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HIGHER PROLONGATIONS AND DISPERSIVENESS IN DYNAMICAL SYSTEMS

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Abstract

In this paper, we discuss prolongation and prolongational limit set. Let X be a locally compact metric space, for a point $x \in X$, we get some properties of the higher positive prolongation $D_\alpha^+(x)$ and higher positive prolongational limit set $J_\alpha^+(x)$. It is shown that if $z \in D_\alpha^+(y)$ and $y \in D_\beta^+(x)$, then there is an ordinal η such that $z \in D_\eta^+(x)$. For a set $M \subset X$, we also discuss two positive prolongations $D_\alpha^+(M)$, $D_u^+(M)$ and two positive prolongational limit sets $J_\alpha^+(M)$, $J_u^+(M)$, and get some relations among them. We also obtain some results about the connectedness of prolongation and prolongational limit set, and a theorem of stability. At last, we discuss dispersive concepts and orbit space.

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1. Introduction

Let (X, d) be a locally compact metric space with metric d, on which there is a flow $\pi : X \times R \to X$. The image $\pi(x, t)$ of a point (x, t) in $X \times R$ will be written simply as xt. If $M \subset X$, $A \subset R$, then MA is the set $\{xt : x \subset M, t \subset A\}$. For any $x \in X$, the set $\gamma^+(x) = xR^+$ is called the *positive semi-trajectory* through x. For a set $A \subset X$, $\overline{A}$, ∂A are respectively denote the closure and boundary of A, and we set $K^+(x) = \overline{\gamma^+(x)}$.

For any $x \in X$, the sets $\omega(x) = \{y \in X$: there is a sequence $\{t_n\}$ in R with $t_n \to +\infty$ and $xt_n \to y\}$, $J^+(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+ such that $x_n \to x$, $t_n \to +\infty$, and $x_n t_n \to y\}$, $D^+(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+ such that $x_n \to x$ and $x_n t_n \to y\}$ are respectively called the *positive limit set*, *positive prolongational limit set* and *positive prolongational* of x. Note that $\omega(x)$, $\alpha(x)$, $J^+(x)$ are closed invariant sets, $D^+(x)$ are closed and positively invariant set. And $\omega(x) \subset J^+(x)$, $D^+(x) = J^+(x) \cup \gamma^+(x)$, $K^+(x) = \gamma^+(x) \cup \omega(x)$.

2. Prolongation and Prolongational Limit Set

Let X be a locally compact metric space, 2^X be the set of all subsets of X. First, we introduct two operations $\mathcal{D}$ and δ on the class of maps from X into 2^X.

If $\Gamma : X \to 2^X$, then we define

(1) $\mathcal{D}\Gamma(x) = \{\bigcap \overline{\Gamma(U)} : U \in \mathcal{N}(x)\}$, where $\mathcal{N}(x)$ denotes the set of all neighborhoods of x.

(2) $\delta\Gamma(x) = \bigcup\{\Gamma^n(x) : n = 1, 2, \dots\}$, where $\Gamma^1(x) = \Gamma(x)$, $\Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$, $n = 2, 3, \dots$.

It is easy to see the following facts: $\mathcal{D}$ and δ are idempotent operators, i.e., $\mathcal{D}^2 = \mathcal{D}$, $\delta^2 = \delta$.

A map $\Gamma : X \to 2^X$ is called *transitive* if $\delta\Gamma = \Gamma$. If $\mathcal{D}\,\Gamma(x) = \Gamma$, then a map $\Gamma : X \to 2^X$ is called a *cluster map*. A map $\Gamma : X \to 2^X$ is called a $c - c$ *map* provided for any compact set $K \subset X$ and $x \in K$, one has either $\Gamma(x) \subset K$, or $\Gamma(x) \cap \partial K \neq \emptyset$.

