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CONNECTIVITY IN *A*-SPACES

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Abstract

Recall that an Alexandroff space (shortly *A*-space) is a topological space such that every point has a smallest neighborhood. These spaces were first introduced by Alexandroff [1] and have become relevant for the study of digital topology. The main purpose of this paper is to study the connectivity properties of these spaces (which we do not assume that they are T_0).

0. Introduction

Topology is the mathematical study of properties of objects which are preserved through deformations, twisting and stretching (mathematically, through functions called *homeomorphism*). Because spaces by themselves are very complicated, they are unmanageable without looking at particular aspects. Topologies on $\mathbb{Z}^2$ are worth study because they have many useful applications in particular in digital image processing. Such topologies are usually used as digitizations of the Euclidean topology on the real plane and they are therefore required to have an analogous

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behavior. For instance the Jordan curve theorem which states that a simple closed curve separates the plane into two connected sets. In fact, there are three appropriate properties. The finiteness of the considered space, its geometric nature and its connectivity. The connectivity concept was firstly developed with a geometric view (the usual Euclidean plane $\mathbb{R}^2$). This point of view is not directly applied for a geometric space obtained throughout a screen equipped with a net ($\mathbb{Z}^2$). One of the topological aspects of a set is its number of path-connected components. Proving that a set is connected is an important problem already considered for robotics (e.g., for path planning) and identifiability applications [7, 8].

Different approaches of this notion appeared in the literature. We mention for sake of examples both of the relational and the C.O.T.S points of view (cf. [2, 4, 6] and [5, 6]). In this note, we are interested with Alexandroff spaces [1], particularly finite spaces which have recently captured people's attention. Since digital processing and image processing start from finite sets of observations and seek to understand pictures that emerge from a notion of nearness of points, Alexandroff topological spaces seem a natural tool in many such scientific applications and permits one to explain the connectivity problems related to our practical study support. In order to make connectivity pertinent, it is obvious that the considered spaces shall be not T_0. In the first part of this work (Sections 1, 2 and 3), we present necessary material needed for the good understanding of our objective. Some basic properties of Alexandroff spaces are developed. Their relationship with binary relations are settled. Recall that our goal is practice, so the digital line $(\mathbb{Z}, \mathcal{K})$ is presented. The second part (Section 4) treats the connectivity concept of Alexandroff spaces. Our main result states that for Alexandroff topological spaces the connectivity concepts under the following points of view are all equivalent.

(i) The set theoretic point of view which is proposed in general topology.

(ii) The path-connectivity point of view.

(iii) The cots path-connectivity point of view.

(iv) The cots arcwise-connectivity point of view.

1. Alexandroff Spaces

First of all, recall the following definition.

Definition 1.1. A topological space (X, τ) is said to be an *Alexandroff space* (for short *A-space*), if any intersection of elements of τ is also an element of τ.

Let (X, τ) be a topological space. Then x be an element of X and denote by $N(x)$ the intersection of all the open neighborhoods of x. If (X, τ) is an A-space, then obviously $N(x)$ is open and it is the smallest open neighborhood of x. Moreover, if $\mathcal{B}$ is a basis of the topology τ, then

$$N(x) = \bigcap_{U \in \tau,\, x \in U} U = \bigcap_{U \in \mathcal{B},\, x \in U} U.$$

Remark 1.2. A topological space (X, τ) is an A-space if and only if every point x in X has a smallest open neighborhood.

Examples 1.3

• If X is a finite topological space, then X is an A-space.

• Let $\mathcal{B} = \{[x, +\infty[, x \in \mathbb{R}\}$. It is obvious that $\mathcal{B}$ is a basis of a topology τ on $\mathbb{R}$, the so-called the *right topology* [1]. Clearly $(\mathbb{R}, \tau)$ is an A-space.

Proposition 1.4. *Let (X, τ) be an A-space and $\mathcal{B} = \{N(x), x \in X\}$. Then $\mathcal{B}$ is a minimal basis of open sets for the topology τ.*

Proof. Denote by $\mathcal{B}'$ a second basis for (X, τ). Since X is an A-space, for each element x in X, $N(x)$ is open. Thus there exists $V_x \in \mathcal{B}'$ such that $x \in V_x$ and $V_x \subset N(x)$. Since $N(x)$ is the smallest open set containing x, $V_x = N(x)$ and $\mathcal{B} \subset \mathcal{B}'$.

