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COUNTABLE ITERATED FUNCTION SYSTEMS

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Abstract

The compact sets invariant with respect to an Iterated (finite) Function System (IFS) have been studied very intensively during last decades by Mandelbrot, Dekking, Hutchinson, Barnsley and many others.

In this paper, it is considered the case of the countable system of contraction maps on a compact metric space X and the contraction map associated to it is defined on the complete metric space of all non-empty compact subset of X endowed with the Hausdorff metric.

We show that the attractor associated to the sequence of contraction maps $(\omega_n)_{n\geq 1}$ is approximated by the attractors of the partial systems $(\omega_n)_{n=1}^{k}$, $k \geq 1$.

1. Preliminary Facts

We shall present some notions and results used in the sequel (more complete and rigorous treatments may be found in [6], [2]).

Let (X, d) be a complete metric space and $\mathcal{K}(X)$ be the class of all compact non-empty subsets of X.

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If we define a function $\delta : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}_+$ by

$$\delta(A, B) = \max\{d(A, B), d(B, A)\}, \tag{1}$$

where

$$d(A, B) = \sup_{x \in A}\left(\inf_{y \in B} d(x, y)\right), \quad \text{for all } A, B \in \mathcal{K}(X),$$

we obtain a metric, namely the *Hausdorff metric*.

The set $\mathcal{K}(X)$ is a complete metric space with respect to this metric δ (see [5], [2]).

Theorem 1.1. *Let* (X, d) *be a complete metric space and* $(A_n)_{n \geq 1}$ *be a sequence of compact subsets of X.*

(a) *If* $A_n \subset A_{n+1}$, *for all* $n \in \mathbb{N}^*$, *and the set* $A := \bigcup_{n=1}^{\infty} A_n$ *is relatively compact, then*

$$\overline{A} = \overline{\bigcup_{n=1}^{\infty} A_n} = \lim_n A_n,$$

the limit is taken with respect to the Hausdorff metric and the bar means the closure;

(b) *If* $A_{n+1} \subset A_n$, $\forall n \in \mathbb{N}^*$, *then*

$$\lim_n A_n = \bigcap_{n=1}^{\infty} A_n.$$

Proof. (a) Let $\varepsilon > 0$. We will find an $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow d(\overline{A}, A_n) = \sup_{b \in \overline{A}}\left(\inf_{a \in A_n} d(b, a)\right) < \varepsilon.$$

Since $n \geq N \Rightarrow A_N \subset A_n \Rightarrow d(\overline{A}, A_n) \leq d(\overline{A}, A_N)$, it suffices to find $N \in \mathbb{N}$ such that

$$\forall\, b \in \overline{A}, \ \exists x \in A_N \ \text{ with } \ d(b, x) < \varepsilon.$$

Indeed, if

$$\forall n \in \mathbb{N}, \exists b_n \in \overline{A} \text{ such that } \forall x \in A_n, \text{ then we have } \mathrm{d}(b_n, x) \geq \varepsilon. \quad (2)$$

Since $\overline{A}$ is a compact set, there exists a convergent, hence Cauchy, subsequence $(b_{n_k})_k$ of $(b_n)_n$. So there exists $k_0 \in \mathbb{N}$ such that

$$l \geq k_0 \Rightarrow \mathrm{d}(b_{n_{k_0}}, b_{n_l}) < \frac{\varepsilon}{2},$$

and

$$b_{n_{k_0}} \in \overline{A} \Rightarrow \exists (a_p^{k_0})_p \subset A, \quad a_p^{k_0} \xrightarrow{p} b_{n_{k_0}}. \quad (3)$$

Let $p \in \mathbb{N}$ be such that

$$\mathrm{d}(a_p^{k_0}, b_{n_{k_0}}) < \frac{\varepsilon}{2}, \quad (4)$$

and let $l \geq k_0$ be such that $a_p^{k_0} \in A_{n_l}$.

Then, using relations (2), (3) and (4), we obtain

$$\varepsilon \leq \mathrm{d}(b_{n_l}, a_p^{k_0}) \leq \mathrm{d}(b_{n_l}, b_{n_{k_0}}) + \mathrm{d}(b_{n_{k_0}}, a_p^{k_0}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This is a contradiction.

Because $A_n \subset \overline{A} \Rightarrow \mathrm{d}(A_n, \overline{A}) = 0$, we have $\delta(A_n, \overline{A}) < \varepsilon, \forall n \geq N$.

