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POSITIVE SOLUTIONS OF A NONLINEAR FOUR-POINT BOUNDARY VALUE PROBLEM IN BANACH SPACES

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Abstract

In this paper, by using the fixed points of strict-set-contractions, we study the existence of at least one or two positive solutions to the nonlinear four-point boundary value problem

$$(p(t)\,y'(t))' - q(t)\,y(t) + a(t)\,f(y(t)) = \theta, \quad 0 < t < 1,$$

$$\alpha y(0) - \beta p(0)\,y'(0) = \mu_1 y(\xi), \quad \gamma y(1) + \delta p(1)\,y'(1) = \mu_2 y(\eta),$$

in Banach space E, where θ is zero element of E, $0 < \xi, \eta < 1$, $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha\gamma + \alpha\delta + \beta\gamma > 0$, $\mu_1, \mu_2 > 0$. As an application, we also give one example to demonstrate our results.

1. Introduction

In the recent ten years, the theory of ordinary differential equations in Banach spaces has become a new important branch (see, for instance,

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[2, 5, 6] and references therein). On the other hand, recently, multi-point boundary value problem for scalar ordinary differential equations have been studied extensively (see, for example, [7, 8, 10] and references therein). However, to the author's knowledge, few papers can be found in the literature for the existence of positive solutions for multi-point boundary value problem in Banach space. Very recently, in [9], the first author studies the existence of positive solutions to the following four-point boundary value problem

$$y''(t) + a(t)f(y(t)) = \theta, \quad 0 < t < 1,$$

$$\alpha y(0) - \beta y'(0) = \mu_1 y(\xi), \quad \gamma y(1) + \delta y'(1) = \mu_2 y(\eta),$$

in Banach space E. So in this paper, we are interested in the existence of positive solutions of the following nonlinear four-point boundary value problem (BVP):

$$(p(t)y'(t))' - q(t)y(t) + a(t)f(y(t)) = \theta, \quad 0 < t < 1, \tag{1.1}$$

$$\alpha y(0) - \beta p(0)y'(0) = \mu_1 y(\xi), \quad \gamma y(1) + \delta p(1)y'(1) = \mu_2 y(\eta), \tag{1.2}$$

in Banach space E. For abstract space, it is here worth mentioning, Guo and Lakshmikantham [4] discuss the multiple solutions of two-point boundary value problems of ordinary differential equations in Banach spaces.

The aim of the present paper is to establish some simple criteria for the existence of at least one or two positive solutions of the BVP (1.1)-(1.2) in Banach space E. The key tool in our approach is the following fixed point theorem of strict-set-contractions [1, 11].

Theorem 1.1 [1, 11]. *Let K be a cone of the real Banach space X and $K_{r,R} = \{x \in K \mid r \leq \|x\| \leq R\}$ with $R > r > 0$. Suppose that $A : K_{r,R} \to K$ is a strict-set-contraction such that one of the following two conditions is satisfied*:

(i) $Ax \not\leq x, \forall x \in K, \|x\| = r$ *and* $Ax \not\geq x, \forall x \in K, \|x\| = R$.

(ii) $Ax \not\geq x, \forall x \in K, \|x\| = r$ *and* $Ax \not\leq x, \forall x \in K, \|x\| = R$.

Then A has at least one fixed point in $K_{r,R}$.

The paper is organized as follows. The preliminary lemmas are in Section 2. In Section 3, we discuss the existence of at least one positive solution and two positive solutions. Finally, in Section 4, we give one example to illustrate our results.

Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a cone P of E, i.e., $x \leq y$ if and only if $y - x \in P$, and P^* denotes the dual cone of P. Denote the normal constant of P by N (see [3]), i.e., $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. For arbitrary $x \in C[I, E]$, evidently, $(C[I, E], \|\cdot\|)$ is a Banach space with $\|x\|_C = \max_{t\in I}\|x(t)\|$. Clearly, $Q = \{x \in C[I, E] | x(t) \geq \theta \text{ for } t \in I\}$ is a cone of the Banach space $C[I, E]$. A function $x \in C^2[I, E]$ is called a positive solution of BVP (1.1)-(1.2) if it satisfies (1.1)-(1.2) and $x \in Q$, $x(t) \neq \theta$.

For a bounded set S in a Banach space, we denote $\alpha(s)$ the Kuratowski measure of noncompactness (see [2, 5, 6], for further understanding). In this paper, we denote $\alpha(\cdot)$ the Kuratowski measure of noncompactness of a bounded set in E and in $C[I, E]$.

From now on, we assume that $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha\gamma + \alpha\delta + \beta\gamma > 0$, $\mu_1, \mu_2 > 0$, $p \in C^1([0, 1], (0, \infty))$, $q \in C([0, 1], (0\ \infty))$. $f \in C(P, P)$, $f(\theta) = \theta$, $\sigma \in \left(0, \frac{1}{2}\right)$ be a constant, $a \in C([0, 1], [0, \infty))$ and there exists $t_0 \in [\sigma, 1-\sigma]$ such that $a(t_0) > 0$.

