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A PROOF THAT CERTAIN CLASSES OF INVARIANT PAINLEVÉ SOLUTIONS OF NLPDEs ARE TRAVELING WAVES AND APPLICATIONS TO FORCED INTEGRABLE AND NONINTEGRABLE SYSTEMS

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Abstract

We consider a technique for deriving exact analytic coherent structure (pulse/front/domain wall) solutions of general NLPDEs via the use of truncated invariant Painlevé expansions, and prove that these solutions satisfy the corresponding traveling wave reduced ODE (as conjectured by Powell et al. [57]). Thus, they not only provide 'partial integrability' in Painlevé's original sense but also a parameterization of the homoclinic or heteroclinic structures of the traveling wave reduced ODEs. Coupling this to Melnikov theory, we then consider the breakdown to chaos of such analytic coherent structure solutions of various long-wave and reaction-diffusion equations under forcing. We also demonstrate that similar treatments are possible for integrable systems (where the soliton/kink solutions represent the homoclinic/heteroclinic structures of the reduced ODEs) using the well-studied forced sine-Gordon equation as the main example. A method of treating the dynamics of the system prior to the onset of chaos by the use of

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intrinsic harmonic balance, multiscale or direct soliton perturbation theory is briefly discussed. It is conceivable that resummation of such perturbation series via the use of Pade approximants or other techniques may enable one to analytically follow the homoclinic or heteroclinic tangling beyond the first transversal intersection of the stable and unstable manifolds and into the chaotic regime.

1. Introduction

There has been considerable interest in coherent structure solutions of nonintegrable nonlinear partial differential equations (NLPDEs) [1, 2, 6, 20-22, 31-34, 44, 50, 51,] since these provide an organizing structure to the space of solutions. In a very rough sense, this is somewhat analogous to the way in which families of soliton solution act as basic building blocks for the solution space of integrable equations. Recent work, primarily in the context of generalized Ginzburg-Landau amplitude equations in pattern-forming systems, has included the existence of pulse (solitary wave), front (shock) and domain wall coherent structures using center manifold techniques [28, 42], as well as investigations of periodic and quasi-periodic solutions [3, 24-26, 29]. Another, more physics-oriented, approach was developed by van Saarloos [65, 66] to investigate linear and nonlinear marginal stability of fronts. This approach has been comprehensively reviewed by van Saarloos and Hohenberg [67] in the context of generalized Ginzburg-Landau equations. Using the idea that spatio-temporal coherent structure solutions of NLPDEs, whether periodic, quasi-periodic, or chaotic must obey the underlying singularity structure, Conte and co-workers [19, 20, 46, 50, 51] have used methods related to the Painlevé test for integrability [17, 62] and its modifications [18] to derive families of solutions of the complex cubic and quintic Ginzburg-Landau equation. Also, using phase-plane techniques on the ordinary differential equation which must be satisfied by any traveling wave solution to the real Ginzburg-Landau equation, Powell et al. [57] have re-derived and significantly elucidated several of van Saarloos' results [65, 66] in a completely different manner. In addition, they use simple analytic solutions of the PDE obtained using truncated Painlevé expansions [13], together with ideas from phase-plane analysis, as well as absolute versus convective instability of waves to (a) show that front/pulse solution of the PDE must satisfy the traveling wave reduced ODE asymptotically, and (b) derive conditions for the accessibility of the solutions from compact support initial conditions.

To date, the approaches to the analysis of coherent structures may broadly be classified into three groups. First, there is the phase-plane/center manifold qualitative analysis of traveling wave reduced ODEs to prove existence and stability of coherent structures [58]. The second approach consists of actual construction of coherent structures via numerical simulation of the traveling wave reduced ODEs. The third approach comprises containment arguments wherein, starting from the correct boundary condition at one end of the interval, one shows that at the other end the solution asymptotes to a constant value and thus corresponds to a coherent structure, rather to shooting off to $\pm\infty$ — these may often involve tricky analysis.

In this paper, we shall primarily consider two things. The first is a proof that certain, constant celerity, classes of solutions (whether pulses (solitary waves) or fronts (shocks or kinks)) obtained using truncated Painlevé expansions (or regular Painlevé expansions) are indeed traveling waves and thus satisfy the corresponding reduced ODE. The motivation for looking for this comes from the conjecture in [57], as well as its applicability in various areas of nonlinear science.

The second main focus in this paper will be to use the above result to develop a new, direct method of treating the behavior of coherent structure solutions of both nonintegrable and integrable NLPDEs under stresses such as forcing. The primary advantage of the method developed here will be its simplicity. However, a key assumption that will go into this approach is the assumption that a coherent structure which has the functional form of a traveling wave would remain a traveling wave under stress, and also not be disrupted by phenomena such as resonances [69] so that one could deal directly with the forced traveling-wave-reduced ODE system. In addition, one needs to be in regimes where the dynamics is dominated by interactions among stable coherent structures [14]. These are motivated by a large body of accumulated evidence. However, the results still need to be validated 'a posteriori'. Note that this approach is clearly not expected to capture the full dynamical behavior of the PDE in all regimes — that would require numerical or perturbative treatment of the PDE. However, it will be seen to treat the evolution of coherent structures accurately and also capture the dynamics in regimes dominated by their interactions.

The approach here is, in a sense, an attempt to connect the first two approaches to coherent structures mentioned above by providing explicit expressions for non-trivial coherent structures, which are indeed traveling waves. These are then considered in the context of modeling forced nonintegrable and integrable systems.

Note also that the modeling aspect considered here is the actual transition to chaos, since this provides a non-trivial and direct application of the coherent structure solutions. However, the bifurcations and dynamics in the pre-chaotic regime may be treated by diverse techniques, including possible extensions into the chaotic regime. This is considered in Section 6 and will be the subject of future work.

The remainder of this paper is organized as follows. Section 2 considers some classes of analytic solutions obtained via invariant Painlevé analysis of various long-wave and reaction-diffusion equations. Section 3 considers a simple proof that the conjecture in [57] is valid for such classes of solutions. In fact, once the class of solutions is identified, the proof is trivial in the regular Painlevé formalism, although it is less so in the invariant Painlevé formalism whence solutions are usually derived. These solutions are then used in Section 4 in the modeling of forced nonintegrable NLPDEs, while Section 5 considers a related treatment of forced integrable systems. Finally, Section 6 summarizes the results and conclusions and also briefly considers possible future work.

2. Some Classes of Invariant Painlevé Solutions

In this section, we first briefly summarize the invariant Painlevé formalism [17].