(b) Since $\bigcap_{k=1}^{\infty} A_k \subset A_n, \forall n$, it follows that $\mathrm{d}\left(\bigcap_{k=1}^{\infty} A_k, A_n\right) = 0, \forall n$.

We shall show that, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$n \geq N \Rightarrow \mathrm{d}\left(A_n, \bigcap_{k=1}^{\infty} A_k\right) < \varepsilon,$$

namely

$$n \geq N, x \in A_n \Rightarrow \exists y \in \bigcap_{k=1}^{\infty} A_k \text{ with } \mathrm{d}(x, y) < \varepsilon.$$

Indeed, assume that there exists $\varepsilon > 0$ such that

$$\forall n \in \mathbb{N}, \exists x_n \in A_n \text{ so that } \forall y \in \bigcap_{k=1}^{\infty} A_k \Rightarrow \mathrm{d}(x_n, y) \geq \varepsilon. \tag{5}$$

The sequence $(x_n)_n$ contains a subsequence $(x_{n_p})_p$ which is convergent to x. Since $(A_n)_n$ is a decreasing sequence of sets, we deduce that $\bigcap_{k=1}^{\infty} A_k = \bigcap_{p=1}^{\infty} A_{n_p}$. It is simple to prove that $x \in \bigcap_{p=1}^{\infty} A_{n_p}$.

Thus, it follows that $x \in \bigcap_{k=1}^{\infty} A_k$.

Next, taking $y = x$, we deduce that $\mathrm{d}(x_n, y) < \varepsilon$, for sufficiently large n. This contradicts (5).

A *contraction* is a map $\omega : X \to X$ for which there exist $0 \leq c < 1$ such that

$$\mathrm{d}(\omega(x), \omega(y)) \leq c\mathrm{d}(x, y) \tag{6}$$

for all $x, y \in X$.

The infimum of all c for which this inequality holds is called the *contraction ratio*.

A set of contractions $(\omega_n)_{n=1}^{k}$, $k \geq 1$, is called an *iterated function system* (IFS), according to M. Barnsley [2]. Such a system of maps induces a map $\mathcal{S}_k : \mathcal{K}(X) \to \mathcal{K}(X)$,

$$\mathcal{S}_k(E) = \bigcup_{n=1}^{k} \omega_n(E) \tag{7}$$

which is a contraction on $\mathcal{K}(X)$ with contraction ratio $r \leq \max_{1 \leq n \leq k} r_n$, r_n being the contraction ratio of ω_n, $n = 1, \ldots, k$.

According to Banach contraction principle, there is a unique set $A_k \in \mathcal{K}(X)$ which is invariant with respect to $\mathcal{S}_k$, that is,

$$A_k = \mathcal{S}_k(A_k) = \bigcup_{n=1}^{k} \omega_n(A_k). \tag{8}$$

We say the set $A_k \in \mathcal{K}(X)$ is the *attractor* of IFS $(\omega_n)_{n=1}^{k}$.

Furthermore, for all $E \in \mathcal{K}(X)$, we have

$$\mathcal{S}_k^p(E) \overset{p}{\to} A_k, \tag{9}$$

where $\mathcal{S}_k^1 = \mathcal{S}_k$, $\mathcal{S}_k^{p+1} = \mathcal{S}_k \circ \mathcal{S}_k^p$ for $p \geq 1$, the convergence being taken with respect to the Hausdorff metric.

For $i_1, \ldots, i_p \in \{1, \ldots, k\}$, $p \geq 1$, denote $\omega_{i_1 \ldots i_p} := \omega_{i_1} \circ \omega_{i_2} \circ \ldots \circ \omega_{i_p}$. In this way one obtains a contraction on X with the contraction ratio $r_{i_1 \ldots i_p} \leq r_{i_1} r_{i_2} \ldots r_{i_p}$.

Using induction, one can show that, for every $E \in \mathcal{K}(X)$ and $p \in \mathbb{N}^*$,

$$\mathcal{S}_k^p(E) = \bigcup_{i_1, \ldots, i_p = 1}^{k} \omega_{i_1 \ldots i_p}(E). \tag{10}$$

We set $e_{i_1 \ldots i_p}$ the unique fixed point of $\omega_{i_1 \ldots i_p}$. Then (according to [6] 3.1(3)) A_k is the closure of set of these fixed points:

$$A_k = \overline{\{e_{i_1 \ldots i_p} : p \in \mathbb{N}^*, 1 \leq i_j \leq k\}}. \tag{11}$$

2. Countable Iterated Function Systems

In this section, (X, d) is supposed to be a compact metric space.