2. The Preliminary Lemmas

Lemma 2.1 [10]. *Let ψ and φ be the solutions of the linear problems*

$$\begin{cases} (p(t)\psi'(t))' - q(t)\psi(t) = 0, \\ \psi(0) = \beta,\ p(0)\psi'(0) = \alpha, \end{cases} \tag{2.1}$$

and

$$\begin{cases} (p(t)\varphi'(t))' - q(t)\varphi(t) = 0, \\ \varphi(1) = \delta,\ p(1)\varphi'(1) = -\gamma, \end{cases} \tag{2.2}$$

respectively. Then

(i) ψ *is strictly increasing on* $[0, 1]$, *and* $\psi(t) > 0$ *on* $(0, 1]$;

(ii) φ *is strictly decreasing on* $[0, 1]$, *and* $\varphi(t) > 0$ *on* $[0, 1)$.

Now, let ψ and φ be the solutions of the linear problems (2.1) and (2.2) respectively, for convenience sake, we set

$$\rho = p(t)\begin{vmatrix} \varphi(t) & \psi(t) \\ \varphi'(t) & \psi'(t) \end{vmatrix} \equiv p(0)\begin{vmatrix} \varphi(0) & \psi(0) \\ \varphi'(0) & \psi'(0) \end{vmatrix}, \quad t \in [0, 1]$$

$$\Delta = \begin{vmatrix} -\mu_1\psi(\xi) & \rho - \mu_1\varphi(\xi) \\ \rho - \mu_2\psi(\eta) & -\mu_2\varphi(\eta) \end{vmatrix},$$

$$G(t, s) = \frac{1}{\rho}\begin{cases} \varphi(t)\psi(s), & 0 \le s \le t \le 1, \\ \varphi(s)\psi(t), & 0 \le t \le s \le 1, \end{cases}$$

$$\Lambda_0 = \min\left\{\frac{\varphi(1-\sigma)}{\varphi(0)}, \frac{\psi(\sigma)}{\psi(1)}\right\}, \quad \Lambda_1 = \min\left\{\Lambda_0, \frac{\Lambda_2}{\Lambda_3}\right\},$$

$$\Lambda_2 = \{\min_{\sigma \le t \le 1-\sigma} \psi(t), \min_{\sigma \le t \le 1-\sigma} \varphi(t), 1\}, \quad \Lambda_3 = \max\{1, \|\varphi\|_C, \|\psi\|_C\},$$

$$\Lambda_4 = \frac{\Lambda_3^2}{\rho}\cdot\left\{1 + \frac{\mu_1\mu_2\varphi(\eta)\Lambda_3}{-\Delta} + \frac{\mu_1\mu_2\psi(\xi)\Lambda_3}{-\Delta} + \frac{\mu_1(\rho - \mu_2\psi(\eta))\Lambda_3}{-\Delta} + \frac{\mu_2(\rho - \mu_1\varphi(\xi))\Lambda_3}{-\Delta}\right\},$$

$$A(h) = -\frac{1}{\Delta}\left[\mu_1\mu_2\varphi(\eta)\int_0^1 G(\xi, s)h(s)ds + \mu_2(\rho - \mu_1\varphi(\xi))\int_0^1 G(\eta, s)h(s)ds\right],$$

and

$$B(h) = -\frac{1}{\Delta}\left[\mu_1\mu_2\psi(\xi)\int_0^1 G(\eta, s)h(s)ds + \mu_1(\rho - \mu_2\psi(\eta))\int_0^1 G(\xi, s)h(s)ds\right],$$

where $h \in C[I, E]$.

Lemma 2.2. *Let* $\Delta \ne 0$. *Then for* $h \in C[I, E]$, *the problem*

$$(p(t)y'(t))' - q(t)y(t) + h(t) = \theta, \quad 0 < t < 1, \tag{2.3}$$

$$\alpha y(0) - \beta p(0)y'(0) = \mu_1 y(\xi),\ \gamma y(1) + \delta p(1)y'(1) = \mu_2 y(\eta), \tag{2.4}$$

has a unique solution

$$y(t) = \int_0^1 G(t, s)h(s)ds + A(h)\psi(t) + B(h)\varphi(t).$$

Proof. The proof of this lemma is easy, so we omit it. □

Lemma 2.3. *Let* $\Delta \neq 0$. *Then*

$$0 \le G(t, s) \le G(s, s), \quad t, s \in I, \tag{2.5}$$

and

$$G(t, s) \ge \Lambda_0 G(s, s), \quad t \in [\sigma, 1-\sigma], \quad s \in I. \tag{2.6}$$

Proof. The inequality (2.5) is obvious. In following, we are going to verify the inequality (2.6). Indeed, when $t \in [\sigma, 1-\sigma]$, we have

$$\begin{aligned}\frac{G(t, s)}{G(s, s)} &= \begin{cases} \dfrac{\varphi(t)}{\varphi(s)}, & 0 \le s \le t \le 1-\sigma, \\ \dfrac{\psi(t)}{\psi(s)}, & \sigma \le t \le s \le 1, \end{cases} \\ &\ge \begin{cases} \dfrac{\varphi(1-\sigma)}{\varphi(0)}, & 0 \le s \le t \le 1-\sigma, \\ \dfrac{\psi(\sigma)}{\psi(1)}, & \sigma \le t \le s \le 1, \end{cases} \\ &\ge \Lambda_0.\end{aligned}$$

This completes the proof. □

In the rest of the paper, we assume that

(H) $\rho > \max\{\mu_1\varphi(\xi), \mu_2\psi(\eta)\} > 0$ and $\Delta < 0$.

