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THE COLORED JONES POLYNOMIALS AND THE ALEXANDER POLYNOMIAL OF THE FIGURE-EIGHT KNOT

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Abstract

The volume conjecture and its generalization state that the series of certain evaluations of the colored Jones polynomials of a knot would grow exponentially and its growth rate would be related to the volume of a three-manifold obtained by Dehn surgery along the knot. In this paper, we show that for the figure-eight knot the series converges in some cases and the limit equals the inverse of its Alexander polynomial.

1. Introduction

Let K be a knot and $J_N(K;t)$ be its colored Jones polynomial corresponding to the N-dimensional irreducible representation of $sl_2(\mathbb{C})$ normalized so that $J_N(U;t)=1$ for the unknot U. The volume conjecture

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[15] states that

$$\lim_{N\to\infty}\frac{\log|J_N(K;\ \exp(2\pi\sqrt{-1}/N))|}{N}=\frac{v_3}{2\pi}\operatorname{Vol}(S^3\setminus K),$$

where v_3 is the hyperbolic volume of the ideal regular hyperbolic tetrahedron, and Vol denotes the simplicial volume. Note that this conjecture was first proposed by Kashaev [10] in a different way. It is generalized by Gukov [8] to a relation of the limit

$$\lim_{N\to\infty}\frac{\log J_N(K;\ \exp(a/N))}{N}, \tag{1.1}$$

with a fixed complex number a to the A-polynomial of K [2], and the volume and the Chern-Simons invariant of a three-manifold obtained by Dehn surgery along K. See also [14, 16] about the generalized volume conjecture for the figure-eight knot.

On the other hand, the author proved also in [14] that the limit (1.1) vanishes for the figure-eight knot if a is real and $|a| < \operatorname{arccosh}(3/2)$ or a is purely imaginary and $|a| < \pi/3$. Garoufalidis and Le proved [5, Theorem 2] that for any knot K, (1.1) vanishes if a is purely imaginary and sufficiently small. This shows that the series $\{J_N(K;\ \exp(a/N))\}_{N=2,3,\ldots}$ grows polynomially when a is small. One may ask whether the series diverges or not.

In this paper, we study the *genuine* limit $\lim_{N\to\infty}J_N(E;\ \exp(a/N))$ for the figure-eight knot E when a is a small complex number, and show that the limit does exist and equals the inverse of its Alexander polynomial. More precisely, we will show the following equality.

Theorem 1.1. *Let E be the figure-eight knot. If a is a complex number with $|2\cosh a - 2| < 1$ and $|\operatorname{Im} a| < \pi/3$, then the series $\{J_N(E;\ \exp(a/N))\}_{N=2,3,\ldots}$ converges and*

$$\lim_{N\to\infty}J_N\left(E;\ \exp\frac{a}{N}\right)=\frac{1}{\Delta(E;\ \exp a)},$$

where $\Delta(E;\ t) = -t + 3 - t^{-1}$ is the Alexander polynomial of E.

Remark 1.2. The range $\{a \in \mathbb{C} \mid |2\cosh a - 2| < 1, |\operatorname{Im} a| < \pi/3\}$ looks like an *oval* (not a mathematical one) around the origin whose boundary goes through the four points $((3+\sqrt{5})/2)$, $\pi\sqrt{-1}/3$, $-\log((3+\sqrt{5})/2)$ and $-\pi\sqrt{-1}/3$ on the Gaussian plane (see Lemma 3.1). The author does not know whether this *oval* is the *circle of convergence* or not.

Remark 1.3. Note that the inequality $|2\cosh a - 2| < 1$ is equal to $|\Delta(E;\, \exp a) - 1| < 1$. This may suggest another relation between the colored Jones polynomials and the Alexander polynomial.

Remark 1.4. Soon after submitting the paper to the mathematics arXiv, Garoufalidis and Le proved that a result similar to Theorem 1.1 holds for any knot [4]. More precisely, they proved that for any knot K, there exists a neighborhood $U_K \subset \mathbb{C}$ of 0 such that if $a \in U_K$, then the limit $\lim_{N\to\infty} J_N(K;\, \exp(a/N))$ exists and equals to $1/\Delta(K;\, \exp a)$.

2. Proof

We first recall the formula of the figure-eight knot due to Habiro and Le ([9], see also [11]).

$$J_N(E;\, t) = \sum_{k=0}^{N-1} \prod_{j=1}^{k} \left(t^{(N+j)/2} - t^{-(N+j)/2}\right)\left(t^{(N-j)/2} - t^{-(N-j)/2}\right).$$

If we replace t with $\exp(a/N)$, then we have

$$J_N\left(E;\, \exp\frac{a}{N}\right) = \sum_{k=0}^{N-1} f_{N,a}(k)$$

with

$$f_{N,a}(k) := \prod_{j=1}^{k} g_{N,a}(j),$$

where

$$g_{N,a}(j) := 4\sinh\left(\frac{a(N+j)}{2N}\right)\sinh\left(\frac{a(N-j)}{2N}\right)$$

$$= 2\cosh a - 2\cosh\frac{aj}{N}.$$

We first show that $J_N\left(E;\ \exp\frac{a}{N}\right)$ converges.