Remark 1.5. (1) The previous property is not a characterization for the Alexandroff spaces. In fact, let $X = [0, +\infty[$ and $\tau = \left\{\left[0, \frac{1}{n}\right[, n \in \mathbb{N}^*\right\}$. The unique basis for (X, τ) is τ itself. Thus it is a minimal one, nevertheless, we have

$$\{0\} = N(0) = \bigcap_{U \in \tau} U \notin \tau.$$

(2) If (X, τ) is an A-space, then

$$\forall (x, y) \in X \times X; x \in \overline{\{y\}} \Leftrightarrow y \in N(x)$$

$\overline{\{y\}}$ is the closure of $\{y\}$.

2. *A*-Spaces and Binary Relations

Recall that given a topology τ on a set X, then the preorder $\leq$ on X defined by (cf. [2]): $x \leq y \Leftrightarrow x \in \overline{\{y\}}$, whenever $x, y \in X$ is called the *specialization preorder* of τ. More generally, if $\mathcal{R}$ is a binary relation defined on a set X, then we can construct on X a topology defined as follows:

Let $\mathcal{R}_d(x) = \{x\} \cup \{y \in X \mid$ there exists $(x_1, ..., x_n) \in X^n$ such that $x_1 = x$, $x_n = y$ and for all i, $x_i \mathcal{R} x_{i+1}\}$. It is easily seen that the family $\{\mathcal{R}_d(x) \mid x \in X\}$ is a basis of a topology $\tau(\mathcal{R}_d)$ on X, we call it the *right-$\mathcal{R}$-topology* on X (cf. [2]).

In the particular case of the A-spaces, we have the following result.

Proposition 2.1 [2]. *The following hold*:

(i) *For an A-space, the topologies on X are in one-to-one correspondence with the preorder $\leq$ on X. The topology corresponding to $\leq$ is T_0 if and only if the relation $\leq$ is a partial order.*

(ii) *Let (X, τ) be an A-space. Then (X, τ) is T_0 if and only if (X, τ) is discrete.*

For the sake of completeness, we recall that a topological space (X, τ) is said to be:

- T_0, if $\forall(x, y) \in X \times X$, then there exists a neighborhood V_x of x such that $y \notin V_x$ or there exists a neighborhood V_y of y such that $x \notin V_y$.

- T_1, if $\forall(x, y) \in X \times X$, then there exists a neighborhood V_x of x such that $y \notin V_x$ and a neighborhood V_y of y such that $x \notin V_y$.

- T_2, if $\forall(x, y) \in X \times X$, then there exists a neighborhood V_x of x and a neighborhood V_y of y such that $V_x \cap V_y = \varnothing$.

Lemma 2.2. *Let* (X, τ) *be an A-space. Then the following hold*:

(i) *For all* $x \in X$, *the neighborhood* $N(x)$ *is a connected set.*

(ii) *Let* $(x, y) \in X \times X$. *If* $x \neq y$, *then the set* $\{x, y\}$ *is connected if and only if* $x \in N(y)$ *or* $y \in N(x)$.

Proof. (i) If we write $N(x) = O_1 \cup O_2$, where the O_i are open sets in (X, τ), then $x \in O_1$ or $x \in O_2$, and since $N(x)$ is the smallest open set containing x, it results that $N(x) \subset O_1$ or $N(x) \subset O_2$. Thus $N(x) = O_1$ or $N(x) = O_2$.

(ii) If $x \in N(y)$, then $y \in \overline{\{x\}}$, and we have $\{x\} \subset \{x, y\} \subset \overline{\{x\}}$. But since $\{x\}$ is a connected set, so is $\overline{\{x\}}$, and consequently $\{x, y\}$ is connected. For the converse, we have $\{x, y\} \subset \overline{\{x\}} \cup \overline{\{y\}}$, and since $\{x, y\}$ is a connected set, we must assume that $\{x, y\} \cap \overline{\{x\}} \neq \varnothing$ and $\{x, y\} \cap \overline{\{y\}} \neq \varnothing$. If $y \notin \overline{\{x\}}$, then $\{x, y\} \cap \overline{\{x\}} = \{x\}$ and since $\{x, y\} \cap \overline{\{x\}} \cap \overline{\{y\}} \neq \varnothing$, we must assume that $x \in \overline{\{y\}}$, which is equivalent to $y \in N(x)$. If $y \in \overline{\{x\}}$, then this is equivalent to $x \in N(y)$.