Lemma 2.4. *Let* (H) *hold. If* $h \in Q$, *then the unique solution* y *of the problem* (2.3)-(2.4) *satisfies* $y(t) \ge \theta$, $t \in I$, *that is,* $y \in Q$.

Proof. In view of $G(t, s) > 0$, $A(h) \ge \theta$, $B(h) \ge \theta$, and Lemma 2.2, we have $y(t) \ge \theta$, $t \in I$. □

Lemma 2.5. *Let* (H) *hold. If* $h \in Q$, *then the unique solution* y *of the problem* (2.3)-(2.4) *satisfies*

$$y(t) \geq \Lambda_1 y(s), \quad \forall t \in [\sigma, 1-\sigma], \quad \forall s \in I.$$

Proof. Since y is the unique solution of the problem (2.3)-(2.4), from Lemmas 2.2 and 2.3, we have

$$\begin{aligned} y(s) &\leq \int_0^1 G(s_1, s_1)h(s_1)ds_1 + A(h)\psi(s) + B(h)\varphi(s) \\ &\leq \int_0^1 G(s_1, s_1)h(s_1)ds_1 + \Lambda_3(A(h) + B(h)), \; s \in I. \end{aligned} \tag{2.7}$$

Thus from Lemma 2.3 and (2.7), for any $t \in [\sigma, 1-\sigma]$, we obtain

$$\begin{aligned} y(t) &= \int_0^1 G(t, s_1)h(s_1)ds_1 + A(h)\psi(t) + B(h)\varphi(t) \\ &= \int_0^1 \frac{G(t, s_1)}{G(s_1, s_1)} G(s_1, s_1)h(s_1)ds_1 + A(h)\psi(t) + B(h)\varphi(t) \\ &\geq \Lambda_0 \int_0^1 G(s_1, s_1)h(s_1)ds_1 + A(h)\psi(t) + B(h)\varphi(t) \\ &\geq \Lambda_0 \int_0^1 G(s_1, s_1)h(s_1)ds_1 + \frac{\Lambda_2}{\Lambda_3} \cdot \Lambda_3(A(h) + B(h)) \\ &\geq \Lambda_1 \left[\int_0^1 G(s_1, s_1)h(s_1)ds_1 + \Lambda_3(A(h) + B(h)) \right] \\ &\geq \Lambda_1 y(s), \quad s \in I. \end{aligned}$$

This completes the proof. □

Now from Lemma 2.2, it is easy to see that the BVP (1.1)-(1.2) has a solution $y = y(t)$ if and only if y is a solution of the operator equation

$$\begin{aligned} y(t) &= \int_0^1 G(t, s)a(s)f(y(s))ds + A(a(\cdot)f(y(\cdot)))\psi(t) + B(a(\cdot)f(y(\cdot)))\varphi(t) \\ &\overset{\Delta}{=} (Ty)(t). \end{aligned} \tag{2.8}$$

In the following, the closed balls in spaces E and $C[I, E]$ are denoted by $G_r = \{x \in E \mid \|x\| \leq r\} (r > 0)$ and $B_r = \{x \in C[I, E] \mid \|x\|_C \leq r\} (r > 0)$, respectively.

Lemma 2.6. *Let* (H) *hold. Suppose that, for any* $r > 0$, f *is uniformly continuous and bounded on* $P \cap G_r$ *and there exists a constant* L_r *with* $0 \le L_r < [2\Lambda_4 \max_{s\in I} a(s)]^{-1}$ *such that*

$$\alpha(f(D)) \le L_r \alpha(D), \quad \forall D \subset P \cap T_r. \tag{2.9}$$

Then, for any $r > 0$, *operator* T *is a strict-set-contraction on* $D \subset P \cap T_r$.

Proof. Since f is uniformly continuous and bounded on $P \cap G_r$, we see from (2.8) that T is continuous and bounded on $Q \cap B_r$. Now, let $S \subset Q \cap B_r$ be arbitrary given. By virtue of (2.8), it is easy to show that the functions $\{Ty \mid y \in S\}$ are uniformly bounded and equicontinuous, and so by [6],

$$\alpha(T(s)) = \sup_{t\in I} \alpha(T(S(t))), \tag{2.10}$$

where $T(S(t)) = \{Ty(t) \mid y \in S,\ t \text{ is fixed}\} \subset P \cap G_r$, for any $t \in I$. Using the obvious formula $\int_0^1 y(t)dt \in \overline{co}\{y(t) | t \in I\}$ for any $y \in C[I, E]$, we find