Definition 2.1. A sequence of contractions $(\omega_n)_{n \geq 1}$ whose contraction ratios are, respectively, r_n, $r_n > 0$, such that $\sup_n r_n < 1$ is called a *countable iterated function system*, for simplicity CIFS.

Let $(\omega_n)_{n \geq 1}$ be a CIFS.

We define the set function $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$\mathcal{S}(E) = \overline{\bigcup_{n=1}^{\infty} \omega_n(E)}, \tag{12}$$

where the bar means the closure of the corresponding set.

Lemma 2.1. *If $(E_n)_n$, $(F_n)_n$ are two sequences of sets in $\mathcal{K}(X)$, then*

$$\delta\left(\overline{\bigcup_{n=1}^{\infty} E_n}, \overline{\bigcup_{n=1}^{\infty} F_n}\right) \leq \sup_n \delta(E_n, F_n).$$

Proof. Obvious.

Theorem 2.1. *The set function $\mathcal{S}$ which is defined in* (12) *is a contraction map on $(\mathcal{K}(X), \delta)$ with contraction ratio $r \leq \sup_n r_n$.*

Proof. Let $A, B \in \mathcal{K}(X)$.

Taking in Lemma 2.1, $E_n = \omega_n(A)$, $F_n = \omega_n(B)$, $n \in \mathbb{N}^*$, we deduce

$$\delta(\mathcal{S}(A), \mathcal{S}(B)) \leq \sup_n \delta(\omega_n(A), \omega_n(B)) \leq \left(\sup_n r_n\right) \cdot \delta(A, B),$$

where, in the last inequality, we take into account the fact that, for $n \in \mathbb{N}^*$, we have

$$\delta(\omega_n(A), \omega_n(B)) \leq r_n \delta(A, B).$$

Theorem 2.2. *There exists a unique set $A \in \mathcal{K}(X)$ which is invariant with respect to $(\omega_n)_{n \geq 1}$, that is,*

$$A = \mathcal{S}(A) = \overline{\bigcup_{n=1}^{\infty} \omega_n(A)}.$$

Furthermore, if $E \in \mathcal{K}(X)$, *then*

$$\mathcal{S}^p(E) \xrightarrow{p} A$$

in the Hausdorff metric, where $\mathcal{S}^1 = \mathcal{S}$, $\mathcal{S}^{p+1} = \mathcal{S} \circ \mathcal{S}^p$, $p \geq 1$.

Proof. The assertions come from Banach contraction principle applied to the contraction $\mathcal{S}$.

Definition 2.2. The non-empty compact invariant set A is called the *attractor* of the countable iterated function system $(\omega_n)_{n \geq 1}$.

According to the Section 1, we denote by A_k and, respectively, by $\mathcal{S}_k$ the attractor and the contraction associated to the partial IFS $(\omega_n)_{n=1}^{k}$, for $k \geq 1$.

The following assertion is obvious, using (11):

Proposition 2.1. *For all* $k \geq 1$, *we have* $A_k \subset A_{k+1}$.

Lemma 2.2. *If* $(E_i)_{i \in \Im}$ *is a family of subsets of a topological space, then*

$$\overline{\bigcup_{i \in \Im} \overline{E_i}} = \overline{\bigcup_{i \in \Im} E_i}.$$

Proof. $E_i \subset \overline{E}_i$, $\forall i \in \Im \Rightarrow \overline{\bigcup_{i \in \Im} E_i} \subset \overline{\bigcup_{i \in \Im} \overline{E_i}}$.

On the other hand, taking into account that $\bigcup_{i \in \Im} \overline{E_i} \subset \overline{\bigcup_{i \in \Im} E_i}$, we deduce the inclusion

$$\overline{\bigcup_{i \in \Im} \overline{E_i}} \subset \overline{\bigcup_{i \in \Im} E_i}.$$

Theorem 2.3. *The set* $A = \overline{\bigcup_{k=1}^{\infty} A_k} \in \mathcal{K}(X)$ *is the attractor of CIFS* $(\omega_n)_{n \geq 1}$.

