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ON THE ASYMPTOTIC BEHAVIOR OF DELAY FIBONACCI EQUATIONS

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Abstract

For Fibonacci sequence, we have several well-known and familiar properties, among which are the facts that the ratio of successive terms approaches a fixed limit, and others. In this paper, we obtain the property of convergence of a ratio of successive value of Fibonacci sequences with delay by applying the elementary number theory.

1. Introduction

We introduce a delay Fibonacci equation:

$$H(n) = H(n - H(n-1)) + H(n - H(n-2)), \quad n \geq 3 \tag{1}$$

with initial value problem $H(1) = H(2) = 1$, where $H(n)$ is a function possessing positive integers for domain as well as for range. This equation was proposed by Hofstadter [5] in his huge book and soon, this equation got treated as an unsolved problems in Number Theory (cf. [2], [3]). Moreover, we consider cousins of Eq. (1)

$$F(n) = F(n - F(n-1)) + F(n - 1 - F(n-2)), \quad n \geq 3 \tag{2}$$

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with $F(1) = 1$ and $F(2) = 1$,

$$T(n) = T(n-1-T(n-1)) + T(n-2-T(n-2)), \quad n \geq 3 \tag{3}$$

with $T(0) = 1$, $T(1) = 1$ and $T(2) = 1$,

$$C(n) = C(C(n-1)) + C(n-C(n-1)), \quad n \geq 3 \tag{4}$$

with $C(1) = 1$ and $C(2) = 1$. Finally, we consider the simple example

$$K(n) = K(K(n-1)) + K(K(n-2)), \quad n \geq 3 \tag{5}$$

with $K(1) = 1$ and $K(2) = 1$ and, more generally, we consider the following equations with delays:

$$K(n) = K(K(n-1)) + K(K(n-2)) + \cdots + K(K(n-m)), \tag{6}$$

$$K(n) = K(K(K(\cdots K(n-1)\cdots))) + K(K(K(\cdots K(n-2)\cdots)))$$

$$+ \cdots + K(K(K(\cdots K(n-m)\cdots))), \quad n > m \geq 2 \tag{7}$$

with $K(1) = K(2) = \cdots = K(m) = 1$ and

$$C(n) = C(C(\cdots C(n-1)\cdots)) + C(C(\cdots C(n-C(n-1))\cdots)), \quad n \geq 3 \tag{8}$$

with $C(1) = C(2) = 1$, and others. We give names to these equations (sequences). Eq. (2) is the *Conolly's equation* (*sequence*) [1], Eq. (3) is called to be the *Tanny's equation* (*sequence*) [7] and Eq. (4) is said to be the *Conway's challenging equation* (*sequence*) [6]. Next, we give the following table for Eq. (1) and this corresponds to the table of standard Fibonacci sequence:

1, 1, 2, 3, 3, 4, 5, 5, 6, 6, 6, 8, 8, 8, 10, 9, 10, 11, 11,

12, 12, 12, 12, 16, 14, 14, 16, 16, 16, 16, 20, 17, 17, 20, 21, …

We then consider three problems for Eq. (1) as follows:

Open Problems

1. Does $H(n)$ miss infinitely many integers such as 7, 13, 15, 18, …? What is $H(s)$'s behavior, in general (cf. [5])?

2. Is there a limit $\alpha < \infty$ such that

$$\frac{H(n+1)}{H(n)} \to \alpha \qquad \text{as} \quad n \to \infty\,?$$

What is this α?

3. Study to make the general theorems as Poincaré and Perron type of recurrence with three terms (and more general terms $m \geq 3$, and also more deep) for Eq. (1). For example,

$$H(n) = H(n - H(n - H(n-1))) + H(n - H(n - H(n-2)))$$
$$+ H(n - H(n - H(n-3))), \quad n \geq 4$$

and we have to set the adequate initial condition, for instance

$$H(1) = H(2) = H(3) = 1 \quad \text{and} \quad H(-k) = 1$$

where k is a nonnegative integer. For Eq. (1), we set up the after conjecture for problem 2, and solve another equation closely related to Eq. (1), especially, we will study in detail with the equation (2) which is cousin of Eq. (1), Eq. (3) and Eq. (4).