Consider the map $\gamma^+(x) : X \to 2^X$ which defines the positive semi-trajectory through each point $x \in X$. We now set $\mathcal{D}\,\delta\gamma^+ \equiv D\gamma^+ = D_1^+$, and call $D_1^+(x)$ as the *first positive prolongation* of *x*. Indeed D_1^+ is a cluster map as $\mathcal{D}$ is idempotent, but it is not transitive. Therefore, we consider the map $\mathcal{D}\,\delta D_1^+$ and denote it by D_2^+ and call it as the *second prolongation* of *x*. We define a prolongation $D_\alpha^+(x)$ for any ordinal number α as follows: If α is a successor ordinal, then having defined $D_{\alpha-1}^+$, we set $D_\alpha^+ = \mathcal{D}\,\delta D_{\alpha-1}^+$. If α is not a successor ordinal, then having defined D_β^+ for every $\beta < \alpha$, we set $D_\alpha^+ = \mathcal{D} \bigcup \{\delta D_\beta^+ : \beta < \alpha\}$. It is easy to see $D^+(x) = D_1^+(x)$. $D_\alpha^+(x)$ are closed $c - c$ map for any ordinal α.

Proposition 2.1. (1) $D_\alpha^+(D_\beta^+(x))$ *is not equal to* $D_{\alpha+\beta}^+(x)$. *Specially,* $D_\alpha^+(D_\beta^+(x)) \neq D_\beta^+(D_\alpha^+(x))$. *There* $D_\alpha^+(M) = \cup\{D_\alpha^+(x) : x \in M\}$.

Example. Consider a dynamical system defined on the real line. The points of the form $\pm \frac{n}{1+n}$, $n = 0, 1, 2, \ldots$ are equilibrium points, and so are the points -1 and $+1$. Between any two successive (isolated) equilibrium points *p*, *q*, such that $p < q$, there is a single trajectory which has *q* as its positive limit point, and *p* as its negative limit point. There is a single trajectory with -1 as its sole positive limit points, and it has no negative limit point, and there is a single trajectory with $+1$ as its negative limit point, and it has no positive limit points. Moreover, between any two such successive equilibrium points, say *q* and *p*, there are two sequences of equilibrium points, say $\{p_n\}$ and $\{q_n\}$, $\cdots q_n \leq$

$q_{n-1} \cdots \leq q_1 \leq p_1 \leq p_2 \leq \cdots$, $p_n \to p$ and $q_n \to q$. Then direction of motion on a trajectory between any two equilibrium points is again from left to right. If we consider the point $P = -1$, then $D_1^+(P) = P$, $D_2^+(P) = P$, but $D_3^+(P) = [-1, 1]$, $D_4^+(P) = [-1, +\infty)$. Then $D_1^+(D_2^+(P)) = P \neq D_3^+(P)$, $D_3^+(D_1^+(P)) = D_3^+(P) \neq D_1^+(D_3^+(P)) = D_1^+([-1, 1])$.

(2) *If* $z \in D_\alpha^+(y)$, $y \in D_\beta^+(x)$, *then there is an ordinal* η *such that* $z \in D_\eta^+(x)$.

Proof. If $z \in D_\alpha^+(y)$, $y \in D_\beta^+(x)$, we let $\eta - 1 = \max\{\alpha, \beta\}$, then $z \in D_{\eta-1}^+(y)$, $y \in D_{\eta-1}^+(x)$. So

$$z \in D_{\eta-1}^+(y) \subset D_{\eta-1}^+(D_{\eta-1}^+(x)) = D_{\eta-1}^{2+}(x) \subset \mathcal{D}\,\delta D_{\eta-1}^+(x) = D_\eta^+(x).$$

The first positive prolongational limit set $x \in X$ is defined by $J_1^+(x) = \{y \in X :$ there are sequences $\{x_n\}$ in X and $\{t_n\}$ in R such that $x_n \to x$, $t_n \to +\infty$, and $x_n t_n \to y\}$. If α is any ordinal number, and J_β^+ has been defined for all $\beta < \alpha$, then we set $J_\alpha^+ = \mathcal{D}(\bigcup\{\delta J^+ : \beta < \alpha\})$. It is easy to see $J^+(x) = J_1^+(x)$.