Lemma 2.1. *For any complex number a with $|2\cosh a - 2| < 1$ and $|\operatorname{Im} a| < \pi/3$, the series $\left\{J_N\left(E;\ \exp\frac{a}{N}\right)\right\}_{N=2,3,\ldots}$ converges.*

Proof. From Lemmas 3.3 and 3.4, we have the following inequalities for $0 < M < N$:

$$|g_{N,a}(j)| < \delta < 1 \quad \text{if } 0 < j < N,$$

$$\left|\frac{g_{M,a}(j)}{g_{N,a}(j)}\right| < 1 \quad \text{if } 0 < j < M,$$

$$\left|\frac{g_{M,a}(j)}{g_{N,a}(j)}\right| > 1 - \frac{j}{M} \quad \text{if } 0 < j < \varepsilon M \text{ for some } \varepsilon > 0,$$

where we put $\delta := |2\cosh a - 2| < 1$. So we have

$$1 > \delta^k > \left|\frac{f_{M,a}(k)}{f_{N,a}(k)}\right| > \prod_{j=1}^{\lfloor \varepsilon M \rfloor - 1}\left(1 - \frac{j}{M}\right)\prod_{j=\lfloor \varepsilon M \rfloor}^{k}\left|\frac{f_{M,a}(k)}{f_{N,a}(k)}\right|$$

for $0 < k < M < N$, where $\lfloor x \rfloor$ is the greatest integer that does not exceed x.

Putting $M' := \lfloor \varepsilon M \rfloor$, we have

$$\left|J_N\left(E;\ \exp\frac{a}{N}\right) - J_M\left(E;\ \exp\frac{a}{M}\right)\right|$$

$$= \left|\sum_{k=0}^{N-1} f_{N,a}(k) - \sum_{k=0}^{M-1} f_{M,a}(k)\right|$$

$$\le \sum_{k=0}^{M-1}|f_{N,a}(k) - f_{M,a}(k)| + \sum_{k=M}^{N-1}|f_{N,a}(k)|$$

$$= \sum_{k=0}^{M-1}|f_{N,a}(k)|\left(1 - \left|\frac{f_{M,a}(k)}{f_{N,a}(k)}\right|\right) + \sum_{k=M}^{N-1}|f_{N,a}(k)|$$

$$< \sum_{k=0}^{M-1} \delta^k \left(1 - \prod_{j=1}^{M'-1} \left(1 - \frac{j}{M}\right) \prod_{j=M'}^{k} \left| \frac{f_{M,a}(k)}{f_{N,a}(k)} \right| \right) + \sum_{k=M}^{N-1} \delta^k$$

$$= \frac{1-\delta^N}{1-\delta} - \sum_{k=0}^{M'-1} \delta^k \prod_{j=1}^{k} \left(1 - \frac{j}{M}\right) - \sum_{k=M'}^{M-1} \delta^k \prod_{j=1}^{M'-1} \left(1 - \frac{j}{M}\right) \prod_{j=M'}^{k} \left| \frac{f_{M,a}(k)}{f_{N,a}(k)} \right|.$$

From Lemma 3.5, this is equal to

$$\frac{1-\delta^N}{1-\delta} - \frac{M'}{\delta} e^{\frac{M'}{\delta}} \int_1^\infty e^{-\frac{M'}{\delta}t} t^{M'-1} dt$$

$$- \prod_{j=1}^{M'-1} \left(1 - \frac{j}{M}\right) \sum_{k=M'}^{M-1} \delta^k \prod_{j=M'}^{k} \left| \frac{f_{M,a}(k)}{f_{N,a}(k)} \right|. \tag{2.1}$$

Note that since

$$\prod_{j=1}^{M'-1} \left(1 - \frac{j}{M}\right) \sum_{k=M'}^{M-1} \delta^k \prod_{j=M'}^{k} \left| \frac{f_{M,a}(k)}{f_{N,a}(k)} \right| < \sum_{k=M'}^{M-1} \delta^k = \delta^{M'} \frac{1-\delta^{M-M'}}{1-\delta},$$

the last term in (2.1) can be arbitrarily small.

Since

$$\int_1^\infty e^{-\frac{M'}{\delta}t} t^{M'-1} dt = \int_1^\infty e^{\left(\log t - \frac{t}{\delta}\right) M'} t^{-1} dt,$$

we can apply Laplace's method to study the asymptotic behavior for large M:

$$\int_1^\infty e^{\left(\log t - \frac{t}{\delta}\right) M'} t^{-1} dt \ \ M \underset{\sim}{\to} \infty \ \frac{1}{M'} \frac{1}{\frac{1}{\delta} - 1} e^{-\frac{M'}{\delta}} = \frac{\delta}{M'} e^{-\frac{M'}{\delta}} \frac{1}{1-\delta}.$$

(See, for example, [17, Chapter 3, Section 7.1]). Therefore, $\left| J_N\left(E; \exp \frac{a}{N}\right) - J_M\left(E; \exp \frac{a}{M}\right) \right|$ can be arbitrarily small if M is sufficiently large, which means that the sequence $\left\{ J_N\left(E; \exp \frac{a}{N}\right) \right\}_{N=2,3,\ldots}$ is a Cauchy sequence and so it converges.