2. The Cousin of $H(n)$

In this section, our Lemmas and Theorem 1 are based on Tanny's idea [7]. First lemma refers to the monotonicity for Eq. (2).

Lemma 1. *For* $F(n)$ *of Eq.* (2),

$$F(n+1) = F(n) \quad \text{or} \quad F(n+1) = F(n) + 1 \tag{9}$$

and if $F(n)$ *is odd for* $n \geq 3$, *then*

$$F(n+1) = F(n) + 1. \tag{10}$$

Proof. We use the mathematical induction for the proofs of (9) and (10) that are true for small n. We proceed by induction. We assume that both (9) and (10) are true for all $k < n$. Then, for $k < n$, we have $F(k+1) - F(k) \in \{0, 1\}$. Thus, for $k = 2, \ldots, n$ we have

$$(k+1-F(k))-(k-F(k-1))=1-(F(k)-F(k-1))\in\{0,1\}.$$

Since $(k-F(k-1))$ and $(k-1-F(k-2))$ differ by at most 1, by the assumption of induction, (9) yields

$$F(k-F(k-1))-F(k-1-F(k-2))\in\{0,1\}. \tag{11}$$

Suppose $F(n)=F(n-1)+1$. Then, by the definition of $F(n)$,

$$\begin{aligned}F(n+1)-F(n) &= F(n+1-F(n))-F(n-1-F(n-2))\\ &= F(n-F(n-1))-F(n-1-F(n-2))\in\{0,1\},\end{aligned}$$

from (11). On the other hand, if $F(n)=F(n-1)$, then by (2)

$$F(n-F(n-1))=F(n-2-F(n-3)).$$

By (11), we must have each of these equals $F(n-1-F(n-2))$. Thus, once again,

$$\begin{aligned}F(n+1)-F(n) &= F(n+1-F(n))-F(n-1-F(n-2))\\ &= F(n+1-F(n))-F(n-F(n-1))\in\{0,1\}.\end{aligned}$$

This completes the induction for (9). For (10), suppose that $F(n)$ is odd. Then, $F(n-1)$ is even. Because, if it is not, by the assumption of induction (10), $F(n)$ would be even. This is a contradiction. We have by (9), $F(n)=F(n-1)+1$. Thus, we obtain $F(n+1)=2F(n-F(n-1))$ so $F(n+1)$ is even. Hence, by (9), $F(n+1)=F(n)+1$. This completes the proof by induction.

Let $\Phi(N)=\{n:F(n)=N\}$. The largest element in $\Phi(N)$ is $\sum_{i=1}^{N}\#\Phi(i)$, where $\#\Phi(N)$ denotes the cardinal number of $\Phi(N)$, that is, the length of the string of consecutive integers whose image under F is N.

Lemma 2. *If* $N>1$ *is odd*, $\#\Phi(N)=1$ *or otherwise*, *if* N *is even*, $\#\Phi(N)\geq 2$.

Proof. It is easy to see that $\Phi(N)$ is not empty for any N. If it is not, let N_0 be the smallest integer with $\Phi(N_0) = \varnothing$. Since $N_0 \geq 3$, it follows from Lemma 1 that there is a unique n_0 such that $F(n_0) = N_0 - 2$ and $F(n_0 + 1) = N_0 - 1$. But, then by the assumption that $\Phi(N_0) = \varnothing$ and (9) of Lemma 1, $F(n) = N_0 - 1$ for every $n > n_0$. We choose n_1 such that $n_1 - N_0 - 2 > n_0$. Then

$$
\begin{aligned}
F(n_1) &= F(n_1 - F(n_1 - 1)) + F(n_1 - 1 - F(n_1 - 2)) \\
&= F(n_1 - N_0 + 1) + F(n_1 - N_0) \\
&= 2(N_0 - 1).
\end{aligned}
$$

