s,1}, \phi_{s,2}, \ldots, \phi_{s,p})'$ and $\theta_s = (\theta_{s,1}, \theta_{s,2}, \ldots, \theta_{s,q})'$, $1 \leq s \leq S$. Suppose the model (4.1) is causal and invertible and the innovation process $\{\varepsilon_t, t \in \mathbb{Z}\}$ is Gaussian. Let $\hat{y}_t$ be the best linear predictor of y_t based on $y_1, y_2, \ldots, y_{t-1}$ and $\hat{\varepsilon}_t = y_t - \hat{y}_t$ be the sample innovation at time t, with mean square error $\Omega_t = E(y_t - \hat{y}_t)^2$. For a given realization $y = (y_1, y_2, \ldots, y_{NS})'$ of the PARMA process given by (4.1), the likelihood of β and σ can be expressed in the innovation form

$$L(\beta, \sigma; y) = (2\pi)^{-\frac{NS}{2}} \prod_{s=1}^{S} \prod_{n=0}^{N-1} (\Omega_{s+nS})^{-1/2} \exp\left\{-\frac{1}{2} \sum_{s=1}^{S} \sum_{n=0}^{N-1} \Omega_{s+nS}^{-1} \hat{\varepsilon}_{s+nS}^2\right\}, \tag{4.2}$$

in which we need to evaluate Ω_{s+nS}, $\hat{y}_{s+nS}$ and hence $\hat{\varepsilon}_{s+nS}$ for $1 \leq s \leq S$ and $0 \leq n \leq N-1$. This can be recursively achieved using the Chandrasekhar filter (3.9) instead of the Kalman one [12], [6] or the innovation algorithm [5], [18], provided that model (4.1) is expressed in a state-space form. Among several state-space forms that can be used to represent a PARMA model [12], [2], we choose the well-known $\max(p, q+1)$ representation [7], which involves fewer operations. Let $r = \max(p, q+1)$ and define the r-variate process $\{\mathbf{x}_t, t \in \mathbb{Z}\}$ as follows:

$$\begin{aligned}\mathbf{x}_{s+nS}(i) = & \sum_{j=i}^{r} \phi_{s+nS+i-1,j}\, y_{s+nS+i-j-1} \\ & - \theta_{s+nS+i-1,j-1} \varepsilon_{s+nS+i-j}, \quad i = 1, \ldots, r. \end{aligned} \tag{4.3}$$

Then model (4.1) can be represented in the following state-space form [12], [2]

$$\begin{cases} \mathbf{x}_{s+nS} = F_s \mathbf{x}_{s+nS-1} + G_s \varepsilon_{s+nS} \\ y_{s+nS} = H' \mathbf{x}_{s+nS} \end{cases}, \tag{4.4}$$

where

$$F_s = \begin{pmatrix} \phi_{s,1} & 1 & \cdots & 0 \\ \phi_{s+1,2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \phi_{s+r-1,r} & 0 & \cdots & 0 \end{pmatrix}, \quad G_s = \begin{pmatrix} 1 \\ -\theta_{s+1,1} \\ \vdots \\ -\theta_{s+r-1,r-1} \end{pmatrix}, \quad H = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

From (4.4), the orthogonal projection $\hat{y}_{s+nS}$ can be expressed as follows:

$$\hat{y}_{s+nS} = H' \hat{\mathbf{x}}_{s+nS}, \tag{4.5}$$

where $\hat{\mathbf{x}}_{s+nS}$ is the best linear predictor of $\mathbf{x}_{s+nS}$ based on $y_1, \ldots, y_{s+nS-1}$, with mean square error covariance matrix

$$\Sigma_{s+nS} = E(\mathbf{x}_{s+nS} - \hat{\mathbf{x}}_{s+nS})(\mathbf{x}_{s+nS} - \hat{\mathbf{x}}_{s+nS})'.$$

The projection $\hat{\mathbf{x}}_{s+nS}$ can be obtained recursively from (2.2d), where Ω_t, K_t, Y_t and M_t are found from Algorithm 3.2 which we rewrite for convenience

$$\begin{aligned} \Omega_{t+S} &= \Omega_t + H_t' Y_t M_t Y_t' H_t, \\ K_{t+S} &= (K_t + F_t Y_t M_t Y_t' H_t), \\ Y_{t+1} &= (F_t - K_t \Omega_t^{-1} H_t') Y_t, \\ M_{t+1} &= M_t - M_t Y_t' H_t \Omega_{t+S}^{-1} H_t' Y_t M_t, \end{aligned} \tag{4.6}$$

with start-up values $\Omega_s = H_s' \Sigma_s H_s$, $K_s = F_s \Sigma_s H_s$, $s = 1, \ldots, S$, where Σ_s, $s = 2, \ldots, S$ are found from (2.2e) and (2.2g), while Y_1 and M_1 are obtained either from (3.11) or (3.12) according to whether $S < r$ or not. On the other hand Σ_1 is given from the DPLE

$$\Sigma_1 = W_1 = F_1 W_0 F_1' + \sigma_1^2 G_1 G_1', \tag{4.7}$$

which may be solved for W_0 (e.g., [27]).

Thus, adopting (4.2)-(4.6), the PARMA likelihood is evaluated with a computational complexity proportional to the sample size, with $O(\min(S, r)^3)$ per iteration.