Proof. Using the notations above, we have

$$\begin{aligned}\mathcal{S}(A_k) &= \overline{\bigcup_{n=1}^{\infty} \omega_n(A_k)} \\ &= \overline{\bigcup_{n=1}^{k} \omega_n(A_k)} \cup \overline{\bigcup_{n=k+1}^{\infty} \omega_n(A_k)} \\ &= A_k \cup \overline{\bigcup_{n=k+1}^{\infty} \omega_n(A_k)}, \qquad k = 1, 2, \ldots. \end{aligned} \tag{13}$$

By Lemma 2.2, it follows that

$$\overline{\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \omega_n(A_k)} = \overline{\bigcup_{k=1}^{\infty} \overline{\bigcup_{n=1}^{\infty} \omega_n(A_k)}}. \tag{14}$$

Then, using (13) and (14),

$$\begin{aligned}\overline{\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \omega_n(A_k)} &= \overline{\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \omega_n(A_k)} \\ &= \overline{\bigcup_{k=1}^{\infty} \overline{\bigcup_{n=1}^{\infty} \omega_n(A_k)}} = \overline{\bigcup_{k=1}^{\infty} \mathcal{S}(A_k)} \\ &= \overline{\bigcup_{k=1}^{\infty} \Big(A_k \cup \overline{\bigcup_{n=k+1}^{\infty} \omega_n(A_k)}\Big)} \\ &= \overline{\bigcup_{k=1}^{\infty} A_k} \cup \overline{\bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k)}. \end{aligned} \tag{15}$$

Let $x \in \bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k)$. Then $\exists k \geq 1$, $\exists n \geq k+1$ such that $x \in \omega_n(A_k)$.

By applying Proposition 2.1, we deduce that $A_k \subset A_n$ and hence $\omega_n(A_k) \subset \omega_n(A_n) \subset A_n$. So $x \in A_n$, and hence $x \in \bigcup_{k=1}^{\infty} A_k$.

It follows that

$$\bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k) \subset \bigcup_{k=1}^{\infty} A_k \Rightarrow \overline{\bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k)} \subset \overline{\bigcup_{k=1}^{\infty} A_k}. \tag{16}$$

Using (15) and (16), we obtain

$$\overline{\bigcup_{k=1}^{\infty} A_k} = \overline{\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \omega_n(A_k)}. \tag{17}$$

So, we have

$$\mathcal{S}\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) = \overline{\bigcup_{n=1}^{\infty} \omega_n\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right)} \subset \overline{\bigcup_{n=1}^{\infty} \overline{\omega_n\left(\bigcup_{k=1}^{\infty} A_k\right)}} = \overline{\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \omega_n(A_k)}$$

$$= \overline{\bigcup_{n=1}^{\infty} \omega_n\left(\bigcup_{k=1}^{\infty} A_k\right)} \subset \overline{\bigcup_{n=1}^{\infty} \omega_n\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right)} = \mathcal{S}\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right),$$

where, in first inclusion, we used the fact that ω_n is continuous, for each $n \geq 1$.

It follows that

$$\mathcal{S}\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) = \overline{\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \omega_n(A_k)}. \tag{18}$$

By (17) and (18), we deduce $\mathcal{S}\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) = \overline{\bigcup_{k=1}^{\infty} A_k}$.

Corollary 2.1. *The attractor of CIFS* $(\omega_n)_{n\geq 1}$ *is the closure of the set of fixed points of the contractions* $\omega_{i_1 \ldots i_p}$, $p \in \mathbb{N}^*$, $i_j \in \mathbb{N}^*$.

Proof. Since

$$\{e_{i_1 \dots i_p} : p \in \mathbb{N}^*, i_j \in \mathbb{N}^*\} = \bigcup_{k=1}^{\infty} \{e_{i_1 \dots i_p} : p \in \mathbb{N}^*, 1 \le i_j \le k\},$$

using (14) and (11), we obtain

$$\overline{\{e_{i_1 \dots i_p} : p \in \mathbb{N}^*, i_j \in \mathbb{N}^*\}} = \overline{\bigcup_{k=1}^{\infty} \overline{\{e_{i_1 \dots i_p} : p \in \mathbb{N}^*, 1 \le i_j \le k\}}} = \overline{\bigcup_{k=1}^{\infty} A_k}.$$

So, by Theorem 2.3, it follows that

$$A = \overline{\{e_{i_1 \dots i_p} : p \in \mathbb{N}^*, i_j \in \mathbb{N}^*\}}.$$

Corollary 2.2. *The attractor of CIFS* $(\omega_n)_{n \ge 1}$ *is*

$$A = \overline{\bigcup_{k=1}^{\infty} A_k} = \lim_k A_k,$$

the limit being taken in $(\mathcal{K}(X), \delta)$.