We close this section by the following corollary.

Corollary 2.3. *Each A-space is a locally connected topological space.*

3. The Digital Line

Recall the following definition.

Definition 3.1. The family $\mathcal{K} = \{\{2n+1\}, \{2n-1, 2n, 2n+1\}\ n \in \mathbb{Z}\}$, is a basis of some topology on $\mathbb{Z}$ the set of integers. It is called the *Khalimsky topology* [6]. The topological space $(\mathbb{Z}, \mathcal{K})$ is called the *digital line*.

Remark 3.2. (1) The odd one-points sets of $(\mathbb{Z}, \mathcal{K})$ are open. The even one-points sets of $(\mathbb{Z}, \mathcal{K})$ are closed and every set $\{x, x+1\}$ of $(\mathbb{Z}, \mathcal{K})$ is locally closed.

(2) The topological space $(\mathbb{Z}, \mathcal{K})$ is an A-space.

(3) The topological space $(\mathbb{Z}, \mathcal{K})$ is a connected topological space.

(4) A set $\{x, y\}$ in the topological space $(\mathbb{Z}, \mathcal{K})$, is connected if and only if $y = x + 1$.

Proof. (1) If $n \in \mathbb{Z}$, then $\{2n+1\}$ is by definition an open set. Let p be an element of $\mathbb{Z}$. On one hand, the half-lines $\{2p+1, 2p+2, 2p+3, \ldots\}$ and $\{\ldots 2p-1, 2p, 2p+1\}$ are open sets. Indeed $\{2p+1, 2p+2, 2p+3, \ldots\}$ $= \{2p+1, 2p+2, 2p+3\} \cup \{2p+3, 2p+4, 2p+5\} \cup \{2p+5, 2p+6, 2p+7\} \cup \ldots$ is a union of open sets. On the other hand, since $\mathbb{Z} - \{2n\}$ is a union of two open half-lines, the one-point set $\{2n\}$ is closed. The set $\{x, x+1\}$ is locally closed, in fact, $\{x, x+1\} = \{\ldots, x+1\} \cap \{x, \ldots\}$ is the intersection of two half-lines, one of them is open and the other is closed.

(2) Let x be an element of $(\mathbb{Z}, \mathcal{K})$, $N(x) = \{x\}$, if x is odd and $N(x)$ $= \{x-1, x, x+1\}$, if x is even.

(3) Suppose $\mathbb{R}$ equipped with its usual topology, and $\mathbb{Z}$ with the Khalimsky topology and consider the map: $f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x)$ $= 2n$ if $x \in \left[2n - \frac{1}{2}, 2n + \frac{1}{2}\right]$ and $f(x) = 2n+1$ if $x \in \left]2n + \frac{1}{2}, 2n + \frac{3}{2}\right[$.

Then it is clear that $f^{-1}(\{2n+1\}) = \left]2n + \frac{1}{2}, 2n + \frac{3}{2}\right[$ and $f^{-1}(\{2n-1, 2n, 2n+1\}) = \left]2n - \frac{3}{2}, 2n + \frac{3}{2}\right[$. Hence f is a continuous map and $(\mathbb{Z}, \mathcal{K})$ is connected, as the image of a connected set by a continuous map.