Now that we know the convergence, we use an *inhomogeneous* recursion formula of $J_N(E;\, t)$ to find the limit. It is known that $J_N(E;\, t)$ satisfies the following formula [6, Section 6.2] (see also [7] for a *homogeneous* recursion formula).

$$\begin{aligned} &J_N(E;\, t) \\ &= \frac{t^{-N-1}(t^N + t)(t^{2N} - t)}{t^N - 1} \\ &\quad + \frac{t^{-2N-2}(t^{N-1} - 1)^2 (t^{N-1} + 1)(t^4 + t^{4N} - t^{N+3} - t^{2N+1} - t^{2N+3} - t^{3N+1})}{(t^N - 1)(t^{2N-3} - 1)} \\ &\quad \times J_{N-1}(E;\, t) - \frac{(t^{N-2} - 1)(t^{2N-1} - 1)}{(t^N - 1)(t^{2N-3} - 1)} J_{N-2}(E;\, t). \end{aligned} \tag{2.2}$$

We want to show that the series $\left\{J_{N-1}\left(E;\, \exp\frac{a}{N}\right)\right\}$ and $\left\{J_{N-2}\left(E;\, \exp\frac{a}{N}\right)\right\}$ also converge and both limits coincide with that of $J_N\left(E;\, \exp\frac{a}{N}\right)$.

For $l = 1$ or 2, put

$$g'_N(j;\, l) := 2\cosh a\left(1 - \frac{l}{N}\right) - 2\cosh\frac{aj}{N},$$

and

$$f'_N(k;\, l) := \prod_{j=1}^{k} g'_N(j;\, l),$$

so that $J_{N-l}\left(E;\, \exp\frac{a}{N}\right) = \sum_{k=0}^{N-l-1} f'_N(k;\, l)$.

Lemma 2.2. *The series* $\left\{J_{N-l}\left(E;\, \exp\frac{a}{N}\right)\right\}_{N=2,3,\dots}$ *converges and shares the limit with* $\left\{J_N\left(E;\, \exp\frac{a}{N}\right)\right\}_{N=2,3,\dots}$.

Proof. From Lemma 3.3 and Corollary 3.2, we have

$$\begin{aligned} |g'_N(j;\, l)| &= 2\left|\cosh a\left(1 - \frac{l}{N}\right) - \cosh\frac{aj}{N}\right| \\ &< 2\left|\cosh a\left(1 - \frac{l}{N}\right) - 1\right| \\ &< 2|\cosh a - 1| \\ &= \delta. \end{aligned}$$

From Lemma 3.6 there exists a positive number ε' such that if $j/N < \varepsilon'$, then

$$1 > \left|\frac{\cosh a\left(1 - \frac{l}{N}\right) - \cosh\frac{aj}{N}}{\cosh a - \cosh\frac{aj}{N}}\right| > 1 - \left|\frac{a\sinh a}{\cosh a - 1}\right|\frac{1}{N}.$$

Putting $c := \left|\frac{a\sinh a}{\cosh a - 1}\right| > 0$, we have

$$1 > \left|\frac{g'_N(j;\, l)}{g_N(j)}\right| > 1 - \frac{c}{N},$$

if $j/N < \varepsilon'$ and so

$$1 > \left|\frac{f'_N(k;\, l)}{f_N(k)}\right| > \left(1 - \frac{c}{N}\right)^k,$$

if $k/N < \varepsilon'$.

Therefore, we have

$$\begin{aligned} &\left|J_N\left(E;\, \exp\frac{a}{N}\right) - J_{N-l}\left(E;\, \exp\frac{a}{N}\right)\right| \\ &= \left|\sum_{k=0}^{\lfloor\varepsilon' N\rfloor - 1}\{f_N(k) - f'_N(k;\, l)\} + \sum_{k=\lfloor\varepsilon' N\rfloor}^{N-1} f_N(k) - \sum_{k=\lfloor\varepsilon' N\rfloor}^{N-l-1} f'_N(k;\, l)\right| \\ &< \sum_{k=0}^{\lfloor\varepsilon' N\rfloor - 1}|f_N(k)|\left\{1 - \left(1 - \frac{c}{N}\right)^k\right\} + \sum_{k=\lfloor\varepsilon' N\rfloor}^{N-1}|f_N(k)| + \sum_{k=\lfloor\varepsilon' N\rfloor}^{N-1}|f'_N(k;\, l)| \end{aligned}$$

$$< \sum_{k=0}^{\lfloor \varepsilon' N \rfloor - 1} \delta^k \left\{ 1 - \left(1 - \frac{c}{N}\right)^k \right\} + 2 \sum_{k=\lfloor \varepsilon' N \rfloor}^{N-1} \delta^k$$

$$= \frac{1 - \delta^{\lfloor \varepsilon' N \rfloor}}{1 - \delta} - \frac{1 - \delta^{\lfloor \varepsilon' N \rfloor}\left(1 - \frac{c}{N}\right)^{\lfloor \varepsilon' N \rfloor}}{1 - \delta\left(1 - \frac{c}{N}\right)} + 2\delta^{\lfloor \varepsilon' N \rfloor} \frac{1 - \delta^{N - \lfloor \varepsilon' N \rfloor}}{1 - \delta},$$

which can be arbitrarily small when N is sufficiently large, since $0 < \delta < 1$. So the series $\left\{ J_{N-l}\left(E;\, \exp \frac{a}{N}\right) \right\}$ $(l = 1 \text{ or } 2)$ converges and its limit is equal to that of $\left\{ J_N\left(E;\, \exp \frac{a}{N}\right) \right\}$.