Hence, the attractor of CIFS $(\omega_n)_{n \ge 1}$ *is approximated by the attractors of partial IFS* $(\omega_n)_{n=1}^{k}$, $k \ge 1$.

Proof. The assertion follows from Theorems 2.3 and 1.1.

Observation. By applying Theorem 1.1 and using the fact that $\mathcal{S}$ is a contraction on $\mathcal{K}(X)$, we have

$$\mathcal{S}(\lim_k A_k) = \lim_k \mathcal{S}(A_k),$$

namely

$$\mathcal{S}\left(\overline{\bigcup_{k=1}^{\infty} A_k}\right) = \lim_k \overline{\bigcup_{n=1}^{\infty} \omega_n(A_k)} = \overline{\bigcup_{k=1}^{\infty} \overline{\bigcup_{n=1}^{\infty} \omega_n(A_k)}}$$

$$= \overline{\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \omega_n(A_k)} = \overline{\bigcup_{k=1}^{\infty} \left[\bigcup_{n=1}^{k} \omega_n(A_k) \cup \bigcup_{n=k+1}^{\infty} \omega_n(A_k)\right]}$$

$$= \overline{\bigcup_{k=1}^{\infty} A_k} \cup \overline{\bigcup_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} \omega_n(A_k)} = \overline{\bigcup_{k=1}^{\infty} A_k},$$

using Lemma 2.2 and (16).

This is an alternative of the proof for the assertion of Theorem 2.3.

Theorem 2.4. *If E is a non-empty compact subset of X, then, for all* $p \in \mathbb{N}^*$,

$$\mathcal{S}^p(E) = \lim_k \mathcal{S}_k^p(E). \tag{19}$$

In particular, if A is the attractor of CIFS $(\omega_n)_{n \geq 1}$, *then*

$$A = \mathcal{S}(A) = \lim_k \mathcal{S}_k(A) = \lim_{k \to \infty} \lim_{p \to \infty} \mathcal{S}_k^p(E)$$

and, also,

$$A = \lim_{p \to \infty} \lim_{k \to \infty} \mathcal{S}_k^p(E),$$

the limits being taken in the space $(\mathcal{K}(X), \delta)$.

So, we have the following diagram:

$$\begin{array}{ccc} \mathcal{S}_k^p(E) & \xrightarrow{p} & A_k \\ \downarrow k & & \downarrow k \\ \mathcal{S}^p(E) & \xrightarrow{p} & A \end{array}$$

Proof. Using (10), for all $p \in \mathbb{N}^*$ and $k \geq 1$, we have

$$\mathcal{S}_k^p(E) = \bigcup_{i_1, \ldots, i_p = 1}^{k} \omega_{i_1 \ldots i_p}(E) \tag{20}$$

(using the notations from part 1).

We shall show, using induction with respect to p, that

$$\mathcal{S}^p(E) = \overline{\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)}.$$

Thus, assume that the equality above holds for $p \geq 1$.

Then, using Lemma 2.2 and the continuity of ω_i,

$$\mathcal{S}^{p+1}(E) = \mathcal{S}\left(\overline{\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)}\right) = \overline{\bigcup_{i=1}^{\infty} \omega_i\left(\overline{\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)}\right)}$$

$$\subset \overline{\bigcup_{i=1}^{\infty} \overline{\omega_i\left(\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)\right)}} = \overline{\bigcup_{i=1}^{\infty} \omega_i\left(\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)\right)}$$

$$\subset \overline{\bigcup_{i=1}^{\infty} \omega_i\left(\overline{\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)}\right)} = \mathcal{S}^{p+1}(E).$$

So,

$$\mathcal{S}^{p+1}(E) = \overline{\bigcup_{i=1}^{\infty} \omega_i\left(\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)\right)}$$

$$= \overline{\bigcup_{i_1, \ldots, i_p, i_{p+1} = 1}^{\infty} \omega_{i_1 \ldots i_{p+1}}(E)}.$$

Let $p \in \mathbb{N}^*$. We apply Theorem 1.1 taking into account the fact that the sequence of compact sets $\left(\bigcup_{i_1, \ldots, i_p = 1}^{k} \omega_{i_1 \ldots i_p}(E)\right)_k$ is, obviously increasing.