(4) Let x and y be two elements of $(\mathbb{Z}, \mathcal{K})$, and suppose that $y > x$. If x and y are odd, then $N(x) = \{x\}$ and $N(y) = \{y\}$, hence $x \notin N(y)$ and $y \notin N(x)$. We conclude that $\{x, y\}$ is not a connected set. If x and y are even, then $N(x) = \{x-1, x, x+1\}$ and $N(y) = \{y-1, y, y+1\}$, and since $y > x$, $x \notin N(y)$ and $y \notin N(x)$. This yields that $\{x, y\}$ is not a connected set. If x is even and y is odd, then $N(x) = \{x-1, x, x+1\}$ and $N(y) = \{y\}$, saying that $x \in N(y)$ or $y \in N(x)$ means that $y = x+1$, so, $\{x, y\}$ is a connected set if and only if $y = x+1$. If x is odd and y is even, then $N(y) = \{y-1, y, y+1\}$ and $N(x) = \{x\}$, saying that $x \in N(y)$ or $y \in N(x)$ means that $x = y-1$, so, $\{x, y\}$ is a connected set if and only if $y = x+1$.

4. Connectivity in A-spaces

We recall the general following definitions:

Definition 4.1. Let (X, τ) be a topological space and A be a nonempty subset of X.

(i) A is said to be *connected*, if for any nonempty open subsets U and V of X, the condition $(U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset)$ implies $(U \cap V \cap A \neq \emptyset)$.

(ii) A is said to be *path-connected* (P.C) (resp., arcwise-connected (A.C)), if any two points x and y of A can be connected by a path (a continuous map γ from $[0, 1]$ to X such that $\gamma(0) = x$ and $\gamma(1) = y$), (resp., an arc) (a homeomorphism γ from $[0, 1]$ to $\gamma([0, 1])$ such that $\gamma(0) = x$ and $\gamma(1) = y$).

(iii) X is said to be a *C.O.T.S* (*connected ordered topological space*, (cf. [6])), if X is a totally ordered connected topological space with the property, for any distinct elements (x_1, x_2, x_3) in X, there is an $i = 1, 2, 3$ such that x_j and x_k lie in different components of $X - \{x_i\}$.

(iv) A C-path (resp., C-arc) in X is the image by a continuous map (resp., a homeomorphism) γ from some C.O.T.S into X.

(v) X is said to be *C.P.C* (*cots path connected*), if any pair of points (x, y) in X is contained in some C-path.

(vi) X is said to be *C.A.C* (*cots arc connected*), if every pair of points (x, y) in X is contained in some C-arc.

We recall from [2], that if (X, τ) is a topological space, we define a τ-cord between two elements x and y of X, as follows: there exists a chain $(x_1, x_2, \ldots, x_n)$ of elements of X such that $x = x_1$, $y = x_n$ and for all $1 \leq i \leq n - 1$, we have $x_{i+1} \in N(x_i)$ or $x_i \in N(x_{i+1})$.

This notion meets the connectivity point of view arising from the theory of relations and graph theory.

At first sight, one might imagine that there are no continuous maps from $[0, 1]$ to an A-space, particularly a finite space, but that is far from the case. The most important feature of A-spaces is that they are surprisingly richly related to the *real* spaces that algebraic topologist care about.

Proposition 4.2. *Let (X, τ) be an A-space. A pair $\{x, y\}$ in X is a connected set if and only if it is P.C.*

Proof. Suppose that $\{x, y\}$ is a connected pair, this is equivalent to: $x \in N(y)$ or $y \in N(x)$. Suppose that $y \in N(x)$ and define $\gamma : [0, 1] \to X$ by $\gamma(0) = x$ and $\gamma(t) = y$ if $t \neq 0$, where $[0, 1]$ is the real interval equipped with its usual topology. We prove that γ is a continuous map. Indeed, let V be an open set in X. If $x \in V$, then $N(x) \subset V$ and since $y \in N(x)$, we have $\gamma^{-1}(V) = [0, 1]$, which is an open set in $[0, 1]$. If $x \notin V$, either $y \notin V$, then $\gamma^{-1}(V) = \varnothing$ which is open in $[0, 1]$, or $y \in V$ and $\gamma^{-1}(V) =]0, 1]$ which is also open in $[0, 1]$. We conclude that $\{x, y\}$ is P.C.

For the converse, we note that the pair $\{x, y\}$ is P.C, as the image by a continuous map of a connected set (the interval I of $\mathbb{R}$).