2.1. Invariant Painlevé formalism

For an NLPDE that is algebraic in U and its derivatives

$$E(U, x, t) = 0,$$

around a movable singular manifold

$$\phi - \phi_0 = 0 \tag{2.1}$$

one looks, in the invariant Painlevé formulation [17], for a solution as an expansion of the form

$$U = \chi^{-\alpha} \sum_{j=0}^{\infty} U_j \chi^j, \tag{2.2}$$

where the coefficients U_j are invariant under a group of homographic (Möbius) transformations on ϕ. The expansion variable χ, which must vanish as $(\phi - \phi_0)$ is chosen to be

$$\chi \equiv \frac{\psi}{\psi_x} = \left(\frac{\phi_x}{\phi - \phi_0} - \frac{\phi_{xx}}{2\phi_x} \right)^{-1},$$

$$\psi = (\phi - \phi_0)\phi_x^{-1/2}.$$

The variable χ satisfies the Ricatti equations

$$\chi_x = 1 + \frac{1}{2} S\chi^2, \tag{2.3a}$$

$$\chi_t = -C + C_x\chi - \frac{1}{2}(CS + C_{xx})\chi^2, \tag{2.3b}$$

while the variable ψ satisfies the linear equations

$$\psi_{xx} = -\frac{1}{2} S\psi, \tag{2.4a}$$

$$\psi_t = \frac{1}{2} C_x \psi - C\psi_x. \tag{2.4b}$$

Note that the systems of Eqs. (2.3) and (2.4) are equivalent to each other. In (2.3) and (2.4), the quantities S (Schwarzian derivative) and C (the "dimension of velocity" or celerity) are defined by

$$S \equiv \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x} \right)^2, \tag{2.5a}$$

$$C \equiv -\phi_t/\phi_x \tag{2.5b}$$

and are invariant under the group of homographic (Möbius) or fractional linear transformations [44]

$$\phi \to \frac{a\phi + b}{c\phi + d}, \quad ad - bc = 1. \tag{2.6}$$

These homographic invariants are linked by the cross-derivative condition $(\phi_{xxxt} = \phi_{txxx})$

$$S_t + C_{xxx} + 2C_xS + CS_x = 0. \tag{2.7}$$

This formalism has been used to derive various classes of exact coherent structure solutions for various equations. The general procedure involves substituting truncated invariant Painlevé expansions into the governing NLPDE(s). In other words, if the leading-order analysis indicates a dominant singularity of order $\phi^{-\alpha}$ about the arbitrary singularity manifold $\phi(x, t) = 0$, then one substitutes a truncated version of (2.2), i.e.,

$$U = \frac{U_0}{\chi^\alpha} + \frac{U_1}{\chi^{\alpha-1}} + \cdots + \frac{U_m}{\chi^{\alpha-m}}. \tag{2.8}$$

This typically yields coupled systems of NLPDEs for the coefficients $U_m(x, t)$ in (2.8). For subsequent use in Section 3, we stress here that, within the invariant Painlevé formalism, these coefficients are thus functions of the homographic invariants C and S and their derivatives. Note also that, unlike for non-invariant Painlevé expansions, no derivatives of χ occur in these equations since these are eliminated in the invariant formalism by the use of the Ricatti equations (2.3). However, quite often, the NLPDEs for the $U_m(x, t)$'s are too hard to solve in general and so solutions are typically (though not always) obtained by making the assumptions that the homographic invariants C and/or S are constants (see [17, 18] and the references therein for further details). Certain other classes of solutions of these equations may sometimes be obtained without this assumption (using Painlevé's method of subequations for instance). It is these classes of invariant Painlevé solutions which we shall mainly consider here. In particular, we shall prove that, as conjectured by Powell et al. [57] for arbitrary solutions obtained from Painlevé analysis, these classes of exact solutions of the NLPDE resulting from a direct application of Painlevé analysis to the PDE are indeed traveling wave solutions. Thus, they satisfy the traveling wave reduced ODE and thereby provide explicit parameterizations of its homoclinic / heteroclinic structures.

In the next two subsections, we summarize such classes of analytic solutions of various long-wave and reaction-diffusion equations obtained by the above procedure and which we shall use subsequently. Following that, we then prove in Section 3 that such solutions must indeed satisfy the traveling wave reduced ODE.

2.2. Solutions for the long-wave equations

Following the above procedure, several classes of solutions have been obtained for various important NLPDEs using truncated invariant Painlevé expansions. We first consider various long-wave equations [5, 27, 70], followed by some reaction-diffusion equations. The Benjamin-Bona-Mahoney (BBM) equation (sometimes known as the regularized long-wave (RLW) equation)

$$U_t + U_x + UU_x - U_{xxt} = 0 \tag{2.9}$$

was derived by Benjamin et al. as a description of long waves in shallow water in lieu of the famous Korteweg-de Vries (KdV) equation [14], which is a soliton equation solvable by inverse scattering. The modified Benjamin-Bona-Mahoney (MBBM) equation

$$U_t + aU_x + bU^2U_x + cU_{xxt} = 0 \tag{2.10}$$

is, similarly, an alternative to the modified KdV (MKdV) equation, which is also an integrable equation solvable by the inverse-scattering transform. The symmetric regularized long-wave (SRLW) equation

$$U_{tt} + aU_{xx} + \frac{1}{2}b(U^2)_{xt} + cU_{xxtt} = 0 \tag{2.11}$$

arises in various physical applications, including ion-acoustic wave propagation in neutral plasmas. It is an alternative to the Boussinesq equation, which also arises in numerous contexts and is also completely integrable by the inverse-scattering transform.

The BBM equation has the analytic solution (see [15, 16, 37, 40, 41]):

$$U_{\mathrm{II}}^{\mathrm{BBM}} = N/D, \tag{2.12a}$$

with

$$N = (c_1^2 + c_2^2)\{C - 1 - 20CQ^2\}$$
$$+ \{4CQ^2 + C - 1\}\{(c_1^2 - c_2^2)\, y_1(x, t) - 2c_1c_2y_2(x, t)\}, \tag{2.12b}$$

$$D = 2\{c_1 \cos(Q\xi) - \beta \sin(Q\xi)\}^2, \tag{2.12c}$$

where

$$\xi \equiv x - CT, \tag{2.12d}$$

$$y_1(x, t) \equiv \cos(2Q\xi), \tag{2.12e}$$

$$y_2(x, t) = \sin(2Q\xi), \tag{2.12f}$$

and C, Q, c_1 and c_2 are arbitrary.

Another class of solutions is:

$$U_{\mathrm{I}}^{\mathrm{BBM}} = \frac{-3(C-1)[c_1y_3(x, t) + c_2y_4(x, t)]^2}{2[c_1y_4(x, t) - c_2y_3(x, t)]^2}, \tag{2.13a}$$

with

$$y_3(x, t) = \sin\left[\left(\frac{C-1}{2C}\right)^{\frac{1}{2}} \frac{\xi}{2}\right], \tag{2.13b}$$

$$y_4(x, t) = \cos\left[\left(\frac{C-1}{2C}\right)^{\frac{1}{2}} \frac{\xi}{2}\right], \tag{2.13c}$$

and

$$\xi \equiv (x - Ct). \tag{2.13d}$$

Similarly, use of an invariant truncated Painlevé expansion yields the following class of solutions of the SRLW equation (2.11) with $a = b = 1$,