Therefore, putting $J_a := \lim_{N \to \infty} J_N\left(E;\, \exp \frac{a}{N}\right)$ and $w := \exp a$, we have from (2.2)

$$\begin{aligned} J_a &= \frac{w^{-1}(w+1)(w^2-1)}{w-1} \\ &\quad + \frac{w^{-2}(w-1)^2(w+1)(1 + w^4 - w - w^2 - w^2 - w^3)}{(w-1)(w^2-1)} J_a \\ &\quad - \frac{(w-1)(w^2-1)}{(w-1)(w^2-1)} J_a. \end{aligned}$$

So we finally have

$$J_a = \frac{1}{-w + 3 - w^{-1}},$$

winch is equal to $1/\Delta(E;\, \exp a)$.

This completes the proof of Theorem 1.1.

Remark 2.3. We used an *inhomogeneous* recursion formula for the colored Jones polynomial of the figure-eight knot. Note that Garoufalidis and Le proved that there always exists a *homogeneous* formula for any knot [6].

The relation between the A-polynomial and the Alexander polynomial [2, Section 6.3 Proposition], the AJ-conjecture proposed by Garoufalidis [3], and Theorem 1.1 suggest that for any knot K if the series $\{J_N(K;\ \exp a/N)\}_{N=2,3,\dots}$ converges for some a, then the limit would be $1/\Delta(K;\ \exp a)$ with $\Delta(K;\ t)$ the Alexander polynomial of K.

In [13] the author proved that for any torus knot T, $\lim_{N\to\infty} J_N(T;\ \exp a/N) = 1/\Delta(T;\ \exp a)$ if a is near $2\pi\sqrt{-1}$ and $\operatorname{Re} a > 0$.

Remark 2.4. Melvin and Morton [12] observed the following formal power series:

$$J_N(K;\ \exp h) = \sum_{j,\,k\geq 0} b_{jk}(K)h^j N^k, \tag{2.3}$$

and conjectured the following (Melvin-Morton-Rozansky conjecture):

(i) $b_{jk}(K) = 0$ if $k > j$, and

(ii) $\sum_{j\geq 0} b_{jj}(K)(hN)^j = \dfrac{1}{\Delta(K;\ \exp hN)}$.

This conjecture was proved by Rozansky [18] non-rigorously, and proved by Bar-Natan and Garoufalidis [1].

Replacing h with a/N, we have from (i) and (ii)

$$J_N\left(K;\ \exp\frac{a}{N}\right) = \sum_{j\geq k\geq 0} b_{jk}(K)a^j N^{k-j},$$

and

$$\sum_{j\geq 0} b_{jj}(K)a^j = \frac{1}{\Delta(K;\ \exp a)}.$$

So we may regard Theorem 1.1 as an analytic version of the Melvin-Morton-Rozansky conjecture.

3. Appendix

In this appendix, we give several technical lemmas used in the paper.

Lemma 3.1. *For a complex number* $a = x + y\sqrt{-1}$ *with* $x, y \in \mathbb{R}$, *the condition* $|2\cosh a - 2| < 1$ *is equivalent to the condition* $\cosh x - \cos y < 1/2$.

Proof. Since

$$\begin{aligned} |\cosh a - 1|^2 &= (\cos y \cosh x - 1)^2 + \sin^2 y \sinh^2 x \\ &= (\cos y \cosh x - 1)^2 + (1 - \cos^2 y)(\cosh^2 x - 1) \\ &= (\cosh x - \cos y)^2, \end{aligned}$$

$\cosh x \geq 1$ and $\cos y \leq 1$, we have $|\cosh a - 1| = \cosh x - \cos y$.

Especially, we have $|x| < \operatorname{arccosh} 3/2 = \log((3 + \sqrt{5})/2) = 0.9642 \cdots < 1$.

Corollary 3.2. *If a complex number a satisfies* $|2\cosh a - 2| < 1$ *and* $|\operatorname{Im} a| < \pi/3$, *then for any real number u with* $0 < u < 1$, *we have* $|\cosh ua - 1| < |\cosh a - 1|$.