Theorem 4.3. *An A-space X is connected if and only if it is P.C (path connected).*

Proof. In [2, Proposition 1.9], it is proved that (X, τ) is connected if and only if for each pair (x, y) of points in X, there exists a τ-cord between x and y, which is equivalent to the existence of a chain $(x_1, x_2, \ldots, x_n)$ of points in X such that $x = x_1$, $y = x_n$ and for all $1 \leq i \leq n-1$, we have $x_{i+1} \in N(x_i)$ or $x_i \in N(x_{i+1})$, which is equivalent to: there exists a chain $(x_1, x_2, \ldots, x_n)$ of points in X such that $x = x_1$, $y = x_n$ and for each $1 \leq i \leq n-1$, $\{x_i, x_{i+1}\}$ is a connected set. Using the previous proposition, $\{x_i, x_{i+1}\}$ is P.C. So that (X, τ) is a connected set if and only if for each pair (x, y) of elements in X, there exists a path from x to y.

Lemma 4.4. *Let X be an A-space and (x, y) be a pair of elements in X. Denote by $C(x, y) = \{(x_0, x_1, x_2, \ldots, x_n) \in X^n$ such that $\forall i \in \{0, \ldots, n-1\}$, $\{x_i, x_{i-1}\}$ is a connected set, $x_0 = x$, and $x_n = y\}$. If X is a connected space, then $C(x, y) \neq \emptyset$ and every minimal element of $C(x, y)$ verify $\forall i \in \{0, 1, \ldots, n-1\}$, we have either $x_i \in N(x_{i+1})$ and $x_{i+1} \notin N(x_i)$ or $x_{i+1} \in N(x_i)$ and $x_i \notin N(x_{i+1})$.*

Proof. In [2, Proposition 1.9], when X is a connected A-space, we prove that $C(x, y)$ is not an empty set. Let $C = (x_0, x_1, x_2, \ldots, x_n)$ be a minimal element of $C(x, y)$ and let u and v be two elements in $C \times C$ satisfying $u \in N(v)$ and $v \in N(u)$. $\forall z \in C$, $\{u, z\}$ is a connected pair if and only if $\{v, z\}$ is a connected pair. Because of the minimality of C, we have necessarily $u = v$ and so, if $u \neq v$ we get, either $u \in N(v)$ and $v \notin N(u)$ or $v \in N(u)$ and $u \notin N(v)$.

Theorem 4.5. *An A-space X is connected if and only if it is C.A.C (cots arc connected).*

Proof. Let x and y be two elements in X. Since X is a connected space, there exists a finite chain

$$C = \{x_0, x_1, \ldots, x_n\},$$

in X such that $x = x_1$, $y = x_n$ and for all $1 \leq i \leq n-1$, $\{x_i, x_{i+1}\}$ is a connected pair. We get C minimal. So that, we have the two possibilities: either $N(x_0) = \{x_0\}$, or $N(x_0) = \{x_0, x_1\}$. Our task is to show that (X, τ) is a C.A.C space. For this end we may construct a homeomorphism ψ from a C.O.T.S on C. The selected C.O.T.S will be an interval I in $(\mathbb{Z}, \mathcal{K})$. To construct ψ, we first of all, describe $N_C(x)$ for every point x in C, $N_C(x)$ is the smallest open neighborhood of x in C. Recall that C is an A-space and it is a minimal chain for the choice of x and y and we have for all $1 \leq i \leq n-1$, $\{x_i, x_{i+1}\}$ is a connected pair. We consider the following two cases:

Case 1. $N_C(x_0) = \{x_0\}$.

Description of $N_C(x_1)$: Since $\{x_0, x_1\}$ is a connected pair and $N_C(x_0) = \{x_0\}$, $x_0 \in N_C(x_1)$ and $N_C(x_1) = \{x_0, x_1\}$ or $\{x_0, x_1, x_2\}$. If $N_C(x_1) = \{x_0, x_1\}$, $x_2 \notin N(x_1)$ and since $\{x_1, x_2\}$ is a connected pair, $x_1 \in N_C(x_2)$, so that $N_C(x_1) \cap N_C(x_2) = \{x_1\}$ or $\{x_0, x_1\}$. This intersection cannot be $\{x_1\}$ in contradiction with $N_C(x_1) = \{x_0, x_1\}$, and cannot be $\{x_0, x_1\}$ because in this case $\{x_0, x_2\}$ will be a connected pair, in contradiction with the minimality, finally $N_C(x_1) = \{x_0, x_1, x_2\}$.