We have, using (20),

$$\mathcal{S}^p(E) = \overline{\bigcup_{i_1, \ldots, i_p = 1}^{\infty} \omega_{i_1 \ldots i_p}(E)} = \overline{\bigcup_{k=1}^{\infty} \bigcup_{i_1, \ldots, i_p = 1}^{k} \omega_{i_1 \ldots i_p}(E)}$$

$$= \lim_k \bigcup_{i_1, \ldots, i_p = 1}^{k} \omega_{i_1 \ldots i_p}(E) = \lim_k \mathcal{S}_k^p(E),$$

hence the equality (19).

The equality $\mathcal{S}(A) = \lim_k \mathcal{S}_k(A)$ follows from (19) for $p = 1$, $E = A$.

Using (9) and Corollary 2.2, we obtain

$$A = \lim_k A_k = \lim_k \lim_p \mathcal{S}_k^p(E).$$

Remark. Let (X, d) be a compact metric space and $(\omega_i)_{i \in I}$ be a set of contraction maps with the ratios r_i, $\sup r_i < 1$, I being at most a countable set.

Repeating the notation above $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$,

$$\mathcal{S}(B) = \overline{\bigcup_{i \in I} \omega_i(B)}, \quad \forall B \in \mathcal{K}(X),$$

we obtain the definitions of the contraction on $(\mathcal{K}(X), \delta)$ and of the attractor associated with the system $(\omega_i)_{i \in I}$ in the case when I is finite (the case of IFS) and (X, d) is a complete metric space.

Moreover, each IFS $(\omega_n)_{n=1}^k$ can be considered like a CIFS according to:

Proposition 2.2. *The set* A_k, $k \in \mathbb{N}^*$, *is the attractor of IFS* $(\omega_n)_{n=1}^k$ *if and only if* A_k *is the attractor of CIFS* $(\omega_n)_{n \geq 1}$, *where* $\omega_n \equiv e_1$ *(the fixed point of* ω_1*), for all* $n > k$.

Proof. The result comes from equalities:

$$A_k = \overline{\bigcup_{n=1}^{\infty} \omega_n(A_k)} = \overline{\bigcup_{n=1}^{k} \omega_n(A_k)} \cup \overline{\bigcup_{n=k+1}^{\infty} \{e_1\}} = \bigcup_{n=1}^{k} \omega_n(A_k)$$

because, of course, $e_1 \in A_k$.

3. Some Examples of the Attractors Associated to a CIFS

First, we shall present two examples which generalize some well-known fractals, like the Cantor set and the Sierpinski triangle.

Example 1 (The attractors of Cantor-infinite type)**.** We shall work in the compact metric space $X = [0, 1]$ with the Euclidean metric.

Let $q \in \left(0, \frac{1}{2}\right]$.

We define, for each $n \in \mathbb{N}^*$, the sequence of contractions $\omega_n : [0, 1] \to [0, 1]$,

$$\omega_n(x) = q^n x + \alpha_n,$$

where $\alpha_1 = 0$, $\alpha_n = q^{n-1} + \left(\frac{1 - 2q}{2 - 3q}\right)^{n-1} + \alpha_{n-1}$, $n \geq 2$.

We put $E_0 = [0, 1]$, $E_1 = \overline{\bigcup_{n=1}^{\infty} \omega_n([0, 1])} = \overline{\bigcup_{n=1}^{\infty} [\alpha_n, \alpha_n + q^n]}$.

We observe that the distance of two consecutive intervals of E_1 has the form

$$\beta^n = \alpha_{n+1} - (\alpha_n + q^n),\ \beta \geq 0, \text{ so } \alpha_n = q^{n-1} + \beta^{n-1} + \alpha_{n-1}, \text{ for } n \geq 2,$$

β being obtained such that the sum of lengths of the intervals of E_1 and the sum of distances of these intervals is worth 1 (the condition that the intervals of E_1 should have disjoint interiors is not necessary).

Thus

$$\sum_{n=1}^{\infty} q^n + \sum_{n=1}^{\infty} \beta^n = 1 \Leftrightarrow \beta = \frac{1 - 2q}{2 - 3q} \geq 0.$$

In this way we justify the form of the contractions ω_n, $n \geq 1$, and that the fact that such maps are well defined.

