Far East J. Appl. Math. **28**(1) (2007), 167-172

A "DIVISION-FREE" FORMULA FOR DIVIDED DIFFERENCES OF POLYNOMIALS

M. POLEZZI and A. SRI RANGA

Universidade Estadual de Mato Grosso do Sul-(UEMS)
Rodovia MS 306, Km 6. Cassilândia, MS, Brazil
e-mail: mpolezzi@ime.unicamp.br

DCCE-IBILCE
Universidade Estadual Paulista
Rua Cristóvão Colombo, 2265, 15054-000
São José do Rio Preto, SP, Brazil
e-mail: ranga@dcce.ibilce.unesp.br

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Abstract

In this note, we provide a formula for the sum of the reciprocals of the derivatives of a rational function at its zeros, in terms of them and of its poles. As a remarkable consequence, we obtain a formula for divided differences of polynomial functions, which does not require the use of divisions, only multiplications and additions.

1. Introduction

Let $r = p/q$ be a rational function. The degree of r, denoted by $\deg(r)$, is given by $\deg(r) = \deg(p) - \deg(q)$. It is easy to see that this definition does not depend upon the choice of p and q (see [3], Section 4.2).

2000 Mathematics Subject Classification: 26C15, 65D05.

Keywords and phrases: rational interpolation, divided differences.

This research is part of the first author's doctorate thesis at IMECC-UNICAMP, with grants from FUNDECT. The research of the second author is supported by grants from FAPESP and CNPq.

Received January 31, 2007

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In this note, we first present a geometrical property of rational functions R with $\deg(R) \geq 2$ and whose zeros are pairwise distinct. For the polynomial case, this property is useful for finding the barycentric weights of polynomial interpolation (see e.g., [1, 5]). As a first application, we point out a short proof of a fundamental result due to Berrut and Mittelmann [2], which is used for finding the barycentric weights of rational interpolants. The barycentric representation is very useful, since it has many advantages in comparison with the canonical one (see e.g., [8]).

Our proof can be found in [7], there we provide a comprehensive account on rational interpolation as well as some new results on the subject. As another application, we have obtained in [6] a new proof of a formula given by Szegö in [9] for the Christoffel numbers (see also [4]).

Next we provide a formula for the sum of the reciprocals of the derivatives of a proper rational function $R = p/q$, with $\deg(p) < \deg(q)$, at the zeros of p, in terms of the zeros of p and q. This result, together with the one for the case $\deg(R) \geq 0$, gives rise to a formula for divided differences of polynomial functions, which does not require any division at all.

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2. Main Results

When R is real and has real zeros, the result we are concerned with is the following: *The sum of the slopes of the normal lines to the graph of the rational function R with $\deg(R) \geq 2$, considered at its zeros, is zero.*

The following lemma, which holds for complex rational functions, is a generalization to the above statement.

Lemma 2.1. *Let* $L(z) = \prod_{j=0}^{l}(z - z_j)$, *where* $z_i \neq z_j$ *for* $i \neq j$. *Also, let* $Q(z) = \sum_{r=0}^{l+1} a_r z^r$, *where* $Q(z_j) \neq 0$ *for* $j = 0 : l$, *and consider the rational function* $R = L/Q$. *Then,*

$$\sum_{j=0}^{l} \frac{1}{R'(z_j)} = a_l + a_{l+1} \sum_{j=0}^{l} z_j.$$

Proof. Let $0 \leq r \leq l.$ Using the Lagrange interpolation formula, we can therefore, write

$$z^r = \sum_{j=0}^{l} \frac{L(z)}{(z - z_j)L'(z_j)} z_j^r.$$

Equating the coefficients of degree l on both sides, we obtain

$$\sum_{j=0}^{l} \frac{z_j^r}{L'(z_j)} = \begin{cases} 0, & \text{if } 0 \leq r \leq l-1, \\ 1, & \text{if } r = l. \end{cases} \tag{2.1}$$

This proves the result when $a_{l+1} = 0.$ Now, applying (2.1) to the equality

$$\sum_{j=0}^{l} \frac{L(z_j)}{L'(z_j)} = 0,$$

one readily finds

$$\sum_{j=0}^{l} \frac{z_j^{l+1}}{L'(z_j)} = \sum_{j=0}^{l} z_j,$$

which completes the proof.

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The next result shows what happens when the denominator degree is larger than the numerator degree.