$$\begin{aligned}
\alpha(T(S(t))) &= \alpha\left(\int_0^1 G(t, s)a(s)f(y(s))ds + A(a(\cdot)f(y(\cdot)))\psi(t) + B(a(\cdot)f(y(\cdot)))\varphi(t)\right) \\
&\le \alpha(\overline{co}\{G(t, s)a(s)f(y(s)) \mid s \in I,\ y \in S\}) \\
&\quad + \frac{\mu_1\mu_2\varphi(\eta)\psi(t)}{-\Delta}\alpha(\overline{co}\{G(\xi, s)a(s)f(y(s)) \mid s \in I,\ y \in S\}) \\
&\quad + \frac{\mu_2(\rho - \mu_1\varphi(\xi))\psi(t)}{-\Delta}\alpha(\overline{co}\{G(\eta, s)a(s)f(y(s)) \mid s \in I,\ y \in S\}) \\
&\quad + \frac{\mu_1\mu_2\psi(\xi)\varphi(t)}{-\Delta}\alpha(\overline{co}\{G(\eta, s)a(s)f(y(s)) \mid s \in I,\ y \in S\}) \\
&\quad + \frac{\mu_1(\rho - \mu_2\psi(\eta))\varphi(t)}{-\Delta}\alpha(\overline{co}\{G(\xi, s)a(s)f(y(s)) \mid s \in I,\ y \in S\}) \\
&\le \frac{\Lambda_3^2}{\rho}\cdot\max_{s\in I} a(s)\cdot\alpha(\{f(y(s)) \mid s \in I,\ y \in S\}) \\
&\quad + \frac{\mu_1\mu_2\varphi(\eta)\Lambda_3^3}{-\Delta\rho}\cdot\max_{s\in I} a(s)\cdot\alpha(\{f(y(s)) \mid s \in I,\ y \in S\})
\end{aligned}$$

$$+\frac{\mu_2(\rho-\mu_1\varphi(\xi))\Lambda_3^3}{-\Delta\rho}\cdot\max_{s\in I}a(s)\cdot\alpha(\{f(y(s))\,|\,s\in I,\ y\in S\})$$

$$+\frac{\mu_1\mu_2\psi(\xi)\Lambda_3^3}{-\Delta\rho}\cdot\max_{s\in I}a(s)\cdot\alpha(\{f(y(s))\,|\,s\in I,\ y\in S\})$$

$$+\frac{\mu_1(\rho-\mu_2\psi(\eta))\Lambda_3^3}{-\Delta\rho}\cdot\max_{s\in I}a(s)\cdot\alpha(\{f(y(s))\,|\,s\in I,\ y\in S\})$$

$$=\Lambda_4\cdot\max_{s\in I}a(s)\cdot\alpha(\{f(y(s))\,|\,s\in I,\ y\in S\})$$

$$=\Lambda_4\cdot\max_{s\in I}a(s)\cdot\alpha(f(B))$$

$$\leq\Lambda_4L_r\cdot\max_{s\in I}a(s)\cdot\alpha(B), \tag{2.11}$$

where $B=\{y(s)\,|\,s\in I,\ y\in S\}\subset P\cap G_r$. For any given $\varepsilon>0$, there exists a partition $S=\bigcup_{j=1}^{n}S_j$ with

$$diam(S_j)<\alpha(S)+\frac{\varepsilon}{3},\quad j=1,2,\ldots,n. \tag{2.12}$$

Now, choose $y_j\in S_j$ $(j=1,2,\ldots,n)$ and a partition $0=t_0<t_1<\cdots<t_i<\cdots<t_m=1$ such that

$$\|y_j(t)-y_j(\bar{t})\|<\frac{\varepsilon}{3},\quad\forall j=1,2,\ldots,n;\ t,\bar{t}\in[t_{i-1},t_i],\ i=1,2,\ldots,m. \tag{2.13}$$

Clearly, $B=\bigcup_{i=1}^{m}\bigcup_{j=1}^{n}B_{ij}$, where $B_{ij}=\{y(t)\,|\,t\in[t_{i-1},t_i],\ y\in S_j\}$. For any two $y(t)$, $\bar{y}(\bar{t})\in B_{ij}(t,\bar{t}\in[t_{i-1},t_i],\ y,\bar{y}\in S_j)$, we have, by (2.12) and (2.13),

$$\begin{aligned}\|y(t)-\bar{y}(\bar{t})\|&\leq\|y(t)-y_j(t)\|+\|y_j(t)-y_j(\bar{t})\|+\|y_j(\bar{t})-\bar{y}(\bar{t})\|\\&\leq\|y-y_j\|_C+\frac{\varepsilon}{3}+\|y_j-\bar{y}\|_C\\&\leq 2\cdot diam(S_j)+\frac{\varepsilon}{3}<2\alpha(S)+\varepsilon,\end{aligned}$$

which implies $diam(B_{ij})\leq 2\alpha(S)+\varepsilon$, and so $\alpha(B)\leq 2\alpha(S)+\varepsilon$. Since ε is arbitrary, we get

$$\alpha(B) \leq 2\alpha(S). \tag{2.14}$$

It follows from (2.10), (2.11) and (2.14) that

$$\alpha(T(S)) \leq 2\Lambda_4 L_r \cdot \max_{s \in I} a(s) \cdot \alpha(S), \quad \forall S \subset Q \cap B_r,$$

and consequently T is a strict-set-contraction on $S \subset Q \cap B_r$ because of

$$2\Lambda_4 L_r \cdot \max_{s \in I} a(s) < 1.$$

3. Main Theorems

In the following for convenience, for any $x \in P$ and $\phi \in P^*$, we set

$$f_0 = \lim_{\| x \| \to 0} \frac{\| f(x) \|}{\| x \|}, \quad f_\infty = \lim_{\| x \| \to \infty} \frac{\| f(x) \|}{\| x \|},$$

$$f_0^\phi = \lim_{\| x \| \to 0} \frac{\phi(f(x))}{\phi(x)}, \quad f_\infty^\phi = \lim_{\| x \| \to \infty} \frac{\phi(f(x))}{\phi(x)},$$

and list some conditions:

(C_1) For any $r > 0$, f is uniformly continuous and bounded on $P \cap G_r$ and there exists a constant L_r with $0 \leq L_r < [2\Lambda_4 \max_{s \in I} a(s)]^{-1}$ such that

$$\alpha(f(D)) \leq L_r \alpha(D), \quad \forall D \subset P \cap T_r.$$

(C_2) There exists $\phi \in P^*$ such that $\phi(x) > 0$ for any $x > \theta$ and $f_0^\phi = \infty$.

(C_3) There exists $\phi \in P^*$ such that $\phi(x) > 0$ for any $x > \theta$ and $f_\infty^\phi = \infty$.

(C_4) There exists $r_0 > 0$ such that

$$\sup_{x \in P \cap G_{r_0}} \| f(x) \| < \frac{r_0}{N\Lambda_4 \int_0^1 a(s) ds}.$$

Theorem 3.1. *Let* (H) *hold, cone P be normal and condition* (C_1) *be satisfied. If conditions* $f_0 = 0$ *and* (C_3) *or* $f_\infty = 0$ *and* (C_2) *are satisfied, then the BVP* (1.1)-(1.2) *has at least one positive solution.*

Proof. Set

$$K = \{y \in Q \mid y(t) \geq \Lambda_1 y(s), \quad \forall t \in [\sigma, 1-\sigma], \forall s \in I\}. \tag{3.1}$$

It is clear that K is a cone of the Banach space $C[I, E]$ and $K \subset Q$. By Lemmas 2.4 and 2.5, we know $T(Q) \subset K$, and so

$$T(K) \subset K. \tag{3.2}$$

We first assume that $f_0 = 0$ and (C_3) are satisfied. Choose $M > \left(\Lambda_1 \int_\sigma^{1-\sigma} a(s)G(t_0, s)ds\right)^{-1}$, by (C_3) there exists $r_1 > 0$ such that

$$\phi(f(x)) \geq M\phi(x), \quad \forall x \in P, \| x \| \geq r_1. \tag{3.3}$$

Now for any

$$R > \frac{Nr_1}{\Lambda_1}, \tag{3.4}$$

we are going to verify that

$$Ty \not\leq y, \quad \forall y \in K, \| y \|_C = R. \tag{3.5}$$

Indeed, if there exists $y_0 \in K$ with $\| y_0 \|_C = R$ such that $Ty_0 \leq y_0$. Then from Lemma 2.5

$$y_0(t) \geq \Lambda_1 y_0(s), \quad \forall t \in [\sigma, 1-\sigma], \forall s \in I,$$

and so

$$N\| y_0(t) \| \geq \Lambda_1 \| y_0(s) \|, \quad \forall t \in [\sigma, 1-\sigma], \forall s \in I,$$

which implies, by (3.4),

$$\min_{t \in [\sigma, 1-\sigma]} \| y_0(t) \| \geq \frac{\Lambda_1}{N} \| y_0 \|_C = \frac{\Lambda_1}{N} R > r_1. \tag{3.6}$$

Then from (2.8), (3.6), (3.3) and Lemma 2.5, we get

$$\begin{aligned} \phi(y_0(t_0)) &\geq \phi(Ty_0(t_0)) \\ &= \int_0^1 G(t_0, s)a(s)\phi(f(y_0(s)))ds + A(a(\cdot)\phi(f(y_0(\cdot))))\psi(t_0) \\ &\quad + B(a(\cdot)\phi(f(y_0(\cdot))))\varphi(t_0) \end{aligned}$$

$$\geq \int_0^1 G(t_0, s)a(s)\phi(f(y_0(s)))ds$$

$$\geq \int_\sigma^{1-\sigma} G(t_0, s)a(s) \cdot \phi(f(y_0(s)))ds$$

$$\geq M\int_\sigma^{1-\sigma} G(t_0, s)a(s) \cdot \phi(y_0(s))ds$$

$$\geq \Lambda_1 M\int_\sigma^{1-\sigma} G(t_0, s)a(s)ds \cdot \phi(y_0(t_0)).$$

This is

$$\phi(y_0(t_0)) \geq \Lambda_1 M\int_\sigma^{1-\sigma} G(t_0, s)a(s)ds \cdot \phi(y_0(t_0)). \tag{3.7}$$

Again, it is easy to see that

$$\phi(y_0(t_0)) > 0. \tag{3.8}$$

In fact, if $\phi(y_0(t_0)) = 0$, since $y_0 \in K$, $0 = \phi(y_0(t_0)) \geq \Lambda_1\phi(y_0(s)) \geq 0$ implies $\phi(y_0(s)) = 0$, $\forall s \in I$, so we have $y_0(s) \equiv \theta$ for any $s \in I$, and consequently $\| y_0 \|_C = 0$, in contradiction with $\| y_0 \|_C = R$. Now, by virtue of (3.7) and (3.8), we find $\Lambda_1 M\int_\sigma^{1-\sigma} G(t_0, s)a(s)ds \leq 1$, which contradicts $M > \left(\Lambda_1\int_\sigma^{1-\sigma} a(s)G(t_0, s)ds\right)^{-1}$, and therefore (3.5) is true.