Proof. From Lemma 3.1, $\cosh x - \cos y < 1/2$ with $a := x + y\sqrt{-1}$. Since $\cosh x$ is increasing (decreasing, respectively) for $x > 0$ ($x < 0$, respectively) and $\cos y$ is decreasing (increasing, respectively) for $0 < y < \pi/3$ ($0 > y > -\pi/3$, respectively), we have

$$\begin{aligned} |\cosh ua - 1| &= |\cosh ux - \cos uy| = \cosh ux - \cos uy \\ &< \cosh x - \cos y = |\cosh x - \cos y| = |\cosh a - 1|. \end{aligned}$$

Lemma 3.3. *For a complex number a with* $|2\cosh a - 2| < 1$ *and* $|\operatorname{Im} a| < \frac{\pi}{3}$, *and real numbers u and v with* $0 \leq u < v < 1$, *we have*

$$|\cosh a - \cosh ua| \geq |\cosh a - \cosh va|.$$

Moreover, the equality holds only when $a = 0$.

Proof. It is clear that both hand sides are equal when $a = 0$. So we assume that $a \neq 0$ and prove the strict inequality.

Put $a := x + y\sqrt{-1}$ with $(x, y) \neq (0, 0)$ and $|y| < \pi/3$.

We will show that $\varphi(x, y, u) := |\cosh a - \cosh ua|^2$ is decreasing with respect to u for $0 < u < 1$. Since $\varphi(x, y, u) = \varphi(-x, y, u) = \varphi(x, -y, u)$, we may assume that $x \geq 0$ and $\pi/3 > y \geq 0$. Since

$$|\cosh a - \cosh ua|^2 = (\cosh x \cos y - \cosh ux \cos uy)^2$$
$$+ (\sinh x \sin y - \sinh ux \sin uy)^2,$$

we have

$$\frac{\partial\varphi(x, y, u)}{\partial u} = -2x\{\sinh x \cosh ux \sin y \sin uy$$
$$+ \cosh x \sinh ux \cos y \cos uy - \sinh ux \cosh ux\}$$
$$- 2y\{\sin uy \cos uy + \sinh x \sinh ux \sin y \cos uy$$
$$- \cosh x \cosh ux \cos y \sin uy\}.$$

Put

$$\varphi_1(x, y, u) := \sinh x \cosh ux \sin y \sin uy + \cosh x \sinh ux \cos y \cos uy$$
$$- \sinh ux \cosh ux,$$

and

$$\varphi_2(x, y, u) := \sin uy \cos uy + \sinh x \sinh ux \sin y \cos uy$$
$$- \cosh x \cosh ux \cos y \sin uy.$$

We will show

(1) $\varphi_1(x, y, u) > 0$ when $x > 0$ and $y \geq 0$, and

(2) $\varphi_2(x, y, u) > 0$ when $x \geq 0$ and $y > 0$.

First we will show (1). Note that if $x > 0$, $\varphi_1(x, 0, u) = \sinh ux (\cosh x - \cosh ux) > 0$, and so we will assume that $y > 0$.

Since $\varphi_1(0, y, u) = 0,$ it is sufficient to show that

$$\frac{\partial\varphi_1(x, y, u)}{\partial x} = u(\cosh x \cosh ux \cos y \cos uy$$

$$+ \sinh x \sinh ux \sin y \sin uy - \sinh^2 ux - \cosh^2 ux)$$

$$+ \cosh x \cosh ux \sin y \sin uy + \sinh x \sinh ux \cos y \cos uy$$

is positive when $x > 0,\ \pi/3 > y > 0$ and $1 > u > 0.$ Note that

$$\frac{\partial\varphi_1(x, y, u)}{\partial x}\Big|_{x=0} = u(\cos y \cos uy - 1) + \sin y \sin uy$$

is positive since its partial derivative with respect to y is $(1 - u^2)$ $\cos y \sin uy,$ which is positive. Moreover, we have

$$\frac{\partial^2\varphi_1(x, y, u)}{\partial x^2}$$

$$= \cosh ux[\sinh x\{2u \cos y \cos uy + (1 + u^2)\sin y \sin uy\} - 2u^2 \sinh ux]$$

$$+ \sinh ux[\cosh x\{2u \sin y \sin uy + (1 + u^2)\cos y \cos uy\} - 2u^2 \cosh ux]$$

$$> \cosh ux[\sinh x\{2u \cos y \cos uy + (1 + u^2)\sin y \sin uy\} - 2u^2 \sinh x]$$

$$+ \sinh ux[\cosh x\{2u \sin y \sin uy + (1 + u^2)\cos y \cos uy\} - 2u^2 \cosh x]$$

$$= \cosh ux \sinh x\{2u \cos y \cos uy + (1 + u^2)\sin y \sin uy - 2u^2\}$$

$$+ \sinh ux \cosh x\{2u \sin y \sin uy + (1 + u^2)\cos y \cos uy - 2u^2\}$$

$$= \sinh x \cosh ux\{2u \cos y \cos uy + (1 - u)^2 \sin y \sin uy$$

$$+ 2u \sin y \sin uy - 2u^2\} + \sinh ux \cosh x\{2u \sin y \sin uy$$

$$+ (1 - u)^2 \cos y \cos uy + 2u \cos y \cos uy - 2u^2\}$$

$$> (\sinh x \cosh ux + \sinh ux \cosh x)$$

$$(2u \sin y \sin uy + 2u \cos y \cos uy - 2u^2)$$

$$= 2u(\sinh x \cosh ux + \sinh ux \cosh x)(\cos(1-u)y - u)$$

$$> 2u(\sinh x \cosh ux + \sinh ux \cosh x)\left\{-\frac{3}{2\pi}(1-u)y + 1 - u\right\}$$

$$= 2u(1-u)(\sinh x \cosh ux + \sinh ux \cosh x)\left(1 - \frac{3y}{2\pi}\right)$$

$$> 0,$$

since $0 < u < 1$ and $\cos z > -\frac{3}{2\pi}z + 1$ for $0 < z < \pi/3$. Therefore, $\partial\varphi_1(x, y, u)/\partial x$ is also positive.