We denote, for $p \geq 1$,

$$E_p = \overline{\bigcup_{n=1}^{\infty} \omega_n(E_{p-1})} = \mathcal{S}(E_{p-1}).$$

Hence $E_p = \mathcal{S}^p(E_0)$, $p = 1, 2, \ldots$.

One can check, by using induction, that the sequence of compact sets $(E_p)_p$ is decreasing.

Now, we shall show that the attractor of CIFS $(\omega_n)_{n\geq 1}$ is

$$A = \bigcap_{p=0}^{\infty} E_p. \tag{21}$$

Indeed, using Theorems 1.1 and 2.2, we have

$$A = \lim_{p\to\infty} \mathcal{S}^p(E_0) = \lim_{p\to\infty} E_p = \bigcap_{p=1}^{\infty} E_p.$$

We denote, for each $k \in \mathbb{N}^*$,

$$E_0^k = E_0, \quad E_p^k = \bigcup_{n=1}^{\infty} \omega_n(E_{p-1}), \; \forall p \in \mathbb{N}^*.$$

It follows, obviously, that $E_p^k \subset E_p^{k+1}$, and $E_{p+1}^k \subset E_p^k$, for all k, p, and, also,

$$\mathcal{S}_k^p(E_0) = \bigcup_{i_1, \ldots, i_p = 1}^{k} \omega_{i_1 \ldots i_p}(E_0) = E_p^k.$$

By applying Theorem 2.4, we deduce

$$A = \lim_p \lim_k \mathcal{S}_k^p(E_0) = \lim_p \lim_k E_p^k = \lim_p \overline{\bigcup_{k=1}^{\infty} E_p^k} = \bigcap_{p=1}^{\infty} \overline{\bigcup_{k=1}^{\infty} E_p^k} \tag{22}$$

and, also,

$$A = \lim_k \lim_p E_p^k = \overline{\bigcup_{k=1}^{\infty} \bigcap_{p=1}^{\infty} E_p^k} = \overline{\bigcup_{k=1}^{\infty} A_k}, \tag{23}$$

where $A_k = \bigcap_{p=1}^{\infty} E_p^k$ is the attractor of partial IFS $(\omega_n)_{n=1}^k$, $k \geq 1$. Moreover, A_k is a set of Cantor-type.

The formulas (21), (22), (23) give some characterizations of the attractor of CIFS $(\omega_n)_{n\geq 1}$.

One can consider the particular case $q = \frac{1}{3}$. We obtain

$$\alpha_1 = 0,\ \alpha_n = \left(\frac{1}{3}\right)^{n-1} + \left(\frac{1}{3}\right)^{n-1} + \alpha_{n-1} = \cdots = 1 - \left(\frac{1}{3}\right)^{n-1},\ n \geq 1$$

and they provide the contractions

$$\omega_n(x) = \frac{1}{3^n} x + 1 - \frac{1}{3^{n-1}},\ \ n \geq 1.$$

Example 2 (The attractors of Sierpinski-infinite type)**.** We denote

$$X = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}$$

the plane surface of the closed triangle having its vertices in the points (0, 0), (0, 1), (1, 0).

Let $p \in \mathbb{N}$, $p \geq 2$, and consider the contractions $\omega_{ij} : X \to X$

$$\omega_{ij}(x, y) = \left(\frac{1}{p^i} x + (j-1)\frac{1}{p^i},\ \frac{1}{p^i} y + \left(\frac{p^i - 1}{p-1} - j\right)\frac{1}{p^i}\right)$$

for all $i = 1, 2, \ldots,\ j = 1, 2, \ldots, \frac{p^i - 1}{p - 1}$.

The attractors associated to the partial IFS for $p = 2$ $\{\omega_{ij} : i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, 2^i - 1\}$ in first cases $k = 3$ (11 contractions), $k = 5$

(57 contractions), $k = 7$ (247 contractions) are, respectively, represented in the figures (Fig. 1, Fig. 2, Fig. 3).