$c = -1$:

$$U_{\mathrm{I}}^{\mathrm{SRLW}} = \frac{12C[c_1 y_5(x, t) + c_2 y_6(x, t)]^2}{(20C-1)[c_1 y_6(x, t) - c_2 y_5(x, t)]^2} \tag{2.14a}$$

$$y_5(x, t) = \sin\left[\frac{\xi}{\sqrt{20C-1}}\right], \tag{2.14b}$$

$$y_6(x, t) = \cos\left[\frac{\xi}{\sqrt{20C-1}}\right], \tag{2.14c}$$

where $\xi = (c - Ct)$, c_1 and c_2 are arbitrary constants, and C is a root of the cubic equation

$$1 - 20C + 9C^2 - 20C^3 = 0,$$

i.e.,

$$C = 0.0510393 \quad \text{or} \quad C = 0.19948 \pm i0.969456.$$

The BBM and SRLW equations also possess other well-known analytic solutions which we summarize briefly here for the sake of comparison. First, there are the well-known families of one-parameter traveling wave pulse solutions

$$U_{\mathrm{III}}^{\mathrm{BBM}}(x, t) = 3(1-C)\,\mathrm{sech}^2\left[\left(\frac{C-1}{4C}\right)^{\frac{1}{2}}(x - Ct) + x_0\right], \tag{2.15}$$

$$U_{\mathrm{III}}^{\mathrm{SRLW}}(x, t) = 3\left(\frac{C^2-1}{C}\right)\mathrm{sech}^2\left[\left(\frac{C^2-1}{4C^2}\right)^{\frac{1}{2}}(x - Ct)\right]. \tag{2.16}$$

The use of non-invariant Painlevé expansions truncated at different levels yields additional solutions

$$U_{\mathrm{IV}}^{\mathrm{BBM}} = -\frac{12c_1^2 c_2 e^{\left(Cx + \frac{Ct}{C^2-1}\right)}}{(C^2)\left\{c_1 e^{Cx + \frac{Ct}{C^2-1}} + c_2\right\}^2}, \quad (c_1 = C^2) \tag{2.17a}$$

$$U_{\mathrm{V}}^{\mathrm{BBM}} = \frac{-12c_1^2}{[c_1(x-t)+c^2]^2}, \tag{2.17b}$$

and

$$U_{\mathrm{III}}^{\mathrm{SRLW}} = \frac{\pm 12C^2 c_1 c_2 e^{C(x-Ct)}}{\sqrt{C^2-1}\left\{c_1 e^{C\left(x\pm\frac{t}{\sqrt{C^2-1}}\right)}+c_2\right\}^2}. \tag{2.18}$$

2.3. Solutions for various reaction-diffusion equations

In a similar manner, one may obtain classes of solutions of various interesting reaction diffusion equations (see [16, 57, 62] for the original references where these equations were considered):

$$U_t = U_{xx} + \frac{U}{b}(b+U)(1-U) \tag{2.19}$$

$$U_t = \beta U^2(1-U) + DU_{xx} \tag{2.20}$$

$$U_t = U(1-U^2) + U_{xx}. \tag{2.21}$$

Of these, (2.20) and (2.21) are the well-known Fisher-Kolmogorov equation and the real Ginzburg-Landau (also known as Newell-Whitehead) equation. Invariant Painlevé expansions truncated at various levels yield the following solutions of the above equations.

For (2.19), we have the solution

$$U^{(1)} = \pm\frac{\sqrt{2b}}{\chi} + \frac{1}{6}(2-2b-C\sqrt{2b}), \tag{2.22}$$

where

$$\chi = \frac{c_1\cos(Q\xi)-c_2\sin(Q\xi)}{-Q[c_2\cos Q\xi + c_1\sin Q\xi]}, \tag{2.23a}$$

$$\xi = x - Ct \tag{2.23b}$$

and C has one of the values

$$C = \frac{-\sqrt{2} - 2\sqrt{2}b}{2\sqrt{b}} \equiv C_1 \tag{2.24a}$$

$$C = \frac{-\sqrt{2} + \sqrt{2}b}{2\sqrt{b}} \equiv C_2 \tag{2.24b}$$

$$C = \frac{2\sqrt{2} + \sqrt{2}b}{2\sqrt{b}} \equiv C_3. \tag{2.24c}$$

A class of solutions of (2.20) is given by

$$U^{(2)} = \pm\frac{\sqrt{2D/\beta}}{\chi} + \left(\frac{2D \mp C\sqrt{2D/\beta}}{6D}\right), \tag{2.25}$$

where χ is given by (2.23), with

$$S = \frac{C^2 - 2D\beta}{6D^2} \equiv 2Q^2 \tag{2.26}$$

and

$$2\beta\left(\frac{1}{3} \mp \frac{C}{3\sqrt{2\beta D}}\right)^3 - 2\beta\left(\frac{1}{3} \mp \frac{C}{3\sqrt{2\beta D}}\right)^2 \pm \sqrt{\frac{2D}{\beta}}C\left(\frac{C^2 - 2D\beta}{6D^2}\right) = 0. \tag{2.27}$$

For instance, with $D = 1, \beta = 2,$ we obtain $C = -1,$ or $C = 2.$

Finally, for (2.21), we have the class of solutions [13]

$$U = \pm\left[\frac{(3 + \sqrt{2}C)c_2 + (-3 + \sqrt{2}C)c_1 e^{(x-Ct)/\sqrt{2}}}{6\{c_2 + c_1 e^{(x-Ct)/\sqrt{2}}\}}\right] \text{ for } C = \pm 3/\sqrt{2}. \tag{2.28}$$

3. Proof that Invariant Painlevé Solutions Satisfy the Traveling Wave Reduced ODE

In this section, we shall show that certain classes of analytic solutions obtained by the use of invariant Painlevé expansions in fact satisfy the traveling wave reduced ODE. Thus, they not only provide ‘partial integrability’ in Painlevé’s original sense, but also provide explicit parameterizations of the homoclinic or heteroclinic structures of the traveling wave reduced ODE. Such solutions directly obtained for the

PDE will therefore satisfy the conjecture of Powell et al. [57] which was discussed in Section 1 (and tested out in [16]). We shall see that this feature of such analytic solutions will also be very useful in our subsequent treatment of forced NLPDEs.

In particular, let us consider the classes of solutions corresponding to $C_x = 0$, i.e., solutions where the celerity is a function of t alone. Equation (2.7) then shows that

$$S = S(x - Ct). \tag{3.1}$$

Using this, both (2.4a) and (2.4b) imply that

$$\psi = \psi(x - Ct). \tag{3.2}$$

Hence

$$\chi \equiv \frac{\psi}{\psi_x} = \chi(x - Ct). \tag{3.3}$$

As stressed in Section 2.1, the expansion coefficients $U_m(x, t)$ in the truncated expansion (2.8) are functions of C, S, and their partial derivatives. Thus, if $C_t = 0$ as well, i.e., for classes of solutions with

$$C = \text{constant} \tag{3.4}$$

these expansion coefficients are functions of $(x - Ct)$ as well. Using this and (3.3) shows that any classes of constant C solutions obtained from truncated invariant Painlevé expansions like (2.8) will explicitly be functions of $(x - Ct)$, and hence satisfy the traveling wave reduced ODE. Note that, as mentioned in Section 2.1, most classes of analytic solutions are derived assuming constant C for operational purposes, although other classes may sometimes also be obtained (for instance by Painlevé's method of subequations or by iterating Bäcklund transformations). However, most classes of smooth solutions obtained by these other methods turn out to have constant C, except for non-smooth solutions such as the cuspons and peakons of the Camassa-Holm equation. They will thus provide explicit expressions or parameterizations for the heteroclinic or homoclinic structures of the traveling wave reduced ODE (corresponding to front (shock) or pulse (solitary wave) solutions of the PDE respectively).

Two generalizations of the above result are immediate. The first is for solutions of equations in more than one spatial dimension and the second is for NLPDEs whose Painlevé analysis yields more than one singularity branch.