Lemma 2.2. *Let* $P(z) = \prod_{j=1}^{n}(z - z_j),$ *where* $z_i \neq z_j$ *for* $i \neq j.$ *Also, let* $Q(z) = \prod_{j=1}^{n+k}(z - y_j),$ *where* $k \geq 0$ *and* $Q(z_j) \neq 0,$ $j = 1 : n,$ *and consider the rational function* $R = P/Q.$ *Then*

$$\sum_{j=1}^{n} \frac{1}{R'(z_j)} = \sum_{t_k=1}^{n} (z_{t_k} - y_{t_k+k}) \sum_{t_{k-1}=1}^{t_k} (z_{t_{k-1}} - y_{t_{k-1}+(k-1)}) \cdots \sum_{t_0=1}^{t_1} (z_{t_0} - y_{t_0}).$$

Proof. We will use double induction on n and k, considering $k = 0,$ $n \geq 1$ and $n = 1,$ $k \geq 0$ as basis of the induction. We have by Lemma 2.1 that the result holds for the case $(n, 0).$ Furthermore, the result is also valid for the case $(1, k)$. Indeed, let $R(z) = (z - z_1)\Big/\prod_{j=1}^{k+1}(z - y_j)$. Then $1/R'(z_1) = \prod_{j=1}^{k+1}(z_1 - y_j)$, and so the result is true in both cases. Thus, suppose the result holds for the cases $(n-1, k)$ and $(n, k-1)$, and let us prove it for the case (n, k), where $n \geq 2$ and $k \geq 1$. We have that

$$\sum_{j=1}^{n} \frac{1}{R'(z_j)} = \sum_{j=1}^{n} \left(\prod_{p=1}^{n+k} (z_j - y_p) \Big/ \prod_{\substack{p=1 \\ p \neq j}}^{n} (z_j - z_p) \right).$$

Applying the case $(n, k-1)$ of the induction hypothesis on the last summand of the right-hand side of the equality, we get

$$\sum_{j=1}^{n} \frac{1}{R'(z_j)} = \sum_{j=1}^{n-1} \frac{\prod_{p=1}^{n+k} (z_j - y_p)}{\prod_{\substack{p=1 \\ p \neq j}}^{n} (z_j - z_p)} + (z_n - y_{n+k}) \left[\sum_{t_{k-1}=1}^{n} (z_{t_{k-1}} - y_{t_{k-1}+(k-1)}) \right.$$

$$\left. \cdots \sum_{t_0=1}^{t_1} (z_{t_0} - y_{t_0}) - \sum_{j=1}^{n-1} \frac{\prod_{p=1}^{n+(k-1)} (z_j - y_p)}{\prod_{\substack{p=1 \\ p \neq j}}^{n} (z_j - z_p)} \right]$$

$$= \left[\frac{z_j - y_{n+k}}{z_j - z_n} - \frac{z_n - y_{n+k}}{z_j - z_n} \right] \sum_{j=1}^{n-1} \frac{\prod_{p=1}^{(n-1)+k} (z_j - y_p)}{\prod_{\substack{p=1 \\ p \neq j}}^{n-1} (z_j - z_p)}$$

$$+ (z_n - y_{n+k}) \sum_{t_{k-1}=1}^{n} (z_{t_{k-1}} - y_{t_{k-1}+(k-1)}) \cdots \sum_{t_0=1}^{t_1} (z_{t_0} - y_{t_0}).$$

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Now, applying the case $(n-1,\ k)$ on the first sum, we get

$$\sum_{j=1}^{n}\frac{1}{R'(x_j)}=\sum_{t_k=1}^{n-1}(z_{t_k}-y_{t_k+k})\sum_{t_{k-1}=1}^{t_k}(z_{t_{k-1}}-y_{t_{k-1}+(k-1)})\cdots\sum_{t_0=1}^{t_1}(z_{t_0}-y_{t_0})$$

$$+(z_n-y_{n+k})\sum_{t_{k-1}=1}^{n}(z_{t_{k-1}}-y_{t_{k-1}+(k-1)})\cdots\sum_{t_0=1}^{t_1}(z_{t_0}-y_{t_0})$$

$$=\sum_{t_k=1}^{n}(z_{t_k}-y_{t_k+k})\sum_{t_{k-1}=1}^{t_k}(z_{t_{k-1}}-y_{t_{k-1}+(k-1)})\cdots\sum_{t_0=1}^{t_1}(z_{t_0}-y_{t_0}).$$

In the light of Lemmas 2.1 and 2.2, a formula for divided differences of polynomial functions can be derived.

Theorem 2.3. *Let* $z_1, ..., z_n$ *be* n *distinct points in* $\mathbb{C}$ *and consider* $q(z)=\prod_{j=1}^{n+k}(z-y_j)$, *where* $k\in\mathbb{Z}$. *Also, denote by* $q[z_1, \ldots, z_n]$ *the* $(n-1)^{th}$ *divided difference of* q *with respect to* $z_1, \ldots, z_n$. *Then*

$$q[z_1, \ldots, z_n]=\begin{cases}0, & \text{if } k\le -2;\\ 1, & \text{if } k=-1;\\ \sum_{j=1}^{n}(z_j-y_j) & \text{if } k=0;\\ \sum_{t_k=1}^{n}(z_{t_k}-y_{t_k+k})\cdots\sum_{t_0=1}^{t_1}(z_{t_0}-y_{t_0}), & \text{if } k\ge 1.\end{cases}$$

Proof. It is a well known fact that $q[z_1, \ldots, z_n]=\sum_{j=1}^{n}q(z_j)/p'(z_j)$, where $p(z)=\prod_{j=1}^{n}(z-z_j)$. Thus, by Lemma 2.1 (if $k\le 0$) or Lemma 2.2 (if $k\ge 0$), the result is straightforward.

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