Definition 2.1. Given any non-compact set $M \subset X$, we define

(1) $\omega(M) = \bigcup\{\omega(x) : x \in M\}$, $D_\alpha^+(M) = \bigcup\{D_\alpha^+(x) : x \in M\}$, $J_\alpha^+(M) = \bigcup\{J_\alpha^+(x) : x \in M\}$.

(2) $\Lambda(M) = \bigcap\{\overline{M[t, +\infty) : t \geq 0}\}$, $J_u^+(M) = \bigcap\{\overline{S(M, \delta)[t, +\infty)} : t \geq 0, \delta > 0\}$, $D_u^+(M) = \bigcap\{\overline{S(M, \delta)[0, +\infty)} : \delta > 0\}$, $S(M, \delta) = \{x \in X : \rho(x, M) < \varepsilon\}(\varepsilon > 0)$.

Property 2.1. Given any non-compact set $M \subset X$, it is easy to see the following facts:

(1) $\omega(M) \subset \Lambda(M) \subset J_u^+(M) \subset D_u^+(M)$, $J_1^+(M) \subset J_u^+(M)$, $D_1^+(M) \subset D_u^+(M)$.

(2) $D_\alpha^+(M) = M[0, +\infty) \cup J_\alpha^+(M)$.

If M is compact, then $D_1^+(M) = D_u^+(M)$. However, $\omega(M) \neq \Lambda(M)$ and $J_1^+(M) \neq J_u^+(M)$, even if M is compact.

Example. (1) Consider a planar flow defined by the differential equations (in polar coordinates) $\dot{r} = r(1-r)$, $\dot{\theta} = 1$. Let $M = \{r : r \leq 1\}$. Then $\Lambda(M) = M$, $\omega(M) = \{r : r = 1\} \cup \{0\}$.

(2) Consider the dynamical system, whose phase portrait is as in Figure 1, let O, P are critical points. Let M be the set $\overline{AB}$. Then $J_u^+(M)$ is the set $\overline{OP}$, $J_1^+(M) = \{O\} \cup \{P\}$.

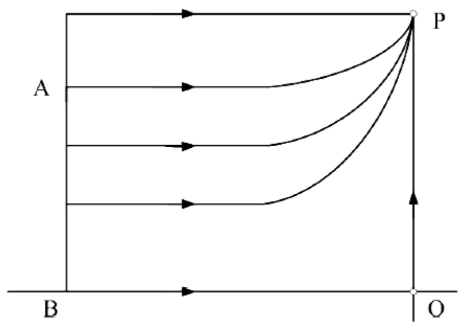

Figure 1

Theorem 2.1. *Let X be a locally compact metric space. $\Gamma(M) = \bigcup\{\Gamma(x) : x \in M\}$, where $\Gamma(x)$ is a $c-c$ map. If M is connected and $\Gamma(M)$ is compact, then $\Gamma(M)$ is connected.*

Proof. If $\Gamma(M)$ is compact, but not connected, then we can write $\Gamma(M) = M_1 \cup M_2$, where M_1, M_2 are non-empty compact disjoint sets. Since X is locally compact, we can choose compact neighborhoods U_1, U_2 of M_1, M_2 respectively such that $U_1 \cap U_2 \neq \varnothing$. Since $M \subset \Gamma(M)$ and M is connected, we have $M \subset M_1$ or $M \subset M_2$. Let $M \subset M_1$. Then there is an $x \in M \subset M_1 \subset U_1$, such that $\Gamma(x) \not\subset U_1$ and $\Gamma(x) \cap \partial U_1 = \varnothing$, contradicting the fact that $\Gamma(x)$ is a $c-c$ map. Thus $\Gamma(M)$ is connected.