This is a contradiction. Thus, for every N, $\Phi(N)$ is not empty, that is F takes every positive integer. It now follows immediately from (10) of Lemma 1 that for N odd, $\#\Phi(N) = 1$. For N even, we proceed by contradiction. Let $N_0 > 2$ be the smallest even number such that $\#\Phi(N_0) = 1$. Then, by assumption, there exists a unique n_0 such that $F(n_0) = N_0$. By Lemma 1, $F(n_0 - 1) = N_0 - 1$ and $F(n_0 + 1) = N_0 + 1$. Furthermore, since $N_0 - 1$ is odd, $F(n_0 - 2) = N_0 - 2$. Now, we have

$$
\begin{aligned}
F(n_0) &= F(n_0 - F(n_0 - 1)) + F(n_0 - 1 - F(n_0 - 2)) \\
&= 2F(n_0 - N_0 + 1).
\end{aligned}
$$

But

$$
\begin{aligned}
F(n_0 + 1) &= F(n_0 + 1 - F(n_0)) + F(n_0 - F(n_0 - 1)) \\
&= 2F(n_0 - N_0 + 1).
\end{aligned}
$$

Then $F(n_0 + 1)$ also equals N_0 which is a contradiction. Thus, $\#\Phi(N_0) \geq 2$ for N is even.

It is easy to guess a simple formula for $\#\Phi(N)$: defined if $N = 2^m N_1$, with $m \geq 0$ and N_1 is odd, then $\#\Phi(N) = m + 1 + B(N_1 = 1)$, where $B(N_1 = 1)$ for $N_1 = 1$ and 0 otherwise. We prove the first half of this formula, namely, where N is a power of 2 as part of the next result. Before we do this, we remark that for N is odd, we get $\#\Phi(N) = 1$ which is true by Lemma 2.

Lemma 3. *For every nonnegative integer* m, $\Phi(2^m) = \{2^{m+1} - m, 2^{m+1} - m + 1, \ldots, 2^{m+1} - 1, 2^{m+1}\}$ *and* $\#\Phi(2^m) = m + 1$.

Proof. We can check that this theorem holds for small values of m. We proceed by induction on m. Suppose that the result is true for all positive integers less than m. Let n_0 be the least positive integer such that $F(n_0) = 2^m$. Then we have $F(n_0 - 1) = 2^m - 1$ by (9) of Lemma 1 and $F(n_0 - 2) = 2^m - 2$ by Lemma 2. Thus, we get

$$\begin{aligned} 2^m &= F(n_0) = F(n_0 - F(n_0 - 1)) + F(n_0 - 1 - F(n_0 - 2)) \\ &= 2F(n_0 - 2^m + 1). \end{aligned}$$

Therefore, $F(n_0 - 2^m + 1) = 2^{m-1}$ so by the induction assumption $n_0 - 2^m + 1$ must be in the set $\{2^m - (m-1), 2^m - m + 2, \ldots, 2^m - 1, 2^m\}$. It is straightforward to show that for n_0 to be the least positive integer such that $F(n_0) = 2^m$, we must have $n_0 - 2^m = 2^m - m$. To see this we claim as follows: Suppose $n_0 - 2^m \geq 2^m - (m-1)$. Since $F(n_0 - 3) = 2^m - 2$, we have

$$\begin{aligned} F(n_0 - 1) &= F(n_0 - 1 - F(n_0 - 2)) + F(n_0 - 2 - F(n_0 - 3)) \\ &= F(n_0 - 2^m + 1) + F(n_0 - 2^m). \end{aligned}$$

$F(n_0 - 1) = 2^m - 1$ is odd and $\#\Phi(2^m - 2)$ is at least 2. But $n_0 - 2^m \geq 2^m - (m-1)$ means that both $n_0 - 2^m$ and $n_0 - 2^m + 1$ are in $\Phi(2^{m-1})$ so $F(n_0 - 1) = 2^{m-1} + 2^{m-1} = 2^m$. However, this contradicts the assumption that n_0 is the least integer with $F(n_0) = 2^m$. Thus, $n_0 - 2^m = 2^m - m$, or $n_0 = 2^{m+1} - m$ as required. From Lemma 2, we have immediately that $F(n_0 + 1) = 2^m$. Furthermore,

$$\begin{aligned} F(n_0 + 2) &= F(n_0 + 2 - F(n_0 + 1)) + F(n_0 + 1 - F(n_0)) \\ &= F(2^{m+1} - m + 2 - 2^m) + F(2^{m+1} - m + 1 - 2^m) \\ &= F(2^m - m + 2) + F(2^m - m + 1) \\ &= 2 \cdot 2^{m-1} = 2^m. \end{aligned}$$