Description of $N_C(x_2)$: Because of the minimality of C, $x_0 \notin N(x_2)$, so $N_C(x_1) \cap N_C(x_2) = \{x_2\}$, which implies that $N_C(x_2) = \{x_2\}$.

Proceeding similarly we obtain $N_C(x_3) = \{x_2, x_3, x_4\}$ and $N_C(x_4) = \{x_4\}$, etc. Finally, if i is even, $N_C(x_i) = \{x_{i-1}, x_i, x_{i+1}\}$ and if i is odd $N_C(x_i) = \{x_i\}$.

We choose $I = [1, n+1] \cap \mathbb{Z}$ and define $\psi : I \to C$ as $\psi(i) = x_{i-1}$. We have $\psi(1) = x = x_0$ and $\psi(1+n) = y = x_n$. Since C is minimal, ψ is an injective map. Then ψ is a bijection from I onto $\psi(I)$. If we denote by

$N_I(i)$ the smallest open neighborhood of i in I, we note that ψ verifies

$$\forall i \in \{0, 1, \dots, n-1\}, \psi(N_I(i)) = N(x_{i-1}).$$

Thus ψ is a homeomorphism from I onto C.

Case 2. $N_C(x_0) = \{x_0, x_1\}$.

On one hand, since $x_1 \in N_C(x_0)$ and C is minimal, $x_0 \notin N(x_1)$. On the other hand, $N_C(x_1)$ is an open set. Hence $N_C(x_0) \cap N(x_1) = \{x_1\}$, and $N_C(x_1) = \{x_1\}$. We are now in the previous case.

We choose $I = [0, n] \cap \mathbb{Z}$ and define $\psi : I \to C$ as $\psi(i) = x_i$. We get a homeomorphism from I onto C.

For the converse, suppose that the topological A-space (X, τ) is not connected, then, there exist two disjoint proper nonempty open subsets U_1 and U_2 of X such that $X = U_1 \cup U_2$. Pick x an element in U_1 and y an element in U_2 and let $C = (x_0, x_1, x_2, \dots, x_n)$ the minimal arc between x and y. Denote p the last index such that $x_p \in U_1$, then $x_{p+1} \in U_2$. This hypothesis permits to write $C = (x_0, \dots, x_p) \cup (x_{p+1}, \dots, x_n)$, a contradiction with the connectivity of C. In conclusion, (X, τ) is a connected space.

The following result generalizes a result of [5].

Theorem 4.6. *Let X be an A-space. Then the following statements are equivalent*:

(i) *X is connected.*

(ii) *X is P.C (path connected).*

(iii) *X is C.P.C (cots path connected).*

(iv) *X is C.A.C (cots arc connected).*

Proof. (ii) $\Leftrightarrow$ (iii) Note that the interval I in $\mathbb{R}$, equipped with its usual order and its usual topology is a C.O.T.S.

(i) $\Leftrightarrow$ (ii) This is Theorem 4.3.

(i) $\Leftrightarrow$ (iv) This is Theorem 4.5.

References

[1] P. Alexandroff, Diskrete Räume, Mat. Sb. (Receul Mathématique) (2) 44(3) (1937), 502-519.

[2] E. Bouassida, O. Echi and E. Salhi, Topologies associées à une relation binaire et relation binaire spectrale, Bollettino U. M. I. (7) 10-B (1996), 417-439.

[3] N. Bourbaki, Topologie Générale Chapitres 1 à 4, Masson, Paris, 1990.

[4] U. Eckhardt and L. Jan Latecki, Topologies for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$, Computer Vision and Image Understanding 90 (2003), 295-312.

[5] E. Khalimsky, R. Kopperman and P. R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology Appl. 36 (1990), 1-17.

[6] T. Y. Kong, R. Kopperman and P. R. Meyer, A topological approach to digital topology, Amer. Math. Monthly 98 (1991), 901-917.

[7] T. Lozano-Pérez, Spatial planning, A configuration space approach, IEEE Trans. Comput. 32(2) (1983), 108-120.

[8] E. Walter and L. Pronzato, Identification of Parametric Models from Experimental Dats, Springer-Verlag, 1997.

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