Next we will show (2). Note that if $\pi/3 > y > 0$, $\varphi_2(0, y, u) = \sin uy (\cos uy - \cos y) > 0$, and so we will assume that $x > 0$. Since $\varphi_2(x, y, 0) = 0$, it is sufficient to show that

$$\frac{\partial\varphi_2(x, y, u)}{\partial x} = \cosh x \sinh ux(\sin y \cos uy - u \sin uy \cos y)$$
$$+ \sinh x \cosh ux(u \sin y \cos uy - \sin uy \cos y)$$

is positive when $x > 0$, $\pi/3 > y > 0$, and $1 > u > 0$. The first term is clearly positive and so we will show that $u \sin y \cos uy - \sin uy \cos y$ is positive. But this can be easily verified since it is 0 when $y = 0$ and its derivative with respect to y is $(1-u^2)\sin y \sin uy$, which is positive.

Lemma 3.4. *There exists a positive number* ε *such that for a complex number* $a \neq 0$ *with* $|\operatorname{Im} a| < \frac{\pi}{3}$ *and* $|\operatorname{Re} a| < \pi$, *and a real number* u *with* $0 < u < \varepsilon$, *we have*

$$\left|\frac{\cosh a - \cosh ua}{\cosh a - 1}\right| > 1 - u.$$

Proof. We will show that

$$|\cosh a - \cosh ua|^2 - (1-u)^2 |\cosh a - 1|^2 > 0,$$

if $0 < u < \varepsilon$. Putting $a := x + y\sqrt{-1}$ with $|x| < \pi$ and $|y| < \frac{\pi}{3}$, the left hand side equals

$$
\begin{aligned}
&(\cosh x \cos y - \cosh ux \cos uy)^2 + (\sinh x \sin y - \sinh ux \sin uy)^2 \\
&- (1-u)^2 \{(\cosh x \cos y - 1)^2 + \sinh^2 x \sin^2 y\} \\
= &\{(2-u)\sinh x \sin y - \sinh ux \sin uy\} \\
&\times (u \sinh x \sin y - \sinh ux \sin uy) \\
&+ \{u - 1 + (2-u)\cosh x \cos y - \cosh ux \cos uy\} \\
&\times (1 - u + u \cosh x \cos y - \cosh ux \cos uy).
\end{aligned}
$$

Since it remains the same if we alter the signs of x or y, we may assume that $\pi > x \geq 0$ and $\pi/3 > y \geq 0\,((x, y) \neq (0, 0))$. Put

$$
\begin{aligned}
\alpha_1(x, y, u) &:= (2-u)\sinh x \sin y - \sinh ux \sin uy, \\
\alpha_2(x, y, u) &:= u \sinh x \sin y - \sinh ux \sin uy, \\
\beta_1(x, y, u) &:= u - 1 + (2-u)\cosh x \cos y - \cosh ux \cos uy, \\
\beta_2(x, y, u) &:= 1 - u + u \cosh x \cos y - \cosh ux \cos uy.
\end{aligned}
$$

We will show that $\alpha_1(x, y, u)$, $\alpha_2(x, y, u)$, $\beta_1(x, y, u)$, and $\beta_2(x, y, u)$ are all positive.

Since $0 < u < 1$, $\sinh x$ is increasing for any x, and $\sin y$ is increasing when $0 \leq y < \pi/3$, we have

$$
\begin{aligned}
\alpha_1(x, y, u) &> (2-u)\sinh ux \sin uy - \sinh ux \sin uy \\
&= (1-u)\sinh ux \sin uy > 0.
\end{aligned}
$$

By the Taylor expansion of $\alpha_2(x, y, u)$ around $x = 0$, we have

$$
\alpha_2(x, y, u) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} u(\sin y - u^{2n} \sin uy) x^{2n+1}.
$$

Since $\sin y$ is increasing for $0 < y < \pi/3$ and $0 < u < 1$, we have $\sin y - u^{2n} \sin uy > 0$. Therefore, $\alpha_2(x, y, u) > 0$.