In a similar manner, we can show successively that if $m \geq 2$, then we have $F(n_0 + k) = 2^m$ for the remaining $k = 3, \ldots, m$. For $k = m + 1$, we have

$$\begin{aligned} F(n_0 + m + 1) &= F(n_0 + m + 1 - F(n_0 + m)) + F(n_0 + m - F(n_0 + m - 1)) \\ &= F(2^{m+1} - m + m + 1 - 2^m) + F(2^{m+1} - m + m - 2^m) \\ &= F(2^m + 1) + F(2^m). \end{aligned}$$

By the induction assumption $F(2^m + 1) = 2^{m-1} + 1$ while $F(2^m) = 2^{m-1}$. This concludes the proof.

To prove the remainder of the formula for $\#\Phi(N)$, we consider the case when N is even but not a power of 2. We first prove an intermediate result.

Lemma 4. *Let M be any even integer, $2 \leq M \leq 2^m - 2$, where $m \geq 2$. Then $\#\Phi(2^m + M) = \#\Phi(2^{m-1} + M/2) + 1$.*

Proof. Let n_0 be the least positive integer such that $F(n_0) = 2^m + M$. Then, $F(n_0 - 1) = 2^m + M - 1$. Since M is even, we have $F(n_0 - 2) = F(n_0 - 3) = 2^m + M - 2$, by Lemmas 1 and 2. Now, we have

$$\begin{aligned} F(n_0) &= F(n_0 - F(n_0 - 1)) + F(n_0 - 1 - F(n_0 - 2)) \\ &= F(n_0 - (2^m + M - 1)) + F(n_0 - 1 - (2^m + M - 2)) \\ &= 2F(n_0 - (2^m + M - 1)). \end{aligned}$$

Thus, we have

$$F(n_0 - F(n_0 - 1)) = F(n_0 - (2^m + M - 1)) = 2^{m-1} + M/2.$$

Furthermore, we obtain

$$F(n_0 - 1) = F(n_0 - 1 - F(n_0 - 2)) + F(n_0 - 2 - F(n_0 - 3)),$$

$$2^m + M - 1 = F(n_0 - (2^m + M - 1)) + F(n_0 - (2^m + M)).$$

Hence, we have

$$F(n_0 - (2^m + M)) = 2^{m-1} + (M/2) - 1,$$

from which we consider that $n_0 - (2^m + M)$ is the smallest integer in $\Phi(2^{m-1} + (M/2) - 1)$. That is, if n_0 is the least integer such that $F(n_0) = 2^m + M$, then $n_0 - (2^m + M)$ is the least integer whose image under F is $2^{m-1} + (M/2) - 1$. Suppose $F(n_0 + 1) = F(n_0 + 2) = \cdots = F(n_0 + R) = 2^m + M$, where $R + 1 = \# \Phi(2^m + M)$. Then,

$$F(n_0 + R + 1) = 2^m + M + 1.$$

If $R = 1$, then $F(n_0 + 2) = 2^m + M + 1$ and $F(n_0 + 2) = F(n_0 + 2 - F(n_0 + 1)) + F(n_0 + 1 - F(n_0)) = F(n_0 + 2 - (2^m + M)) + 2^{m-1} + M/2$. Thus

$$F(n_0 + 2 - (2^m + M)) = 2^{m-1} + (M/2) + 1,$$

so $\# \Phi(2^{m-1} + M/2) = 1$, and this theorem holds in this case. We assume $R \geq 2$. Using the recursion for $F(n_0 + 1)$ and the fact that $F(n_0 - F(n_0 - 1)) = 2^{m-1} + M/2$, we get

$$F(n_0 + 1 - (2^m + M)) = 2^{m-1} + M/2.$$

Proceeding iteratively we readily show that

$$F(n_0 + k - (2^m + M)) = 2^{m-1} + M/2$$

for $k = 1, 2, \ldots, R - 1$, while

$$F(n_0 + R - (2^m + M)) = 2^{m-1} + M/2 + 1.$$

Thus, for $R \geq 2$, $\# \Phi(2^m + M) = \# \Phi(2^{m-1} + M/2) + 1$, and the proof is complete.

