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CONTINUOUS DEFORMATIONS FROM C^*-ALGEBRAS TO THEIR DIAGONALS

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Abstract

In this paper, we study continuous deformations from C^*-algebras to their diagonals by some examples. Furthermore, we obtain existence theorems of certain types of continuous deformation concerning C^*-algebras.

Introduction

It is customary that C^*-algebras (with operator norm topology) are viewed as noncommutative (topological) spaces since commutative ones $C_0(X)$ correspond to locally compact Hausdorff spaces X, where $C_0(X)$ is the C^*-algebra of continuous functions on X vanishing at infinity. Also, continuous fields of C^*-algebras are assumed as a noncommutative analogue to complex vector bundles over spaces. Especially, a continuous deformation from a C^*-algebra $\mathfrak{A}$ to another $\mathfrak{B}$ is a continuous field

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C^*-algebra on the closed interval $[0, 1]$ with fibers $\mathfrak{A}_t$ given by $\mathfrak{A}_t = \mathfrak{A}$ for $0 < t \leq 1$ and $\mathfrak{A}_0 = \mathfrak{B}$ at $t = 0$ (cf. [1]).

In this paper, we study continuous deformations from C^*-algebras to their diagonals (that are commutative C^*-algebras) by some examples. Furthermore, we obtain existence theorems of certain types of continuous deformation concerning C^*-algebras. We also obtain their consequences related with K-theory of C^*-algebras (see [1, 2, 3], for K-theory of continuous deformations of C^*-algebras).

1. Continuous Deformations of C^*-algebras

Let $M_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ matrices over $\mathbb{C}$ of complex numbers.

Theorem 1.1. *There exists a continuous deformation from* $M_n(\mathbb{C})$ *to* $\mathbb{C}^n$.

Proof. Define an $M_n(\mathbb{C})$-valued function on $[0, 1]$:

$$X_t = \begin{pmatrix} a_{11} & ta_{12} & \cdots & ta_{1n} \\ ta_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,t} \\ ta_{n,1} & \cdots & ta_{n,n-1} & a_{nn} \end{pmatrix} = D + tN,$$

for any $X \in M_n(\mathbb{C})$ and $t \in [0, 1]$, where D is the diagonal matrix (or part) of X and $N = X - D$ is the off-diagonal matrix (or part) of X. It is clear that $X_1 = X$ and $X_0 = D$. We identify $X_0 = D$ with the element $a_{11} \oplus a_{22} \oplus \cdots \oplus a_{nn} \in \mathbb{C}^n$. The C^*-algebra generated by the functions X_t for $X \in M_n(\mathbb{C})$ gives a continuous deformation from $M_n(\mathbb{C})$ to $\mathbb{C}^n$. Indeed, note that

$$|\| X_t \| - \| X_s \|| \leq \| X_t - X_s \| = |t - s| \| N \|,$$

for $t, s \in [0, 1]$, and

$$\| D \| = \max_{1 \leq j \leq n} |a_{jj}| = \| a_{11} \oplus a_{22} \oplus \cdots \oplus a_{nn} \|,$$

where $M_n(\mathbb{C})$ has the operator norm and $\mathbb{C}^n$ has the maximum norm. Also, for $X = D_1 + N_1$ and $Y = D_2 + N_2$ in $M_n(\mathbb{C})$ the same decomposition as above,

$$X_t + Y_t = D_1 + D_2 + t(N_1 + N_2),$$

$$X_t Y_t = (D_1 + tN_1)(D_2 + tN_2) = D_1 D_2 + t(D_1 N_2 + N_1 D_2 + tN_1 N_2),$$

from which addition and multiplication (and involution) are well defined.

Let $\mathbb{K}(H)$ be the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space H and $C_0(\mathbb{N})$ be the C^*-algebra of all functions on the set $\mathbb{N}$ of all natural numbers vanishing at infinity (where $C_0(\mathbb{N}) = c_0(\mathbb{N})$ by another notation).

Theorem 1.2. *There exists a continuous deformation from* $\mathbb{K}(H)$ *to* $C_0(\mathbb{N})$.

Proof. The similar proof as that of Theorem 1.1 is valid in this case. For a compact operator $T \in \mathbb{K}(H)$, we have the decomposition $T = D + N$ into its diagonal operator D and off-diagonal part N with respect to an orthogonal basis of H. Since T is compact, D can be identified with an element of $C_0(\mathbb{N})$ by spectral theory.

Let $\mathbb{B}(H)$ be the C^*-algebra of all bounded operators on a separable infinite dimensional Hilbert space H, and $C^b(\mathbb{N})$ be the C^*-algebra of all bounded functions on the set $\mathbb{N}$ (where $C^b(\mathbb{N}) = l^\infty(\mathbb{N})$ by another notation).

Theorem 1.3. *There exists a continuous deformation from* $\mathbb{B}(H)$ *to* $C^b(\mathbb{N})$.

Proof. The similar proof as that of Theorem 1.1 is also valid in this case. For a bounded operator $T \in \mathbb{B}(H)$, we have the decomposition $T = D + N$ into its diagonal operator D and off-diagonal part N with respect to an orthogonal basis of H. Since T is bounded, D can be identified with an element of $C^b(\mathbb{N})$ by spectral theory.

Remark. Note that the centers of $M_n(\mathbb{C})$, $\mathbb{K}(H)$ and $\mathbb{B}(H)$ are all trivial.

In a general situation, we obtain

Theorem 1.4. *Let $\mathfrak{A}$ be a C^*-algebra. Suppose that $\mathfrak{A}$ has a sequence of mutually orthogonal projections p_j such that either their finite sums form an approximate identity for $\mathfrak{A}$ nonunital, or their sum is the identity element of $\mathfrak{A}$ unital. Then there exists a continuous deformation from $\mathfrak{A}$ either to the direct sum $\oplus_{j=1}^{\infty} p_j \mathfrak{A} p_j$ (vanishing at infinity), or to the direct product $\prod_{j=1}^{\infty} p_j \mathfrak{A} p_j$, respectively.*

Proof. It is standard that such mutually orthogonal projections give the decomposition of $\mathfrak{A}$ (nonunital or unital) into an (infinite) matrix algebra with its diagonal given by the direct sum or the direct product as in the statement.

In other words,

Corollary 1.5. *A partition of unity (or non-unity) for a C^*-algebra gives rise to its continuous deformation.*

As for K-theory of C^*-algebras,

Corollary 1.6. *Continuous deformations of C^*-algebras do not induce continuity with respect to K_0-groups of their fibers.*

Proof. Indeed, $K_0(M_n(\mathbb{C})) \cong \mathbb{Z} \cong K_0(\mathbb{K})$ but $K_0(\mathbb{C}^n) \cong \mathbb{Z}^n$ and $K_0(C_0(\mathbb{N})) \cong \oplus^{\infty}\mathbb{Z}$ (while K_1-groups of them are all trivial). Also, $K_0(\mathbb{B}(H)) \cong 0$ but $K_0(C^b(\mathbb{N})) = K_0(l^{\infty}(\mathbb{N})) \cong \Pi^{\mathbb{R}}\mathbb{Z}$ the direct product over $\mathbb{R}$ since the cardinality of $\mathbb{R}$ is $2^{\mathbb{N}}$ (while K_1-groups of them are trivial since they are Von Neumann algebras).

On the other hand, the data in the proof above also says

Proposition 1.7. *A partition of unity (or non-unity) for a C^*-algebra can be represented by (or viewed as) K_0-groups of fibers coming from its continuous deformation obtained by us.*

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