The Taylor expansions of $\beta_1(x, y, u)$ and $\beta_2(x, y, u)$ around $u = 0$ gives

$$\beta_1(x, y, u) = 2(\cosh x \cos y - 1) - (\cosh x \cos y - 1)u + \frac{1}{2}(y^2 - x^2)u^2$$

$$+ \frac{1}{24}(-x^4 + 6x^2y^2 - y^4)u^4 + O(u^6),$$

$$\beta_2(x, y, u) = (\cosh x \cos y - 1)u + \frac{1}{2}(y^2 - x^2)u^2$$

$$+ \frac{1}{24}(-x^4 + 6x^2y^2 - y^4)u^4 + O(u^6),$$

and so

$$\begin{aligned}\beta(x, y, u) &:= \beta_1(x, y, u)\beta_2(x, y, u)\\ &= 2(\cosh x \cos y - 1)^2 u - (\cosh x \cos y - 1)\\ &\quad (\cosh x \cos y - 1 + x^2 - y^2)u^2\\ &\quad + \frac{1}{12}(4(x^4 - 3x^2y^2 + y^4)\\ &\quad - (x^4 - 6x^2y^2 + y^4)\cosh x \cos y)u^4 + O(u^6).\end{aligned}$$

Therefore, $\beta(x, y, u)$ is positive for small u if $\cosh x \cos y \neq 1$.

When $\cosh x \cos y = 1$, we have

$$\beta(x, y, u) = \frac{1}{4}(x^2 - y^2)^2 u^4 + O(u^6). \tag{3.1}$$

Since

$$\cosh x \cos x|_{x=0} = 1,$$

$$\left.\frac{d \cosh x \cos x}{dx}\right|_{x=0} = (\sinh x \cos x - \cosh x \sin x)|_{x=0} = 0,$$

and

$$\frac{d^2 \cosh x \cos x}{dx^2} = -2 \sinh x \sin x < 0 \quad \text{if } 0 < x < \pi,$$

$\cosh x \cos x < 1$ for $0 < x < \pi$, which means that $x \neq y$ when $\cosh x \cos y = 1$. So $\beta(x, y, u) > 0$ for small u since the coefficient of u^4 in (3.1) is positive.

Thus we have concluded that $\beta(x, y, u) > 0$ for small u.

Lemma 3.5. *For a positive integer m and a positive real number a, we have*

$$\int_1^\infty e^{-at} t^m dt = \frac{e^{-a}}{a} \sum_{k=0}^{m} \frac{m!}{a^k (m-k)!}.$$

Proof. Integration by parts gives

$$\begin{aligned}
&\int_1^\infty e^{-at} t^m dt \\
&= \left[-\frac{1}{a} e^{-at} t^m\right]_1^\infty + \frac{m}{a} \int_1^\infty e^{-at} t^{m-1} dt \\
&= \frac{1}{a} e^{-a} + \frac{m}{a} \int_1^\infty e^{-at} t^{m-1} dt \\
&= \frac{1}{a} e^{-a} + \frac{m}{a^2} e^{-a} + \frac{m(m-1)}{a^2} \int_1^\infty e^{-at} t^{m-2} dt \\
&= \frac{1}{a} e^{-a} + \frac{m}{a^2} e^{-a} + \cdots + \frac{m(m-1) \times \cdots \times (m-k+1)}{a^{k+1}} e^{-a} \\
&\quad + \cdots + \frac{m(m-1) \times \cdots \times 2}{a^m} e^{-a} + \frac{m!}{a^m} \int_1^\infty e^{-at} dt \\
&= \frac{1}{a} e^{-a} + \frac{m}{a^2} e^{-a} + \cdots + \frac{m(m-1) \times \cdots \times (m-k+1)}{a^{k+1}} e^{-a} \\
&\quad + \cdots + \frac{m(m-1) \times \cdots \times 2}{a^m} e^{-a} + \frac{m!}{a^{m+1}} e^{-a},
\end{aligned}$$

and the proof is complete.

Lemma 3.6. *For a complex number a with* $|2\cosh a - 2| < 1$, $|\operatorname{Im} a| < \pi/3$, *and* $a \neq 0$, *there exists a positive number* $\varepsilon' > 0$ *such that if* $0 < x < \varepsilon'$ *and* $0 < u < \varepsilon'$, *then*

$$1 > \left|\frac{\cosh a(1-x) - \cosh ua}{\cosh a - \cosh ua}\right| > 1 - \left|\frac{a \sinh a}{\cosh a - 1}\right| x.$$

Proof. Using the Taylor expansion with respect to x and u around $x = u = 0$, we have

$$\frac{\cosh a(1-x) - \cosh ua}{\cosh a - \cosh ua} = 1 - \frac{a \sinh a}{\cosh a - 1} x + \frac{a^2 \cosh a}{2(\cosh a - 1)} x^2 + R_a(u, x),$$

where $R_a(u, x)$ the terms with total degrees of u and x are greater than two.

From Lemma 3.7, we have

$$\left|\frac{\cosh a(1-x) - \cosh ua}{\cosh a - \cosh ua}\right| < 1,$$

if x and u are sufficiently small.