We now have the formula for $\# \Phi(N)$ by above Lemma 4. If $N = 2^r \cdot N_1$ is even but not a power of 2, so $N_1 > 1$, then N can be written as $2^m + M$, with $M = 2^r \cdot p$, $m > r \geq 1$, p is odd, and $2 \leq M \leq 2^m - 2$. Applying Lemma 4, r times readily yields the desired result, namely, $\# \Phi(N) = r + 1 + B(N_1)$.

Lemma 5. *For* $N = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_p}$, $0 \leq m_1 < m_2 < \cdots < m_p$, *the greatest element in* $\Phi(N)$ *is given by*

$$\sum_{i=1}^{N} \# \Phi(i) = 2N - p + 1.$$

Proof. If $p = 1$, then this formula gives the greatest element of $\Phi(N)$ as 2^{m_1+1}, which also follows directly from Lemma 3. The simplest proof, once the formula has been guessed, is by induction. The result is true for $N = 1, 2$. Assume the result for N. For $N + 1$, we consider three cases:

(i) Suppose $m_1 \geq 1$. Then we can write

$$N + 1 = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_{p+1}},$$

where $0 = l_1 < l_2 = m_1 < l_3 = m_2 < \cdots < l_{p+1} = m_p$. Since $N + 1$ is odd, $\# \Phi(N + 1) = 1$ so

$$\sum_{i=1}^{N+1} \# \Phi(i) = 1 + \sum_{i=1}^{N} \# \Phi(i) = 1 + (2N - p + 1)$$

$$= 2(N + 1) - (p + 1) + 1,$$

as required.

(ii) Suppose $m_1 = 0$, and that k exists, with $1 < k < p$, where k is the smallest such that $m_p \geq k$. Then it is readily seen that

$$N + 1 = 2^{k-1} + 2^{m_k} + 2^{m_{k+1}} + \cdots + 2^{m_p},$$

with $k - 1 < m_k < m_{k+1} < \cdots < m_p$. This can be rewritten as

$$N + 1 = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_{p-k+2}},$$

where $k - 1 = l_1 < m_k = l_2 < m_{k+1} = l_3 < \cdots < m_p = l_{p-k+2}$. Thus, $\#\Phi(N+1) = k$ and

$$\sum_{i=1}^{N+1} \#\Phi(i) = k + \sum_{i=1}^{N} \#\Phi(i) = k + (2N - p + 1)$$

$$= 2(N + 1) - (p - k + 2) + 1,$$

which is of the required form.

(iii) The only remaining case has $m_k = k - 1$ for $k = 0, 1, 2, \ldots, p$. Then, $N + 1 = 2^{l_1}$, $l_1 = p$ and $\#\Phi(N + 1) = p + 1$.

$$\sum_{i=1}^{N+1} \#\Phi(i) = p + 1 + \sum_{i=1}^{N} \#\Phi(i) = p + 1 + (2N - p + 1)$$

$$= 2(N + 1),$$

which completes this induction.

The main result of this section is the following:

Lemma 6. *Let* $N = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_p}$, $0 \leq m_1 < m_2 < \cdots < m_p$. *If* $p = 1$, *then* $F(n) = N$ *for all* $(m_1 + 1)$ *consecutive integers* $n = 2^{m_1+1} - m_1, 2^{m_1+1} - m_1 + 1, \ldots, 2^{m_1+1} - 1, 2^{m_1+1}$. *If* $p \geq 2$, *then* $F(n) = N$ *for all* $(m_1 + 1)$ *consecutive integers from* $n = 2N - m_1 - p + 1$ *to* $n = 2N - p + 1$ *inclusive.*

It is clear that the proof of this lemma essentially follows from the same argument as in the proof of Lemma 5 by using the mathematical induction and Lemmas 1 and 3. So we will omit this. If $p = 1$, then the largest n for which $F(n) = N$ is always $2N$ and $F(2N) = N$. This characterizes precisely those n for which $F(n)/n = 1/2$.