Corollary 2.1. *Let X be a locally compact metric space. If M is connected and $D_\alpha^+(M)$ is compact, then $D_\alpha^+(M)$ is connected.*

Let X be a locally compact metric space, M be a non-empty compact subset of X. Then it is easy to see $\overline{M[0, +\infty)} = M[0, +\infty) \cup \Lambda(M)$. $\overline{M[0, +\infty)}$ is compact if and only if $\Lambda(M)$ is a non-empty compact set.

Lemma 2.1. *If M is non-empty and closed, $J_u^+(M)$ is non-empty and compact, then $\Lambda(M)$ is non-empty and compact. If M is not closed, then $J_u^+(M)$ is non-empty and compact, $\Lambda(M)$ may be empty.*

Proof. Let $y \in J_u^+(M)$. Then there is a $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+, such that $t_n \to +\infty$, $\rho(x_n, M) \to 0$, $x_n t_n \to y$. Since M is closed, we will have $x_n \to x \in M$, so $y \in J_1^+(x) \subset J_1^+(M) \subset J_u^+(M)$. And $J_u^+(M)$ is compact, so $J_1^+(x)$ is compact. And so $\omega(x)$ is non-empty and compact. Since $\omega(x) \subset \omega(M) \subset \Lambda(M)$ and $\Lambda(M)$ is closed, $\Lambda(M)$ is empty and compact.

Example. Consider a dynamical system whose phase portrait is as in Figure 2, and O, P are critical points. Let M be the set $\overline{AB} \setminus (\{A\} \cup \{B\})$. Then $J_u^+(M)$ is the set $\overline{QP} \setminus \{P\}$, $\Lambda(M) = \varnothing$.

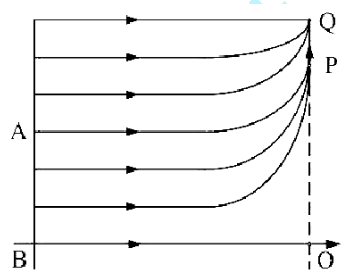

Figure 2

Lemma 2.2. *Let X be a locally compact metric space, M be a non-empty compact subset of X. Then $D_u^+(M)$ is compact if and only if $J_u^+(M)$ is non-empty and compact.*

Proof. Since $J_u^+(M)$ is non-empty and compact, $\Lambda(M)$ is non-empty and compact, then $\overline{M[0, +\infty)}$ is compact. And M is compact, then $D_u^+(M)$ $= D_1^+(M) = M[0, +\infty) \cup J_1^+(M) \subset \overline{M[0, +\infty)} \cup J_u^+(M)$, and $\overline{M[0, +\infty)} \cup$

$J_u^+(M) \subset D_u^+(M)$. So $D_u^+(M) = \overline{M[0, +\infty)} \cup J_u^+(M)$ is compact. The converse is trivial.

Theorem 2.2. *Let X be a locally compact metric space, M be a non-empty compact and connected subset of X. If $J_u^+(M)$ is compact, then it is connected.*

Proof. Let $J_u^+(M) \neq \varnothing$ and $J_u^+(M)$ be compact but disconnected. Then there are two compact non-empty sets P and Q such that $J_u^+(M) = P \cup Q$ and $P \cap Q = \varnothing$. We can see that $\Lambda(M)$ is non-empty and compact, then $\overline{M[0, +\infty)}$ is compact and $\Lambda(M)$ is connected [4, Lemma 4.1]. So $\Lambda(M) \subset P$ or $\Lambda(M) \subset Q$. Let $\Lambda(M) \subset P$. Then $M[0, +\infty) \cup \Lambda(M) \cup P = \overline{M[0, +\infty)} \cup P$ is compact. We assume that $(M[0, +\infty) \cup P) \cap Q \neq \varnothing$, then there is an $x \in M[0, +\infty) \cap Q$, Q must be invariant. This will show that $\Lambda(M) \subset Q$, a contradiction. Since M is compact, $D_u^+(M) = M[0, +\infty) \cup J_u^+(M) = M[0, +\infty) \cup P \cup Q$, $M[0, +\infty) \cup P$ and Q are disjoint compact sets, we have a contradiction to the fact that $D_u^+(M)$ is connected whenever compact. This proves the theorem.