Moreover, we have

$$\left|\frac{\cosh a(1-x) - \cosh ua}{\cosh a - \cosh ua}\right| + \left|\frac{a \sinh a}{\cosh a - 1}\right| x$$

$$\geq \left|\frac{\cosh a(1-x) - \cosh ua}{\cosh a - \cosh ua} + \frac{a \sinh a}{\cosh a - 1} x\right|$$

$$= \left|1 + \frac{a^2 \cosh a}{2(\cosh a - 1)} x^2\right| + R_a(u, x).$$

From Lemma 3.8, we have $\operatorname{Re} \dfrac{a^2 \cosh a}{2(\cosh a - 1)} > 0$ if $|2\cosh a - 2| < 1$, and the other inequality follows.

Lemma 3.7. *For a complex number* $a \neq 0$ *with* $|\operatorname{Im} a| < \pi$, *we have*

$$\operatorname{Re} \frac{a \sinh a}{\cosh a - 1} > 0.$$

Proof. We put $a := x + \sqrt{-1}y$ with $|y| < \pi$. Then we have

$$\operatorname{Re}\frac{a\sinh a}{\cosh a - 1} = \frac{(\cosh x - \cos y)(x\sinh x + y\sin y)}{(\cos y\cosh x - 1)^2 + \sin^2 y\sinh^2 x} > 0,$$

if $|y| < \pi$ and $(x, y) \neq (0, 0)$.

Lemma 3.8. *For a complex number* $a \neq 0$ *with* $|2\cosh a - 2| < 1$, *and* $|\operatorname{Im} a| < \pi/3$, *we have*

$$\operatorname{Re}\frac{a^2\cosh a}{\cosh a - 1} > 0.$$

Proof. Putting $a := x + \sqrt{-1}y$ with $x, y \in \mathbb{R}$, we have

$$\operatorname{Re}\frac{a^2\cosh a}{\cosh a - 1} = \frac{f(x, y)}{(\cos y\cosh x - 1)^2 + \sin^2 y\sinh^2 x},$$

with

$$f(x, y) := (x^2 - y^2)(\cos^2 y + \sinh^2 x - \cos y\cosh x) + 2xy\sin y\sinh x.$$

We will show that $f(x, y) > 0$. We may assume that $x \geq 0$, $y \geq 0$ ($(x, y) \neq (0, 0)$) as before. Since

$$f(x, 0) = x^2(1 + \sinh^2 x - \cosh x) = x^2(\cosh^2 x - \cosh x) > 0,$$

when $x > 0$, we will assume that $y > 0$.

Since $f(x, y)$ is analytic, it is sufficient to prove that every nth derivative of f at $x = 0$ is positive, when n is even and zero when n is odd.

Since

$$\left.\frac{\partial^k (x^2 - y^2)}{\partial x^k}\right|_{x=0} = \begin{cases} -y^2 & \text{if } k = 0, \\ 2 & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left.\frac{\partial^k (\sinh^2 x - \cos y\cosh x)}{\partial x^k}\right|_{x=0} = \begin{cases} -\cos y & \text{if } k = 0, \\ 2^{k-1} - \cos y & \text{if } k \text{ is even and positive,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\frac{\partial^k (x \sinh x)}{\partial x^k} = \begin{cases} k & \text{if } k \text{ is even and positive,} \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\left.\frac{\partial^n f(x,\ y)}{\partial x^n}\right|_{x=0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left.\frac{\partial^{n-k}(x^2 - y^2)}{\partial x^{n-k}}\right|_{x=0} \times \left.\frac{\partial^k(\cos^2 y + \sinh^2 x - \cos y \cosh x)}{\partial x^k}\right|_{x=0}$$

$$+ 2y \sin y \left.\frac{\partial^n (x \sinh x)}{\partial x^n}\right|_{x=0}$$

$$= \begin{cases} -y^2(\cos^2 y - \cos y) & \text{if } n = 0, \\ -y^2(2 - \cos y) + 2(\cos^2 y - \cos y) + 4y \sin y & \text{if } n = 2, \\ -y^2(2^{n-1} - \cos y) + n(n-1)(2^{n-3} - \cos y) + 2ny \sin y & \text{if } n > 3 \text{ and even,} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $-y^2(\cos^2 y - \cos y) > 0$ since $y < \pi/3$. If n is even and $n \geq 4$, then since $y < \pi/3$, we have

$$\left.\frac{\partial^n f(x,\ y)}{\partial x^n}\right|_{x=0}$$

$$= 2^{n-4}\{n(n-1) - 8y^2\} + n(n-1)(2^{n-4} - \cos y) + y^2 \cos y + 2ny \sin y$$

$$> 2^{n-4}\left\{12 - 8\left(\frac{\pi}{3}\right)^2\right\} + n(n-1)(1 - \cos y) > 0.$$

To show that

$$\left.\frac{\partial^2 f(x,\ y)}{\partial x^2}\right|_{x=0} = -2y^2 + 2(\cos^2 y - \cos y) + 4y \sin y + y^2 \cos y$$

is positive, we will consider the function $g(y) := -2y^2 + 2(\cos^2 y - \cos y) + 4y \sin y$. Since

$$\frac{dg(y)}{dy} = 2(3 \sin y - 2y) + 4 \cos y(y - \sin y)$$

is easily verified to be positive, we have $g(y) > 0$. So $\partial^2 f(x, y)/\partial x^2 |_{x=0}$ is also positive.

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