On the other hand, since $f_0 = 0$ and $f(\theta) = \theta$, for any $\varepsilon \in \left(0, \left(N\Lambda_4\int_0^1 a(s)ds\right)^{-1}\right)$, there exists $r_2 \in (0, R)$ such that

$$\| f(x) \| \leq \varepsilon\| x \|, \quad \forall x \in P, \quad \| x \| < r_2. \tag{3.9}$$

We now prove that for any $r \in (0, r_2)$,

$$Ty \not\geq y, \quad \forall y \in K, \quad \| y \|_C = r. \tag{3.10}$$

In fact, if there exists $y_1 \in K$ with $\| y_1 \|_C = r$ such that $Ty_1 \geq y_1$, then from (2.8), we get

$$\theta \le y_1(t) \le Ty_1(t)$$

$$= \int_0^1 G(t, s)a(s)f(y_1(s))ds + A(a(\cdot)f(y_1(\cdot)))\psi(t) + B(a(\cdot)f(y_1(\cdot)))\varphi(t)$$

$$\le \frac{\Lambda_3^2}{\rho}\int_0^1 a(s)f(y_1(s))ds + \frac{\mu_1\mu_2\varphi(\eta)\Lambda_3^3}{-\Delta\rho}\int_0^1 a(s)f(y_1(s))ds$$

$$+ \frac{\mu_2(\rho - \mu_1\varphi(\xi))\Lambda_3^3}{-\Delta\rho}\int_0^1 a(s)f(y_1(s))ds$$

$$+ \frac{\mu_1\mu_2\psi(\xi)\Lambda_3^3}{-\Delta\rho}\int_0^1 a(s)f(y_1(s))ds$$

$$+ \frac{\mu_1(\rho - \mu_2\psi(\eta))\Lambda_3^3}{-\Delta\rho}\int_0^1 a(s)f(y_1(s))ds$$

$$= \Lambda_4\int_0^1 a(s)f(y_1(s))ds, \quad \forall t \in I.$$

Hence, by virtue of (3.9) and the cone P is normal, we have

$$\| y_1(t) \| \le N\Lambda_4\int_0^1 a(s)\| f(y_1(s)) \| ds$$

$$\le N\varepsilon\Lambda_4\int_0^1 a(s)\| y_1(s) \| ds$$

$$\le N\varepsilon\Lambda_4\int_0^1 a(s)ds \cdot r, \quad \forall t \in I,$$

and so

$$\| y_1 \|_C \le N\varepsilon\Lambda_4\int_0^1 a(s)ds \cdot r < r,$$

which contradicts with $\| y_1 \|_C = r$. Thus, (3.10) is true.

By Lemma 2.6, T is a strict-set-contraction on $K_{r,R} = \{y \in K \mid r \le \| y \|_C \le R\}$. Observing (3.2), (3.5), (3.10) and using Theorem 1.1, we see that T has a fixed point on $K_{r,R}$, which is a positive solution of BVP (1.1)-(1.2).

Next, in case when $f_\infty = 0$ and (C_2) are satisfied, the proof is similar. In the same way as establishing (3.5) we can assert from (C_2) that there exists $r_2 > 0$ such that for any $0 < r < r_2$,

$$Ty \not\leq y, \quad \forall y \in K, \quad \| y \|_C = r. \tag{3.11}$$

On the other hand, since $f_\infty = 0$, for any $\varepsilon \in \left(0, \left(N\Lambda_4 \int_0^1 a(s)ds\right)^{-1}\right)$, there exists $l > 0$ such that

$$\| f(x) \| \leq \varepsilon \| x \|, \quad \forall x \in P, \quad \| x \| \geq l. \tag{3.12}$$

Also, by (C_1),

$$\sup_{x \in P \cap G_l} \| f(x) \| = b < \infty. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\| f(x) \| \leq \varepsilon \| x \| + b, \quad \forall x \in P. \tag{3.14}$$

Taking $R > \max\left\{ r_2, \dfrac{bN\Lambda_4 \int_0^1 a(s)ds}{1 - \varepsilon N\Lambda_4 \int_0^1 a(s)ds} \right\}$, we now prove that

$$Ty \not\geq y, \quad \forall y \in K, \quad \| y \|_C = R. \tag{3.15}$$

In fact, if there exists $y_2 \in K$ with $\| y_2 \|_C = R$ such that $Ty_2 \geq y_2$, then from (2.8), we get

$$\theta \leq y_2(t) \leq Ty_2(t) = \Lambda_4 \int_0^1 a(s) f(y_2(s)) ds, \quad \forall t \in I.$$

Hence, by virtue of (3.14) and the cone P is normal, we have

$$\begin{aligned} \| y_2(t) \| &\leq N\Lambda_4 \int_0^1 a(s) \| f(y_2(s)) \| ds \\ &\leq N\Lambda_4 \int_0^1 a(s)(\varepsilon \| y_2(s) \| + b) ds \\ &\leq N\Lambda_4 \int_0^1 a(s)ds \cdot (\varepsilon \| y_2 \|_C + b), \quad \forall t \in I, \end{aligned}$$

and so

$$\| y_2 \|_C \leq N\Lambda_4 \int_0^1 a(s)ds \cdot (\varepsilon R + b) < R,$$

which contradicts with $\| y_2 \|_C = R$. Thus, (3.15) is true.