For NLPDEs in $(2+1)$ for instance, (2.7) is supplemented by the conditions

$$S_y + K_{xxx} + 2K_x S + KS_x = 0, \tag{3.5}$$

$$C_y - K_t + C_x K - CK_x = 0 \tag{3.6}$$

and (2.4) is supplemented by

$$\psi_y = \frac{1}{2} K_x \psi - K\psi_x, \tag{3.7}$$

where

$$K \equiv -\phi_y / \phi_x. \tag{3.8}$$

Now, for classes of solutions with C and K constant, (3.6) is trivially satisfied, while (2.7) and (3.5) show that

$$S = S(x - Ky - Ct). \tag{3.9}$$

Using this, (2.4) and (3.7) show that

$$\psi = \psi(x - Ky - Ct) \tag{3.10}$$

and hence

$$\chi \equiv \frac{\psi}{\psi_x} = \chi(x - Ky - Ct). \tag{3.11}$$

Thus, by the same reasoning as above, all coefficients as well as χ in the truncated invariant Painlevé expansion for the solution (the series is similar to (2.8) except that all quantities occurring in it are also functions of y) are functions of $(x - Ky - Ct)$. Thus such classes of solutions corresponding to constant C and K will indeed be traveling wave reductions and satisfy the traveling wave reduced ODE.

The second extension corresponds to solutions which are constructed using more than one singularity branch (if the PDE admits more than one branch of singularities). There are of course a very large number of such cases, such as the bright soliton of the Nonlinear Schrödinger Equation which is most easily constructed using two branches (while the dark soliton requires only a single branch [19, 56]. Say that a solution is constructed based on a truncated invariant expansion with two singularity manifolds χ_i and coefficients $U_k^{(i)}$ for branches $i = 1, 2$, i.e.,

$$U = \sum_{k=0}^{m_1} \frac{U_k^{(1)}}{\chi_1^{\alpha+k}} + \sum_{k=0}^{m_2} \frac{U_k^{(2)}}{\chi_2^{\alpha+k}}. \tag{3.12}$$

As for the cases above with one singularity branch, the coefficients depend on the homographic invariants S_i and C_i and their derivatives. For classes of solutions where the C_i's are constants, the analogs of (2.7) for each branch show that

$$S_i = S_i(x - C_i t), \quad i = 1, 2, \ldots \tag{3.13}$$

while the analogs of (2.4) for each branch reveal that

$$\psi_i = \psi_i(x - C_i t), \quad i = 1, 2, \ldots \tag{3.14a}$$

$$\chi_i = \chi_i(x - C_i t), \quad i = 1, 2, \ldots. \tag{3.14b}$$

Thus, when (3.13-3.14) and C_i = constant are used, the coefficients in (3.12) take the form

$$\begin{aligned} U_k^{(i)} &= U_k^{(i)}(C_1, S_1(x - C_1 t), S_{1t}(x - C_1 t), S_{1x}(x - C_1 t), \ldots; \\ &\qquad C_2, S_2(x - C_2 t), S_{2t}(x - C_2 t), S_{2x}(x - C_2 t) \cdots) \\ &= U_k^{(i)}(x - C_1 t, x - C_2 t). \end{aligned} \tag{3.15}$$

From (3.12), (3.14) and (3.15), it is now apparent that such classes of solutions corresponding to the C_i's constant will be functions of $(x - C_i t)$, and thus will satisfy the traveling wave reduced ODE. Also, as is quite well known, this also indicates that such solutions may correspond to bidirectional waves for any NLPDE, such as the Boussinesq equation, where two of the singularity branches may have celerities (C_i's) of opposite signs.

This completes the proof. In the next section, we consider forced NLPDEs where the classes of invariant Painlevé solutions we have considered in this section will be seen to be important.

4. Forced Nonintegrable Systems

Forced systems arise in many areas (see [36, 38, 52, 54, 61] for instance), such as the famous Stokes layer problem [68], surface waves due to time-dependent pressure [60, 74], surface waves in viscous fluids moving down an inclined plane with an uneven bottom [8, 59], or a very large variety of configurations involving forced systems of oscillators ([36, 38, 52, 54, 61] for example). In addition, there have also been marginally related studies of forced integrable systems, such as the forced KdV, forced Nonlinear Schrödinger Equation, and forced Sine-Gordon systems (see [9-11, 30, 55, 63] for instance). While studies of forced integrable systems have made extensive use of the geometry, studies of forced non-integrable systems have usually involved numerical simulations supplemented by perturbative/geometric perturbative approaches.

In this section, we shall see that use of the ideas we have considered earlier allows us to analytically treat the breakdown (to chaos) under forcing of coherent structure solutions of both integrable and non-integrable systems. For the sake of accuracy, this statement should be supplemented with the coda that we shall be able derive the threshold for the onset of chaos under forcing. However, the analytical consideration of the actual dynamics prior to the onset of chaos would still require the use of multiscale perturbation theory (for forcings which are fast compared to the intrinsic system timescale) [54], or direct soliton perturbation theory (see [71]-[73] and the references therein for instance for recent reviews) for forced integrable systems. It is also conceivable that resummation of these perturbation series [4, 7] may enable one to analytically follow the homoclinic or heteroclinic tangling beyond the first transversal intersection of the stable and unstable manifolds and into the chaotic regime (see [64] for instance).

In order to set the stage, we shall first consider a forced reaction diffusion system which is of relevance in modeling stirred-tank open reactor systems [23, 35] used in studies of nonlinear chemical systems, two-dimensional Turing pattern formation and so on. It will be seen that the analysis goes through relatively smoothly in this example, although even here a particular ordering of the various terms is essential. However, subsequent examples will illustrate the possible complications, and show how these may be circumvented by the use of the ideas and classes of coherent structure solutions which were considered in Sections 2 and 3.

Consider the forced reaction diffusion system

$$U_t = U_{xx} + f(U) + F \cos \omega t \tag{4.1}$$

for various possible nonlinear reaction terms $f(U)$. If we consider coherent structure (pulse (or solitary wave), and front (or kink or shock)) solutions of this equation in the form of traveling waves (this is the typical situation, see [49] for example), then we have

$$-c \frac{dU}{dz} = \frac{d^2U}{dz^2} + f(U) + F \cos\left[\frac{\omega(x-z)}{c}\right] \tag{4.2}$$

for the ODE governing them and where $z = x - ct$. If the parameters c and F, corresponding to the wave speed and the forcing amplitude, are ordered as (the motivation for this is to ensure that the unforced ODE may be integrated relatively simply, and similar orderings are thus fairly common for various forced systems, including forced integrable systems such as in [30]):

$$c = \varepsilon C_0 \tag{4.3a}$$

$$F = \varepsilon F_0, \tag{4.3b}$$

where ε is a small parameter, equation (4.2) becomes

$$\frac{d^2U}{dz^2} + f(U) = -\varepsilon\left[F_0 \cos\left\{\frac{\omega(x-z)}{\varepsilon c_0}\right\} + c_0 \frac{dU}{dz}\right]. \tag{4.4}$$