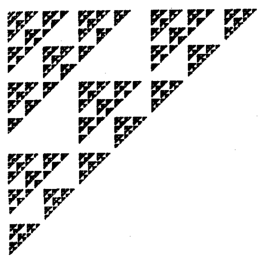

Fig. 1

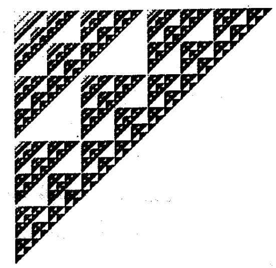

Fig. 2

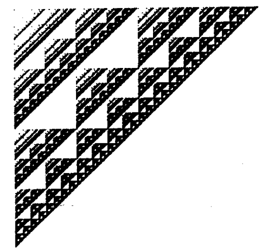

Fig. 3

We may consider, here, a more general case:

Let $q \in \left(0, \frac{1}{2}\right]$ and the contractions

$$\omega_{ij} = (q^i x + (j-1)q^i, q^i y + (k_i - j)q^i)$$

for all $i = 1, 2, \ldots, j = 1, 2, \ldots, k_i$, where $k_i = \left[\frac{1-q^i}{q^{i-1}(1-q)}\right]$, and the brackets symbolize the integer part.

Example 3 (The attractors of von-Koch-infinite type)**.** Let $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. We consider the contractions which are defined as follows: for every $n \in \mathbb{N}^*$, there exist uniquely $p \in \{0, 1, \ldots\}$, $k \in \{1, 2, 3, 4\}$ such that $n = 4p + k$. Then

$$\omega_n(x, y) := \begin{cases} \frac{1}{2^{p+1}}\left(\frac{1}{3}x + 2^{p+1} - 2, \frac{1}{3}y\right), & \text{if } k = 1; \\ \frac{1}{2^{p+1}}\left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + 2^{p+1} - \frac{5}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y\right), & \text{if } k = 2; \\ \frac{1}{2^{p+1}}\left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + 2^{p+1} - \frac{3}{2}, -\frac{\sqrt{3}}{6}x + \frac{1}{6}y + \frac{\sqrt{3}}{6}\right), & \text{if } k = 3; \\ \frac{1}{2^{p+1}}\left(\frac{1}{3}x + 2^{p+1} - \frac{4}{3}, \frac{1}{3}y\right), & \text{if } k = 4. \end{cases}$$

(see Fig. 4)

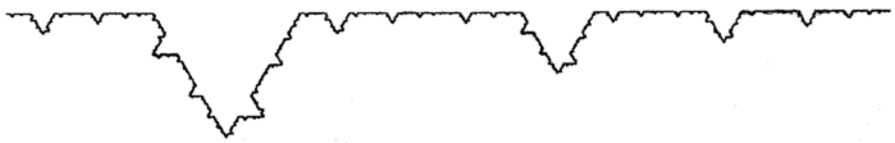

Fig. 4

Example 4 (The attractors *circles-infinite*). Let $X = B\left[0, \frac{3}{2}\right] = \left\{x \in \mathbb{R}^2 : \| x \| \leq \frac{3}{2}\right\}$ the closed ball centered at 0 of radius $\frac{3}{2}$ ($\| \cdot \|$ represent the Euclidean norm on $\mathbb{R}^2$).

Let $C > 0$ be a constant and we consider the points

$$P_n = \left(\cos \frac{2n\pi}{n + C}, \sin \frac{2n\pi}{n + C}\right), n = 0, 1, \ldots$$

on the unitary circle, $P_0 = (1, 0)$.

Also, one can consider a sequence $(r_n)_{n \geq 0} \subset \left(0, \frac{1}{3}\right]$ and the maps $\omega_n : X \to X$, defined by

$$\omega_n(x, y) = \left(r_n x + \cos \frac{2n\pi}{n + C}, r_n y + \sin \frac{2n\pi}{n + C}\right), n = 0, 1, \ldots.$$

It is not difficult to observe that the definition is correct and $(\omega_n)_{n \geq 0}$ is a sequence of contractions on the compact metric space X.

In Figure 5, we have presented an example for $n = 0, 1, 2, \ldots, 600$, $C = 30$, $r_n = \frac{1}{\sqrt{2n + 9}}$, $n \geq 0$.

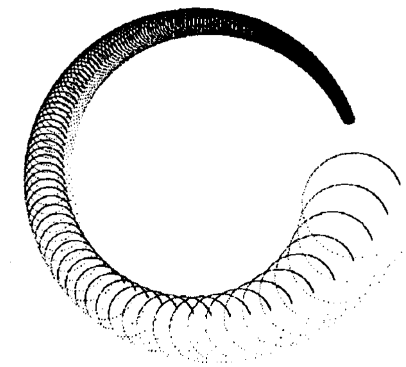

Fig. 5

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