By Lemma 2.6, T is a strict-set-contraction on $K_{r,R} = \{y \in K \mid r \leq \| y \|_C \leq R\}$. Observing (3.2), (3.11), (3.15) and using Theorem 1.1, we see that T has a fixed point on $K_{r,R}$, which is a positive solution of BVP (1.1)-(1.2). □

Theorem 3.2. *Let* (H) *hold and cone* P *be normal. Suppose that conditions* (C_1)-(C_4) *are satisfied. Then the BVP* (1.1)-(1.2) *has at least two positive solutions* y_1 *and* y_2 *which satisfy*

$$0 < \| y_2 \|_C < r_0 < \| y_1 \|_C. \tag{3.16}$$

Proof. Taking the same cone $K \subset C[I, E]$ as in Theorem 3.1. As in the proof of Theorem 3.1, we can show that

$$T(K) \subset K, \tag{3.17}$$

and we can choose r, R with $R > r_0 > r > 0$ such that

$$Ty \not\leq y, \quad \forall y \in K, \quad \| y \|_C = R, \tag{3.18}$$

$$Ty \not\leq y, \quad \forall y \in K, \quad \| y \|_C = r. \tag{3.19}$$

On the other hand, it is easy to see that

$$Ty \not\geq y, \quad \forall y \in K, \quad \| y \|_C = r_0. \tag{3.20}$$

In fact, if there exists $y_0 \in K$ with $\| y_0 \|_C = r_0$ such that $Ty_0 \geq y_0$, then from (2.8), we get

$$\theta \leq y_0(t) \leq Ty_0(t) = \Lambda_4 \int_0^1 a(s) f(y_0(s))ds, \quad \forall t \in I.$$

Hence, by virtue of (C_4) and the cone P is normal, we have

$$\| y_0(t) \| \leq N\Lambda_4 \int_0^1 a(s) \| f(y_0(s)) \| ds$$

$$\leq N\Lambda_4 \int_0^1 a(s)ds \cdot \sup_{x \in P \cap G_{r_0}} \| f(x) \|$$

$$< r_0, \quad \forall t \in I,$$

and so $\| y_0 \|_C < r_0$, in contradiction with $\| y_0 \|_C = r_0$. Thus, (3.20) is true.

By Lemma 2.6, T is a strict-set-contraction on $K_{r_0, R} = \{y \in K \mid r_0 \leq \| y \|_C \leq R\}$, and also on $K_{r, r_0} = \{y \in K \mid r \leq \| y \|_C \leq r_0\}$. Now observing (3.17)-(3.20) and applying Theorem 1.1 to T, $K_{r_0, R}$ and T, K_{r, r_0}, respectively, we assert that there exists $y_1 \in K_{r_0, R}$ and $y_2 \in K_{r, r_0}$ such that $Ty_1 = y_1$ and $Ty_2 = y_2$, which are two positive solutions of BVP (1.1)-(1.2). Finally, (3.20) implies $\| y_1 \| \neq r_0$, $\| y_2 \| \neq r_0$, and so (3.16) holds. □

4. One Example

In this section, in order to illustrate our results, we consider an example.

Example 4.1. Consider the boundary value problem in $E = R^n$ (n-dimensional Euclidean space and $\| x \| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$)

$$((1 + t)x_i'(t))' - \frac{1}{1 + t} x_i(t) + a(t) f_i(x_1, x_2, \ldots, x_n) = 0, \quad 0 < t < 1, \tag{4.1}$$

$$x_i(0) - x_i'(0) = \frac{1}{2} x_i\left(\frac{1}{2}\right), \quad \frac{1}{2} x_i(1) + x_i'(1) = x_i\left(\frac{1}{4}\right), \quad i = 1, 2, \ldots, n, \tag{4.2}$$

where $a(t) = \dfrac{(1 - t)^4}{10(n + 1)\sqrt{n}}$, and

$$f_i(x_1, x_2, \ldots, x_n) = \sqrt{x_{i+1}} + x_{i+2}^2, \quad i = 1, 2, \ldots, n - 2,$$

$$f_{n-1}(x_1, x_2, \ldots, x_n) = \sqrt{x_n} + x_1^2, \quad f_n(x_1, x_2, \ldots, x_n) = \sqrt{x_1} + x_2^2.$$