Picking the reaction function to be the real Ginzburg-Landau (Newell-Whitehead) one (see [13, 27])

$$f(U) = \pm(U^2 - 1)U \tag{4.5}$$

and defining

$$x_1 \equiv U \tag{4.6a}$$

$$x_2 \equiv dU/dz\,, \tag{4.6b}$$

equation (4.4) may be written in the standard perturbed Hamiltonian form suited to setting up a Melnikov integral [36, 52]

$$\dot{\vec{x}} = \vec{f} + \varepsilon\vec{g}, \tag{4.7}$$

where

$$\vec{f} = [x_2, \mp x_1(x_1^2 - 1)]^T \tag{4.8}$$

$$\vec{g} = \left[0, -F_0 \cos\left\{\frac{\omega(x-z)}{\varepsilon c_0}\right\} - c_0 \frac{dU}{dZ}\right]^T. \tag{4.9}$$

Hence, we may set up the Melnikov integral as

$$\begin{aligned}
M(\theta) &= \int_{-\infty}^{\infty} |\vec{f}(z) \times \vec{g}(z+\theta)|\, dz \\
&= \mp\int_{-\infty}^{\infty} 2\mathrm{sech}^2(2z)\left[F_0 \cos\left\{\frac{\omega(x-z-\theta)}{\varepsilon c_0}\right\} \pm 2c_0\mathrm{sech}^2(2z)\right] dz \\
&= \mp 2F_0 \cos[\alpha(x-\theta)]\int_{-\infty}^{\infty} \cos(\alpha z)\,\mathrm{sech}^2(2z)\, dz \\
&\mp 2F_0 \sin[\alpha(x-\theta)]\int_{-\infty}^{\infty} \sin(\alpha z)\,\mathrm{sech}^2(2z)\, dz - 4c_0\int_{-\infty}^{\infty} \mathrm{sech}^4 2z dz \\
&= \mp 2F_0 \cos[\alpha(x-\theta)]\int_{-\infty}^{\infty} \cos(\alpha z)\,\mathrm{sech}^2(2z)\, dz - \frac{8}{3}c_0,
\end{aligned} \tag{4.10}$$

where

$$\alpha \equiv \omega/\varepsilon c_0 \tag{4.11}$$

and we have used [36, 52] the standard closed-form expressions for the heteroclinic orbits of the *unperturbed* Hamiltonian system in (4.7) and associated values of various integrals in (4.10) which occur commonly in Melnikov analyses. For typical parameter values $\omega = 0.1$, $\varepsilon = 0.1$, $c_0 = 1$ for instance, (4.10) yields

$$M(\theta) = \mp 0.188276 \cos(x-\theta) - \frac{8}{3}. \tag{4.12}$$

Thus, for $\cos(x-\theta) = \pm 1$ for the upper/lower signs respectively, the Melnikov function has a simple zero. As is well known, this corresponds to the first transversal intersection of the stable and unstable manifolds of the saddle points of the unperturbed Hamiltonian system in (4.7) and corresponds to the onset of chaos. For the above parameters, this occurs at a value of the forcing $F_0 = 1.4747$. Of course, as the forcing amplitude is turned up, one could numerically track the actual breakdown of the heteroclinic half-orbit in the usual way [36, 52,]. If we consider parametric forcing, for instance if F in (4.2) is replaced by uF, the above computation may be redone yielding

$$M(\theta) = 2F_0 \sin[\alpha(x-\theta)]\int_{-\infty}^{\infty} \sin(\alpha z)\tanh(2z)\operatorname{sech}^2(2z)\,dz$$

$$- 4c_0\int_{-\infty}^{\infty}\operatorname{sech}^2 2z dz$$

$$= -0.452068 F_0 \sin\alpha(x-\theta) - \frac{8}{3}c_0. \tag{4.13}$$

For the same parameters as above, the onset of chaos now occurs at larger values of forcing, i.e., $F_0 > 5.899$.

Before we leave this first, elementary example, we should stress two features. First, consideration of traveling wave coherent structures of the PDE (4.1) effectively reduces our problem to an ODE. However, the unforced ODE in this example is easily integrable in terms of elementary functions, and this yields expressions for the heteroclinic orbits of the unforced system in (4.7). Secondly, the original PDE is an infinite-dimensional system which couples all x values in the domain of the problem. However, in considering the forced traveling-wave reduced ODE (4.2/4.7), we are in fact assuming that the coherent structures retain their traveling wave form well into the forced regime. Strong support for this comes from the robustness of the coherent structures under perturbation ([8, 12, 38, 43, 47, 54, 59-61, 68, 74,] for instance), but results need 'a posteriori' validation. Note that this is provided that we are in regimes dominated by stable coherent structures [14] in the absence of resonances [69]. In computing and interpreting the Melnikov function, we thus take the point of view that we are in fact dealing with an ODE, with the x taking *one*, *but arbitrary*, *value*. Carrying this further, *if the Melnikov function goes to zero now at any* x, then there will be onset of chaos. Note that the important point above is that x may take any value. In a very imprecise sense, this is analogous to the 'method of normal modes' in hydrodynamic stability analysis, where one allows the wavenumber k to take one, but arbitrary, value and then sees whether the system may be unstable at *any* k (except that linear systems are dealt with there). This k value is then the one mode among the infinity of wavenumber values for the various modes which is most unstable, or goes unstable first. Note that the traveling wave reduced ODE is not expected to predict full dynamical or bifurcation behavior of the PDE accurately. As discussed, this needs numeric or perturbative work on the PDE. However, as we shall check 'a posteriori', it captures the qualitative evolution of coherent structures accurately.

Let us consider a second example next in order to see possible complications, as well as ways around them. Considering traveling wave solutions of the real Ginzburg-Landau (Newell-Whitehead) [13, 27] system (2.21) [47, 57, 62] as above, it is straightforward to verify that the resulting ODE cannot be integrated in terms of elementary functions, but one requires the use of elliptic functions. Of course, elliptic functions solutions are widely used in different contexts, such as water waves and various other areas in general (see [45, 48] for instance). However, the solutions obtained from invariant Painlevé analysis give direct 'partial integrability' in terms of elementary functions, and, as discussed in Section 3, they are in fact traveling wave reductions. Hence, we may use these direct solutions in (2.28) for the PDE (2.21) as a simple explicit parameterization for the heteroclinic orbits of its traveling wave reduced ODE (connecting the saddle points at $(\pm 1, 0)$. Notice too that one need not order some of the terms to be smaller than others, as was needed in the earlier example. Hence, for the forced version of (2.21), i.e., (4.1) with

$$f(U) = U(1 - U^2) \tag{4.14}$$

and forcing function of the form $F_1 \cos(\omega t) + F_2$, the corresponding traveling wave reduced ODE (4.2) may be written in the form (4.7) (using (4.6) and setting $\varepsilon = 1$) with

$$\vec{f} = \left[x_2,\ -x_1 + x_1^3 - \frac{3}{\sqrt{2}}\, x_2\right]^T \tag{4.15}$$

$$\vec{g} = \left[0, -F_1 \cos\left\{\frac{\omega(x-z)\sqrt{2}}{3}\right\} - F_2 \sin\left\{\frac{\omega(x-z)\sqrt{2}}{3}\right\}\right]^T. \tag{4.16}$$