5. Conclusion

In this paper the discrete-time Chandrasekhar recursions have been generalized to the periodic time-varying state-space case through several forms. These recursions allow in a large range of cases to solve the periodic Riccati difference equation with a considerable reduction in the computational complexity. Along similar lines to the standard case [21], a square root version of these recursions can be easily derived in order to improve the numerical stability of the proposed algorithms. On the other hand, we have shown how the Chandrasekhar filter can be applied to efficiently evaluate the PARMA likelihood as an attractive alternative to the existing methods. Other useful applications for time series analysis as well as for the periodic system theory can be given, in particular, we mention the calculation of exact Fisher information matrix for PARMA models and the development of fast RLS algorithms for periodic systems.

Acknowledgement

The authors are deeply grateful to an anonymous referee and to Professor Mohamed Bentarzi for their helpful comments.

References

[1] A. Aknouche, Causality conditions and autocovariance calculation in PVAR models, J. Statist. Comput. Simulation, Forthcoming.

[2] R. Azrak and G. Mélard, The exact quasi-likelihood of time-dependent ARMA models, J. Statist. Plann. Inference 68 (1998), 31-46.

[3] S. Bittanti, P. Colaneri and G. D. De Nicolao, The difference periodic Riccati equation for the periodic prediction problem, IEEE Trans. Automat. Control 33 (1988), 706-712.

[4] A. W. Bojanczyk, G. Golub and P. Van Dooren, The Periodic Schur Decomposition: Algorithms and Applications, F. T. Luk, ed., Proceedings SPIE Conference 1770 (1992), 31-42.

[5] P. J. Brockwell and R. A. Davis, Time Series: Theory and Methods, Springer-Verlag, New York, 1991.

[6] J. Casals, S. Sotoca and M. Jerez, Un algoritmo rápido para evaluar la verosimilitud exacta de modelos V ARMAX periódicos, Estadistica Espanola 40 (1998), 269-291.

[7] P. De Jong and J. Penzer, The ARMA model in state space form, Statist. Probab. Lett. 70 (2004), 119-125.

[8] B. Friedlander, T. Kailath, M. Morf and L. Ljung, Extended Levinson and Chandrasekhar recursions for general discrete-time linear estimation problems, IEEE Trans. Automat. Control 23 (1978), 750-760.

[9] J. J. Hench and A. J. Laub, Numerical solution of the discrete-time periodic Riccati equation, IEEE Trans. Automat. Control 39 (1994), 1197-1210.

[10] A. Houacine, Regularized fast recursive least squares algorithms for adaptive filtering, IEEE Trans. Signal Process. 39 (1991), 860-870.

[11] A. Houacine and G. Demoment, Chandrasekhar adaptive regularizer for adaptive filtering, Proc. IEEE ICASSP, Tokyo, Japan, 1986.

[12] C. Jymenez, A. I. McLeod and K. W. Hipel, Kalman filter estimation for periodic autoregressive-moving average models, Stochastic Hydrology and Hydraulics 3 (1989), 227-240.

[13] T. Kailath, Some new algorithms for recursive estimation in constant, linear systems, IEEE Trans. Automat. Control 18 (1973), 750-760.

[14] E. Kalman, A new approach to linear filtering and predicting problems, Transactions of the ASME, J. Basic Engineering Series 82 (1960), 35-45.

[15] A. Klein, G. Mélard and T. Zahaf, Computation of the exact information matrix of Gaussian dynamic regression time series models, Ann. Statist. 26 (1998), 1636-1650.

[16] R. Kohn and C. F. Ansley, Computing the likelihood and its derivatives for a Gaussian ARMA models, J. Stat. Comput. Simul. 22 (1985), 229-263.

[17] R. Lund and I. V. Basawa, Modeling and inference for periodically correlated time series, Asymptotics, Nonparametrics and Time Series, S. Gosh, ed., Marcel-Dekker, New York, 1999, pp. 37-62.

[18] R. Lund and I. V. Basawa, Recursive prediction and likelihood evaluation for periodic ARMA models, J. Time Ser. Anal. 21 (2000), 75-93.

[19] G. Mélard, Algorithm AS197: A fast algorithm for the exact likelihood of autoregressive-moving average models, Appl. Stat. 33 (1984), 104-114.

[20] G. Mélard and A. Klein, On a fast algorithm for the exact information matrix of a Gaussian ARMA time series, IEEE Trans. Signal Process. 42 (1994), 2201-2203.

[21] M. Morf and T. Kailath, Square root algorithms for least squares estimation, IEEE Trans. Automat. Control 20 (1975), 487-497.

[22] M. Morf, G. S. Sidhu and T. Kailath, Some new algorithms for recursive estimation in constant, linear, discrete-time systems, IEEE Trans. Automat. Control 19 (1974), 315-323.

[23] J. G. Pearlman, An algorithm for the exact likelihood of a high-order autoregressive-moving average process, Biometrika 67 (1980), 232-233.

[24] A. H. Sayed and T. Kailath, Extended Chandrasekhar recursions, IEEE Trans. Automat. Control 39 (1994), 619-623.

[25] A. H. Sayed, T. Kailath and H. Lev-Ari, Generalized Chandrasekhar recursions from the generalized Schur algorithm, IEEE Trans. Automat. Control 39 (1994), 2265-2269.

[26] B. L. Shea, The exact likelihood of a vector autoregressive moving average model, J. Roy. Stat. Soc. C 38 (1989), 161-184.

[27] A. Varga, Periodic Lyapunov equations: some applications and new algorithms, Int. J. Control 67 (1997), 69-87.

[28] A. Varga and P. Van Dooren, Computational methods for periodic systems-an overview, Proc. of IFAC Workshop on Periodic Control Systems, Como, Italy, 2001, pp. 171-176.

■