Theorem 1. *We have* $F(n)/n \to 1/2$ *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = 1. \tag{12}$$

Proof. By using Lemma 6, it is easy to see that the values of n for which $F(n) = N$ satisfy

$$2N - \log_2 N \leq n \leq 2N.$$

From the second inequality we get $N \geq n/2$. Then, the first half of the inequality gives $N \leq n/2 + (\log_2 n)/2$. Thus, for any n with $F(n) = N$ we have $n/2 + (\log_2 n)/2 \geq F(n) \geq n/2$ from which the desired result follows. Then, we have (12). This completes the proof.

We demonstrate theorem of another type for the asymptotic behavior for Eq. (2).

Theorem A. *For equation* (2), *let* $n \geq 2$ *and let* m *be the largest integer such that*

$$n = 2^m + k, \quad \textit{for} \quad m \geq 1, \quad 2^m - 1 \geq k \geq 0. \tag{13}$$

Then, *with* $F(1) = 0$,

$$F(n) = 2^{m-1} + F(k+1), \quad \textit{for} \quad n \geq 2. \tag{14}$$

See [1] for the proof of Theorem A.

Next, we compute the asymptotic behavior of the ratio of successive terms of solution $H(n)$ of Eq. (1) (cf. [4]).

For $H(n)$ of Eq. (1), we guess

$$\lim_{n\to\infty}\frac{H(n+1)-[(n+1)/2]}{H(n)-[n/2]}=1,$$

where $[n]$ means the greatest integer $\le n$.

In a supplementary of above, we only describe the following sequence of $H(n+1)/H(n)$:

$$1,\ 1,\ 2,\ \frac{3}{2},\ 1,\ \frac{4}{3},\ \frac{5}{4},\ 1,\ \frac{6}{5},\ 1,\ 1,\ \frac{4}{3},\ 1,\ 1,\ \frac{5}{4},\ \frac{9}{10},\ \frac{10}{9},\ \frac{11}{10},$$

$$1,\ \frac{12}{11},\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{7}{8},\ 1,\ \frac{8}{7},\ 1,\ 1,\ 1,\ \ldots.$$

We have a conjecture for problems 1 and 2 as follows:

Conjecture. For Eq. $H(n)$,

$$\lim_{n\to\infty}\frac{H(n+1)}{H(n)}=1.$$

Remark. For Eq. (3), Tanny [7] has shown that $T(n)/n\to 1/2$ as $n\to\infty$, that is $\lim_{n\to\infty}T(n+1)/T(n)=1$. In [6], Mallows studied that Conway's challenging sequence (4) has the similar property of $C(n)/n\to 1/2$ as $n\to\infty$, that is $\lim_{n\to\infty}C(n+1)/C(n)=1$ making the new function $A(n)=2C(n)-n$. Eq. (5) has simple structure for all $n\ge 3$, that is $K(n)\equiv 2$. Thus, in the theorem below, we get the asymptotic behavior of submarine equations. The very important and strong common property in Eqs. (2), (3) and (4) is monotonicity. However, the sequence $H(n)$ of Hofstadter's equation (1) is not so. The values of this $H(n)$ express highly erratic behavior with no discernible regularities. This behavior may seem like a "chaotic" as in the discrete case of the theory of strange attracters. Unfortunately, we cannot now prove it directly, and to our best belief it has not been proved. We are working for it.

For Eq. (1), we impose the expression (13). Then, by the same argument as in Theorem A, we have

$$H(n) = 2^{m-1} + H(k+l), \quad \text{for} \quad n \geq 2, \tag{15}$$

where $l = l(n)$ is some integer and must depend on n [1].