Thus, the corresponding Melnikov function is

$$M(\theta) = -F_1 \cos\alpha(x-\theta) I_1 - F_1 \sin\alpha(x-\theta) I_2 - F_2 \tag{4.17a}$$

$$I_1 = \int_{-1}^{0} \cos\left[\alpha\sqrt{2}\, n \ln\left\{-\frac{c_2}{c_1}\left(\frac{1+U}{U}\right)\right\}\right] dU \tag{4.17b}$$

$$I_2 = \int_{-1}^{0} \sin\left[\alpha\sqrt{2}\, n \ln\left\{-\frac{c_2}{c_1}\left(\frac{1+U}{U}\right)\right\}\right] dU. \tag{4.17c}$$

Here, $\alpha = \sqrt{2}\,\omega/3,$ and we have chosen U for the case $C = 3/\sqrt{2}$ in (2.28) for x_1, and its derivative (with respect to $z = x - Ct$) for x_2. Thus, for the typical parameter values below, the Melnikov integral goes through a simple zero for

a. $F_1/F_2 \geq 9.567$ for $\omega = \frac{3}{\sqrt{2}}$, $\alpha = 1$, $c_1 = c_2 = 1$, (4.18)

b. $F_1/F_2 \geq 1.9092$ for $\omega = 1$, $\alpha = \frac{\sqrt{2}}{3}$, $c_1 = c_2 = 1$. (4.19)

The above approach may clearly be used for any nonintegrable PDE where nontrivial families of coherent structure solutions are obtained by invariant Painlevé analysis, e.g., for the long-wave equations. Thus, for the solutions $U_{\mathrm{I}}^{\mathrm{BBM}}$ in (2.13) of the BBM equation (2.9), under forcing of the form $F_1 \cos(\omega t) + F_2$, for typical parameter values c_1 arbitrary, $c_2 = 0$, the onset of chaos occurs at:

a. $F_2/F_1 \geq 2.839$ for $\omega = 1/2\pi$ (4.20)

b. $F_2/F_1 \geq 657.4$ for $\omega = 1$. (4.21)

Clearly, the other solutions to the long-wave equations in Section 2 may be treated in a similar fashion.

At this point, before concluding the discussion of forced nonintegrable systems, it is worth briefly mentioning one other related point, which is of crucial importance in actual physical modeling applications. In [22] and [34], the typical approach taken to invariant Painlevé analytical solutions of nonintegrable PDEs was to consider their real parts for real values of the arbitrary constants in the solutions (e.g., the c_i's in (2.12)). However, for these to be meaningful physical solutions of the physical PDEs, these constants need to be chosen such that the full analytical solutions are themselves real. For instance, the solutions $U_{\mathrm{I}}^{\mathrm{BBM}}$ in (2.12) may be shown to be real if

$$c_{1r}c_{2r} = -c_{1i}c_{2i}, \tag{4.22}$$

where the r and i subscripts denote the real and imaginary parts of the constants c_1 and c_2. An example is shown in Figure 1 for the real solution $u_{\mathrm{I}}^{\mathrm{BBM}}$ for parameter values $C = 0.5$, $c_1 = 5(1 - i\sqrt{2})$, $c_2 = \sqrt{2} + i$ satisfying these conditions. The solution is a real solitary wave or pulse with asymptotic value $3(C-1)/2$ at both ends.

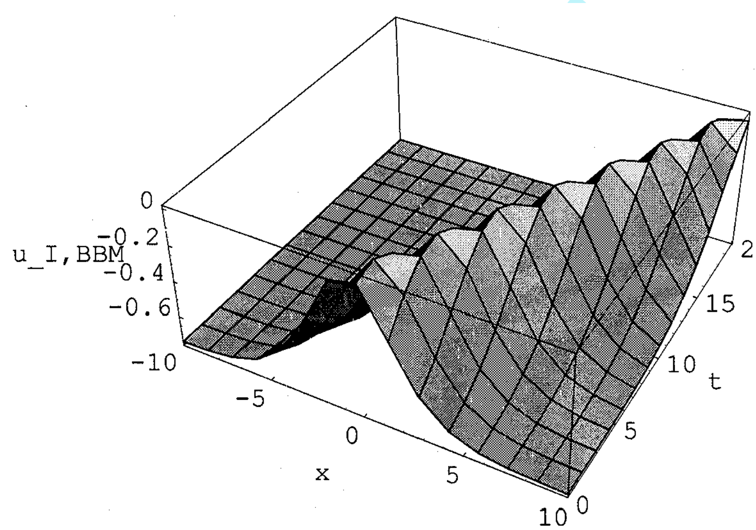

Figure 1. $U_{\mathrm{I}}^{\mathrm{BBM}}$ in (2.13) for $C = 0.5$, $c_1 = 5(1 - i\sqrt{2})$, and $c_2 = \sqrt{2} + i$ satisfying (4.22). In the $(\xi, U_{\mathrm{I}}^{\mathrm{BBM}})$ plane, it is a pulse with asymptotic values $3(C-1)/2$ as $\xi \to \pm\infty$.

Similarly, a fairly involved calculation reveals that, for

$$Q = q_r + iq_i, \tag{4.23}$$

the solutions $U_{\mathrm{II}}^{\mathrm{BBM}}$ are real for:

$$q_r = 0 \tag{4.24a}$$

<u>and</u>

$$c_{2r} = -c_{1i}, \quad c_{1r} = -c_{2i}$$

or

$$c_{2r} = c_{1i}, \quad c_{1r} = c_{2i}$$

or

$$c_{2r} = -c_{2i}, \quad c_{1r} = c_{1i}$$

or

$$c_{2r} = c_{2i}, \quad c_{1r} = -c_{1i}. \tag{4.24b}$$

<u>OR</u>

$$q_i = 0 \tag{4.25a}$$

and

$$c_{2r} = c_{1i} = 0, \quad c_{1r} = \pm c_{2i}$$

or

$$c_{2r} = -c_{2i}, \quad c_{1i} = \mp c_{2i}, \quad c_{1r} = \pm c_{2i}$$

or

$$c_{2r} = c_{2i}, \quad c_{1i} = \mp c_{2i}, \quad c_{1r} = \mp c_{2i}$$

or

$$c_{2r} = \mp c_{1r}, \quad c_{1i} = \mp c_{2i}$$

or

$$c_{2r} = \mp c_{2i}, \quad c_{1i} = \mp c_{1r}$$

or

$$c_{1r} = \mp c_{2r}, \quad c_{2i} = \mp c_{1i}. \tag{4.25b}$$

In (4.25b), either the upper or the lower signs apply, but one should not mix an upper and a lower sign. Note too that the cases in (4.24) have $q_r = 0$ and will be genuine real and spatially confined coherent structure solutions of the BBM equation, while those in (4.25) with $q_i = 0$ really correspond to solutions which are periodic and thus not spatially confined coherent structures. An example of the former is shown in Figure 2 for parameters $C = 0.5$, $Q = i$, $c_1 = (1 - i)\pi$, $c_2 = (1 + i)\pi$ satisfying (4.24). The solution is a solitary wave with asymptotic values

$$\lim_{z \to \pm\infty} U_{\rm II}^{\rm BBM} = 4CQ^2 + C - 1, \tag{4.26}$$

and this would be readily apparent in a $(\xi, U_{\rm II}^{\rm BBM})$ plot. Figure 3 shows a periodic, non-coherent structure wave-train for $C = 2$, $Q = 1$, $q_i = 0$, and $c_1 = c_2 = (1 - i)\pi$ corresponding to (4.25).