Definition 2.2. A closed invariant set $M \subset X$ is said to be *orbital stability*, if for each point $x \in M$ and $\varepsilon > 0$, there exists a $\delta = \delta(x, \varepsilon) > 0$, such that $d(yt, M) < \varepsilon$ for all $t > 0$ whenever $d(y, x) < \delta$.

We recall that a point $x \in X$ is said to be *Lyapunov stable in positive direction* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $t \geq 0$, then $d(xt, yt) < \varepsilon$. It is easy to see that if a dynamical system is positive Lyapunov stable at a point $x \in X$, then $\omega(x) = J^+(x)$.

Theorem 2.3. *Let a dynamical system (X, R, π) is positive Lyapunov stable at each point $x \in X$, and $J^+(\omega(x)) = \omega(x)$, $\omega(x) \neq \varnothing$, then $D^+(x)$ is orbital stability.*

Proof. Let $y \in \omega(x)$. Then $y \in J^+(y)$. Since $J^+(x)$ is invariant, $\gamma(y) \subset J^+(x) \subset J^+(\omega(x))$. So $D^+(\omega(x)) = \omega(x)R^+ \cup J^+(\omega(x)) = J^+(\omega(x)) = \omega(x)$.

Since the dynamical system is positive Lyapunov stable at each point $x \in X$, $\omega(x) = J^+(x)$. So $D^+(x) = K^+(x)$. Then it is sufficient to prove that $K^+(x)$ is orbital stability whenever $D^+(\omega(x)) = \omega(x)$. We assume that $K^+(x)$ is not orbital stability, then there is a $y \in K^+(x)$, $\varepsilon > 0$, a sequence $\{x_n\} \subset X$ with $x_n \to y$ and $\{t_n\} \subset R^+$ such that $d(x_n t, K^+(x)) < \delta$ $(0 \le t \le t_n)$, $d(x_n t_n, K^+(x)) = \varepsilon$. If $y \in \gamma^+(x)$, then this contradicts to that the dynamical system is positive Lyapunov stable at each point $x \in X$. If $y \in \omega(x)$, then $\varepsilon = d(x_n t_n, K^+(x)) \le d(x_n t_n, \omega(x))$, this contradicts to that $D^+(\omega(x)) = \omega(x)$. Hence $K^+(x)$ is orbital stability.

3. Dispersiveness

This section is devoted to the dynamical systems which are in general lack of recursiveness. There we set the X is a locally compact T_2 space. Let $x \in X$. Then the point x is called *positively Poisson unstable* if $x \notin \omega(x)$, *negatively Poisson unstable* if $x \notin \alpha(x)$, and *Poisson unstable* if it is both positively and negatively Poisson unstable. If each $x \in X$ is Poisson unstable, then the dynamical system is said to be *Poisson unstable*. If for every $x \in X$, $\omega(x) \cup \alpha(x) = \emptyset$, then the dynamical system is said to be *divergent*. The point x is called *wandering* whenever $x \notin X$. If every $x \in X$ is wandering, then the dynamical system is said to be *completely unstable*. If for every pair of points $x, y \in X$ there exist neighborhoods U_x of x and U_y of y such that U_x is not positively recursive with respect to U_y, then the dynamical system is said to be *dispersive*. This is equal to $J^+(x) = \emptyset$ for each point $x \in X$.

We know that each of the concepts above implies the preceding one. The set of all points $x \in X$ such that $x \in J^+_\alpha(x)$ will be denoted by R_α. And we set $R = \bigcup\{R_\alpha : \alpha$ an ordinal number$\}$. We see that if the dynamical system is dispersive, then $J^+_\alpha(x) = \emptyset$ for all $x \in X$, i.e., $R = \emptyset$. And if $R = \emptyset$, then $R_1 = \emptyset$, this is equal to the dynamical system is completely unstable.