Set $\theta = (0, 0, \ldots, 0) \in R^n$, $\alpha = \beta = 1$, $\gamma = \delta = \frac{1}{2}$, $\mu_1 = \frac{1}{2}$, $\mu_2 = 1$, $\xi = \frac{1}{2}$, $\eta = \frac{1}{4}$, $p(t) = 1 + t$, $q(t) = \frac{1}{1+t}$, and $f = (f_1, f_2, \ldots, f_n)$, then

$$p(0) = 1, \quad p(1) = 2, \quad \psi(t) = 1 + t, \quad \varphi(t) = \frac{1}{1+t}, \quad \rho = 2,$$

$$\Lambda_3 = \max\{1, \|\varphi\|_C, \|\psi\|_C\} = \max\{1, 1, 2\} = 2, \quad \Delta = \begin{vmatrix} -\frac{3}{4} & \frac{5}{3} \\ \frac{3}{4} & -\frac{4}{5} \end{vmatrix} = -\frac{13}{20},$$

$$\Lambda_4 = \frac{\Lambda_3^2}{\rho} \cdot \left\{1 + \frac{\mu_1\mu_2\varphi(\eta)\Lambda_3}{-\Delta} + \frac{\mu_1\mu_2\phi(\xi)\Lambda_3}{-\Delta} + \frac{\mu_1(\rho - \mu_2\phi(\eta))\Lambda_3}{-\Delta}\right.$$
$$\left. + \frac{\mu_2(\rho - \mu_1\varphi(\xi))\Lambda_3}{-\Delta}\right\} = \frac{844}{39}.$$

Taking

$$P = \{x = (x_1, x_2, \ldots, x_n) \in R^n \mid x_i \geq 0, i = 1, 2, \ldots, n\},$$

then P is a normal cone and normal constant $N = 1$. $f : P \to P$ is continuous, $f(0, 0, \ldots, 0) = 0$, and the condition (H) holds. Moreover, in this case, condition (C_1) is automatically satisfied since $\alpha(f(D))$ is identical to zero for any $D \subset P \cap T_r$. It is clear that $P^* = P$ in this case, so we choose $\phi = (1, 1, \ldots, 1)$, and then

$$\frac{\phi(f(x))}{\phi(x)} = \frac{\sum_{i=1}^{n} f_i(x_1, x_2, \ldots, x_n)}{\sum_{i=1}^{n} x_i}.$$

We now prove that the conditions (C_2) and (C_3) are satisfied. In fact, for any $x \in P$, $x \neq \theta$, we have $\phi(x) > 0$ and

$$\frac{\phi(f(x))}{\phi(x)} = \frac{\sum_{i=1}^{n}(\sqrt{x_i} + x_i^2)}{\sum_{i=1}^{n} x_i} \geq \frac{\sum_{i=1}^{n}\sqrt{x_i}}{\sum_{i=1}^{n} x_i} \geq \frac{\max_{1\leq i\leq n}\sqrt{x_i}}{n\max_{1\leq i\leq n} x_i}$$

$$= \frac{1}{n} \cdot \frac{1}{\max_{1 \le i \le n} \sqrt{x_i}} \to \infty, \left(\because \| x \| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \to 0 \right),$$

and

$$\frac{\phi(f(x))}{\phi(x)} = \frac{\sum_{i=1}^{n} (\sqrt{x_i} + x_i^2)}{\sum_{i=1}^{n} x_i} \ge \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i} \ge \frac{\max_{1 \le i \le n} x_i^2}{n \max_{1 \le i \le n} x_i}$$

$$= \frac{1}{n} \cdot \max_{1 \le i \le n} x_i \to \infty, \left(\because \| x \| = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \to \infty \right),$$

so the conditions (C_2) and (C_3) hold. Finally, we are going to verify that (C_4) is satisfied. Indeed,

$$\Lambda_4 \int_0^1 a(s) ds < \frac{17}{39(n+1)\sqrt{n}} < \frac{1}{2(n+1)\sqrt{n}}.$$

Again, taking $r_0 = 1$, since $G_{r_0} = \{x \in R^n \,|\, \| x \| \le 1\}$, $0 \le \sum_{i=1}^{n} x_i^2 \le 1$ and $0 \le \sqrt{x_i} \le 1$ $(i = 1, 2, \dots, n)$ for any $x \in P \cap G_{r_0}$. Hence (notice $N = 1$, $r_0 = 1$)

$$\sup_{x \in P \cap G_{r_0}} \| f(x) \| = \sup_{x \in P \cap G_{r_0}} \left(\sum_{i=1}^{n} f_i^2 \right)^{\frac{1}{2}} \le \sqrt{n} \cdot \sup_{x \in P \cap G_{r_0}} \left(\max_{1 \le i \le n} f_i \right)$$

$$\le \sqrt{n} \cdot \sup_{x \in P \cap G_{r_0}} \left(\sum_{i=1}^{n} f_i \right) = \sqrt{n} \cdot \sup_{x \in P \cap G_{r_0}} \left(\sum_{i=1}^{n} \sqrt{x_i} + \sum_{i=1}^{n} x_i^2 \right)$$

$$\le (n+1)\sqrt{n} < 2(n+1)\sqrt{n} < \frac{r_0}{N \Lambda_4 \int_0^1 a(s) ds},$$

which implies the condition (C_4) holds. Hence, by Theorem 3.2, the BVP (4.1)-(4.2) has at least two positive solutions $x, y \in C^2[I, R^n]$ such that

$$0 < \max_{t \in I} \sum_{i=1}^{n} x_i^2(t) < 1 < \max_{t \in I} \sum_{i=1}^{n} y_i^2(t).$$

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