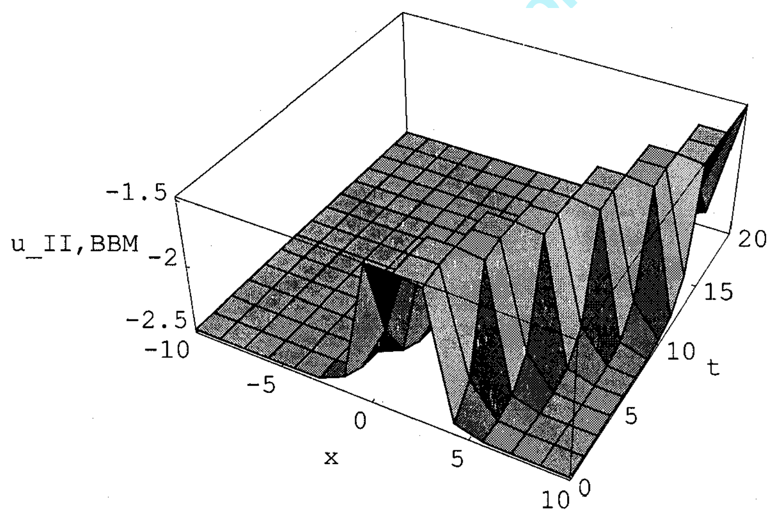

Figure 2. $U_{\rm II}^{\rm BBM}$ in (2.12) for $C = 0.5$, $Q = i$, $c_1 = (1 - i)\pi$, and $c_2 = (1 + i)\pi$ satisfying (4.24). In the $(\xi, U_{\rm II}^{\rm BBM})$ plane it is a pulse with asymptotic values $(4CQ^2 + C - 1)$ as $\xi \to \pm\infty$.

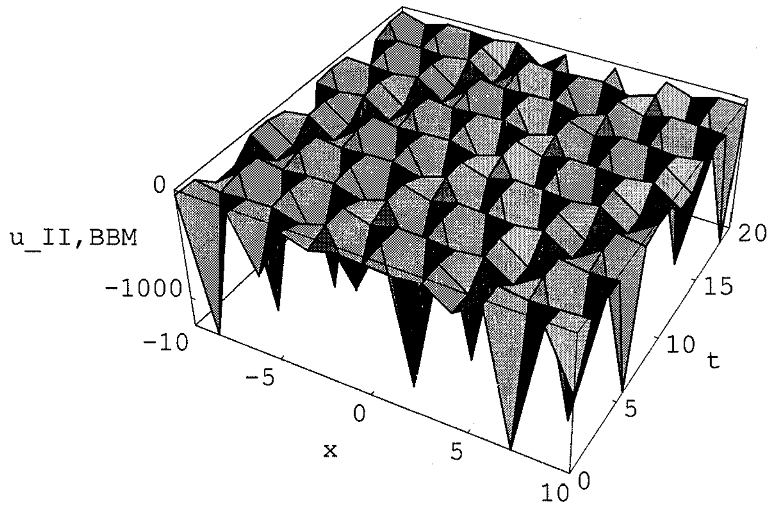

Figure 3. Periodic case of $U_{\text{II}}^{\text{BBM}}$ in (2.12) for $C = 2$, $Q = 1$, $q_i = 0$, and $c_1 = c_2 = (1 - i)\pi$ corresponding to (4.25). In the ξ, $U_{\text{II}}^{\text{BBM}}$ plane, it is a periodic wave-train.

This concludes our discussion of forced nonintegrable systems, and we next briefly consider forced integrable systems in a similar fashion.

5. Forced Integrable Systems

Given the above, the effects of forcing on multisoliton solutions of integrable PDEs may now be treated in a manner analogous to Section 4. Note that there is an extra issue here, viz. the compatibility of the forcing and the initial conditions [54].

Consider the forced damped driven nonlinear Sine-Gordon equation which has been considered extensively before, although from a completely different perspective using its rich geometry and also using direct numerical simulations [9-11, 30, 55, 63]

$$U_{tt} - U_{xx} + \sin U = \varepsilon[-\alpha U_t + \Gamma \sin \omega t]. \tag{5.1}$$

Note that we have ordered the terms as in earlier studies, i.e., both the damping and driving (the perturbation or stress terms) are weak or ordered to be $O(\varepsilon)$ for the reasons discussed in Section 4, i.e., so that the unperturbed or unstressed system is easily integrable. Several earlier studies focused on cases where the initial unperturbed, i.e., undamped and unforced, configuration was the breather solution of the Sine-Gordon equation. Since our approach is very direct, we shall consider cases where this unperturbed configuration is either the 1-kink, 2-kink, or the breather solution of the SG equation.

For the one-soliton (kink) solution of the unperturbed system (the Sine-Gordon equation)

$$U = 4\tan^{-1}\left[\exp\left\{\frac{(x - ct - z_0)}{\sqrt{1 - c^2}}\right\}\right] \tag{5.2}$$

we may, exactly as in Section 4, cast the traveling wave reduced ODE in Hamiltonian form and hence set up the Melnikov integral

$$\begin{aligned} M(\theta) = \frac{\varepsilon}{c^2 - 1}\Bigg\{&-\frac{4\alpha c}{(c^2 - 1)^2}\int_{-\infty}^{\infty}\operatorname{sech}^2\left[\frac{z - z_0}{\sqrt{1 - c^2}}\right]dz \\ &+ \frac{2\Gamma}{\sqrt{1 - c^2}}\sin\frac{\omega(x - \theta)}{c}\int_{-\infty}^{\infty}\cos\left(\frac{\omega z}{c}\right)\operatorname{sech}\left[\frac{z - z_0}{\sqrt{1 - c^2}}\right]dz \\ &- \frac{2\Gamma}{\sqrt{1 - c^2}}\cos\frac{\omega(x - \theta)}{c}\int_{-\infty}^{\infty}\sin\left(\frac{\omega z}{c}\right)\operatorname{sech}\left[\frac{z - z_0}{\sqrt{1 - c^2}}\right]dz\Bigg\}. \end{aligned} \tag{5.3}$$

Here, z is the usual traveling wave reduced variable $(x - ct)$. Picking parameters $\omega = 0.87$, $\varepsilon\alpha = 0.04$, $\varepsilon = 0.1$ corresponding to [9-11, 30, 55, 63], the Melnikov function has a simple zero corresponding to the onset of chaos for

a. $\dfrac{\Gamma}{\alpha} \geq 16.98$ for $c = 0.9$, $z_0 = 0$ (5.4)

b. $\dfrac{\Gamma}{\alpha} \geq 164.48$ for $c = 0.98$, $z_0 = 0.$ (5.5)