We rewrite Eqs. (6), (7) and (8) to simplify the procedure to Eqs. (16), (17) and (18), respectively,

$$K_n = K_{K_{n-1}} + K_{K_{n-2}} + \cdots + K_{K_{n-m}} \tag{16}$$

and

$$K_n = K_{K^{\cdot^{\cdot^{\cdot}}}{}_{K_{n-1}}} + K_{K^{\cdot^{\cdot^{\cdot}}}{}_{K_{n-2}}} + \cdots + K_{K^{\cdot^{\cdot^{\cdot}}}{}_{K_{n-m}}}, \quad n > m \geq 2 \tag{17}$$

with $K_1 = K_2 = \cdots = K_m = 1$,

$$C_n = C_{C^{\cdot^{\cdot^{\cdot}}}{}_{C_{n-1}}} + C_{C^{\cdot^{\cdot^{\cdot}}}{}_{C_{n-C_{n-1}}}}, \quad n \geq 3 \tag{18}$$

with $C_1 = C_2 = 1$. Then, we obtain the following proposition:

Proposition. *For equations* (16), (17) *and* (18),

$$\lim_{n\to\infty} \frac{K_{n+1}}{K_n} = 1, \tag{19}$$

and

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 1, \tag{20}$$

respectively.

Proof. It is sufficient to take up Eq. (18) on behalf of these equations. We first prove that

$$C_n \equiv 2 \quad \text{for} \quad n \geq 3. \tag{21}$$

To do this, we use the mathematical induction. It is clear that (21) holds in case $n = 3$. We assume that (21) holds when $n = k$, that is $C_k = 2$. Then, we get

$$C_{k+1} = C_{C^{\cdot^{\cdot^{\cdot}}}{}_{C_{C_k}}} + C_{C^{\cdot^{\cdot^{\cdot}}}{}_{C_{C_{k+1-C_k}}}}$$

$$= C_{C^{\cdot^{\cdot^{\cdot}}}{}_{C_2}} + C_{C^{\cdot^{\cdot^{\cdot}}}{}_{C_{C_{k-1}}}} \qquad \text{(by assumption)}$$

$$= C_{C_{\cdot_{\cdot_{C_1}}}} + C_{C_{\cdot_{\cdot_{C_1}}}}$$

$$= \cdots = C_1 + C_1 = 2$$

by initial value $C_1 = C_2 = 1$. This completes the induction for (21). Thus, we can easily prove (20). We also have $K_n \equiv m$ for $n > m \geq 2$ to Eqs. (16) and (17), and we easily have (19).

Finally, we consider the following equation for Eq. (3):

$$T_n = T_{n-1-T_{n-1-T_{n-1}}} + T_{n-2-T_{n-2-T_{n-2}}}, \qquad n \geq 3 \tag{22}$$

with $T_0 = T_1 = T_2 = 1$ and $T_{-k} = 1$, for all $k \in Z^+$.

Then, we have the result by direct computing.

Theorem 2. *The solution sequence of Eq.* (22) *is equivalent to the solution sequence of Eq.* (23):

$$T_n = T_{n-2} + T_{n-3}, \qquad n \geq 3 \tag{23}$$

with $T_0 = T_1 = T_2 = 1$, *and*

$$\lim_{n \to \infty} \frac{T_{n+1}}{T_n} = \rho_1,$$

where ρ_1 *satisfies the following characteristic equation of* (23):

$$\rho^3 - \rho - 1 = 0.$$

Proof. We drive Eq. (23) from Eq. (22). We can check that this theorem holds for small values of n. Put positive integers m and l so that $m = T_{n-1}$ and $l = T_{n-2}$. For these integers m and l, we can take nonnegative integers i and j so that $n - 1 - m = -i$ and $n - 2 - l = -j$ for $n \geq 11$. Thus, $T_{-i} = T_{-j} = 1$. Then, we have

$$\begin{aligned} T_n &= T_{n-2} + T_{n-3} = T_{n-1-1} + T_{n-2-1} \\ &= T_{n-1-T_{-i}} + T_{n-2-T_{-j}} = T_{n-1-T_{n-1-m}} + T_{n-2-T_{n-2-l}} \\ &= T_{n-1-T_{n-1-T_{n-1}}} + T_{n-2-T_{n-2-T_{n-2}}}, \end{aligned}$$

as required. It is easy to drive Eq. (23) from Eq. (22) by the above argument.

It is clear that Theorem 2 is definitely not true if initial value $T_{-k} \neq 1$ for all $k \in Z^{+}$.

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