Definition 3.1. Let π be a dynamical system on X. The relation C of being on the same orbit (i.e., xCy iff $x \in \gamma(y)$) is an equivalence relation on X. The set of equivalence classes modulo C will standardly be endowed with the quotient topology, denoted by X/C, and called the *orbit space*. The canonical quotient map $X \to X/C$ will consistently be denoted by $e : X \to X/C$, thus $e(x) = \gamma(x) \in X/C$ for $x \in X$.

We see that the quotient map e is a continuous open surjection, so the orbit space is locally compact, but may not be T_2.

Lemma 3.1 [3]**.** *$K(x) = \gamma(x)$ iff X/C is a T_1 space*; *$D(x) = \gamma(x)$ iff X/C is a T_1 space.*

Theorem 3.1. *If the dynamical system is divergent*, *then the orbit space is a T_1 space.*

Proof. If the dynamical system is divergent, $\omega(x) \cup \alpha(x) = \varnothing$ for every $x \in X$, then $K(x) = \gamma(x)$, so X/C is a T_1 space.

If the dynamical system is Poisson unstable, then X/C may not be a T_1 space. For example, consider a dynamical system in euclidean (x_1, x_2)-plane. The unit circle contains a rest point p and a trajectory γ such that for each point $q \in \gamma$, we have $\omega(x) = \alpha(x) = p$. All trajectories in the interior of the unit circle $(= \{p\} \cup \gamma)$ have the same property as γ. All trajectories in the exterior of the unit circle are spiral to the unit circle as $t \to +\infty$, so that for each point q in the exterior of the unit circle we have $\omega(q) = \{p\} \cup \gamma$, and $\alpha(x) = \varnothing$. Notice that if we consider the dynamical system obtained from this one by deleting the rest point p, then this system is Poisson unstable, but for every $q \in X$, $\gamma(q) \neq K(q)$, then the orbit space is not a T_1 space.

Theorem 3.2. *If the dynamical system is dispersive*, *then the orbit space is a T_2 space.*

Proof. If the dynamical system is dispersive, $J^+(x) = \varnothing$ for every $x \in X$, then the orbit space is a T_2 space.

If $R = \varnothing$, then X/C may not be a T_2 space. For example, consider the dynamical system defined by the system $\dot{x} = \sin y$, $\dot{y} = \cos^2 y$. The orbit are the curves $x = c + \sec x$, and the lines $y = (2k+1)(\pi/2)$; $(k = 0, \pm 1, \ldots)$. Let the dynamical system obtained by restricting to the strip $-(\pi/2) \leq y \leq \pi/2$. In this case, $R = \varnothing$ and the orbit space is not a T_2 space, but the dynamical system is not dispersive.

Theorem 3.3. *Let X be a locally compact T_2 space, if the dynamical system (X, R, π) is Lagrange unstable and is positively Lyapunov stable on the set $M = \{x : J^+_\alpha(x) \neq \varnothing\}$ for each ordinal α, then $R = \varnothing$.*

Proof. Let $R \neq \varnothing$. Then there exists a $y \in X$ such that $J^+(y) \neq \varnothing$. So $J^+_\alpha(y) \neq \varnothing$ for every α. From the condition, we know that the dynamical system is Lyapunov stable at the point y, so $\omega(y) = J^+(y) \neq \varnothing$. Let $z \in \omega(y)$. Then $J^+(y) \subset J^+(z)$ [1, Lemma 6.17]. So $z \in \omega(y) \subset J^+(y) \subset J^+(z)$. Then $z \in M$, $\omega(z) = J^+(z)$. So $\omega(y) \subset \omega(z)$. Since $\omega(z)$ is a closed invariant set, $\omega(z) \subset \omega(y)$, then we have $\omega(z) = \omega(y)$. So $\omega(x)$ is positively minimal [1, Lemma 12.3]. We know that X is locally compact, then $\omega(y)$ is compact [1, Lemma 12.8]. So $\overline{\gamma^+(y)}$ is compact [2, Theorem 3.93]. This contradicts to the fact that the dynamical system is Lagrange unstable. So $R = \varnothing$.

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