Repeating the process when the initial unperturbed configuration is the two-kink solution

$$U_2 = 4\tan^{-1}\left[\left(\frac{a_1 + a_2}{a_2 - a_1}\right)\tan\frac{1}{4}(\phi_2 - \phi_1)\right] \tag{5.6a}$$

$$\phi_i = 4\tan^{-1}\left(\exp\left[-a_i\xi - \frac{\tau}{a_i}\right]\right), \quad i = 1, 2, \ldots \tag{5.6b}$$

$$\xi = \frac{(x-t)}{2}, \quad \tau = \frac{(x+t)}{2} \tag{5.6c}$$

for $a_1 = 1.5$, $a_2 = -2$ (so that $(a_1 + a_2)/(a_2 - a_1) > 0$) shown in Figure 3, the simplest way to formulate the Melnikov integral now is to recast the system in terms of the characteristic variables ξ and τ of (5.6c) using the chain rule. Once this is done, one may then consider the equation

$$\tau = x - \xi \tag{5.7}$$

resulting from (5.6c) at *fixed, but arbitrary* x (as in Section 4) to obtain the damped, forced system (4.6)/(4.7) with

$$\vec{f} = [x_2, -\sin x_1]^T \tag{5.8}$$

$$\vec{q} = \left[0, \frac{\varepsilon\alpha v}{2} + \varepsilon\Gamma\sin\omega(x - 2\xi)\right]^T \tag{5.9}$$

and the overdot representing $d/d\xi$. The Melnikov integral may now be readily evaluated as

$$M(\theta) = \varepsilon\left\{\frac{\alpha}{2}\int_{-\infty}^{\infty} U_{2\xi}^2 d\xi + \Gamma\sin\omega(x - 2\theta)\int_{-\infty}^{\infty} U_{2\xi}\cos(2\omega\xi)\,d\xi - \Gamma\cos\omega(x - 2\theta)\int_{-\infty}^{\infty} U_{2\xi}\sin(2\omega\xi)\,d\xi\right\}, \tag{5.10}$$

where the subscript ξ indicates a partial derivative with respect to ξ. Evaluating this at both ends and the midpoint of an interval of length 24 (as in [30]) for the parameters mentioned above yields the onset of chaos for

$$\frac{\Gamma}{\alpha} \geq 13.605 \text{ at } x = 0 \tag{5.11a}$$

$$\frac{\Gamma}{\alpha} \geq 6.113 \text{ at } x = 12 \tag{5.11b}$$

$$\frac{\Gamma}{\alpha} \geq 3.94 \text{ at } x = 24. \tag{5.11c}$$

Clearly, as for the numerical studies, one must take slices at different x values and see where the lowest $\frac{\Gamma}{\alpha}$ threshold occurs.

Finally, repeating the above for the case where the initial unstressed configuration is the breather solution

$$U_3 = 4\tan^{-1}[(1-\omega^2)\,\omega^{-1}\sin(\omega\gamma)(t - vx)\,\mathrm{sech}\gamma(1-\omega^2)^{1/2}(x - vt)] \quad (5.12)$$

with

$$\gamma = (1 - v^2)^{-1/2} \quad (5.13)$$

and we pick $v = 0.95$, so that the solution is close to a pure traveling wave. The Melnikov integral may be computed in a manner analogous to that for the simple one-kink case yielding

$$M(\theta) = \frac{\varepsilon}{c^2 - 1}\left\{\alpha c\int_{-\infty}^{\infty} x_2^2 dz + \Gamma \sin\frac{\omega(x-\theta)}{c}\int_{-\infty}^{\infty} x_2 \cos\left(\frac{\omega z}{c}\right) dz\right.$$
$$\left. - \Gamma\cos\frac{\omega(x-\theta)}{c}\int_{-\infty}^{\infty} x_2 \sin\left(\frac{\omega z}{c}\right) dz\right\}. \quad (5.14)$$

For $\omega = 0.87$ as above, the onset of chaos under forcing occurs at

$$\frac{\Gamma}{\alpha} \geq 3.779 \text{ for } c = 0.3 \quad (5.15a)$$

$$\frac{\Gamma}{\alpha} \geq 14.047 \text{ for } c = 0.6 \quad (5.15b)$$

$$\frac{\Gamma}{\alpha} \geq 36.0445 \text{ for } c = 0.9. \quad (5.15c)$$

This concludes our treatment of the damped, driven Sine-Gordon equation and we proceed next to briefly discuss our results, consider the features, validity, and scope of the treatment we have developed here, and also comment on possible future extensions.

6. Summary and Conclusions

In this paper, we have primarily done two things. The first is a proof that certain, constant celerity, classes of solutions (whether pulses (solitary waves) or fronts (shocks or kinks)) obtained using truncated Painlevé expansions (or regular Painlevé expansions) are indeed traveling waves and thus satisfy the corresponding reduced ODE. As mentioned earlier, the motivation for looking for this comes from its applicability in various areas of nonlinear science.

The second main focus in this paper was to use the above result to develop a new, direct method of treating the behavior of coherent structure solutions of both nonintegrable and integrable NLPDEs under stresses such as forcing. The primary merit of this approach was of course its simplicity. However, a key assumption that went into this approach was the assumption that a coherent structure which had the functional form of a traveling wave would indeed remain a traveling wave under stress so that one could deal directly with the forced traveling-wave-reduced ODE system. As mentioned earlier, this was motivated by a body of accumulated evidence to this effect [8, 12, 38, 43, 47, 54, 59-61, 68, 74]. Let us now briefly consider how this assumption stacks up, i.e., whether the results obtained from it validate it *a posteriori* or not.

In particular, note how close the agreement obtained using this point of view is qualitatively between the result in (5.15a) and that in [9-11, 30, 55, 63] where the onset of chaos occurs at $\varepsilon\Gamma = 0.105$ and $\varepsilon\alpha = 0.04$, or for

$$\frac{\Gamma}{\alpha} \geq \frac{0.105}{0.04} = 2.625. \tag{6.1}$$

When comparing these, note that we have chosen a breather initial condition as in [9-11, 30, 55, 63]; however, the breather parameters could not be exactly matched to those used in [9-11, 30, 55, 63] since the latter are not completely specified save the localization length and number of peaks per period). This may be considered to be strong *a posteriori* evidence that the direct approach developed in this paper in treating the evolution of coherent structure solutions under forcing (or other stresses or perturbations) is accurate. In particular, this does indicate that the key assumption which was made in this treatment, i.e., coherent traveling wave pulse and front structures retaining their traveling wave forms well into the forced regime, is indeed valid. In addition, they control the dynamics and are not disrupted by resonance effects [14, 69].

Finally, let us conclude by making some remarks regarding possible extensions of this work which are in progress. We have considered thresholds for the onset of chaos in this paper. However, the consideration of the actual dynamics prior to the onset of chaos is something that is worth studying further analytically and checking against existing numerical studies of forced NLPDEs. Possible approaches that could be tried include multiscale perturbation theory (for forcings which are fast compared to the intrinsic system timescale) [6, 53], or direct soliton perturbation theory (see [7, 72, 73] and the references therein for instance for recent reviews) for forced integrable systems. Another fruitful approach may be to consider both static and dynamic (Hopf or Flutter) bifurcations in the pre-chaotic regime via the intrinsic Harmonic Balance Method developed by Huseyin [39]. It is also conceivable that resummation of these perturbation series [4, 64] may enable one to analytically follow the homoclinic or heteroclinic tangling beyond the first transversal intersection of the stable and unstable manifolds and into the chaotic regime (see [64] for instance). One other simple possibility, although unlikely to yield results correlated to those for the actual forced NLPDE and hence of questionable value, would be to integrate the forced traveling wave reduced ODEs numerically, and study the results using the standard numerical diagnostics [52].

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