*Far East J. Appl. Math.* **28**(1) (2007), 37-52

# A NEW NUMERICAL ALGORITHM FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH INITIAL AND BOUNDARY CONDITIONS

**BAKARI ABBO*, OUSSÉNI SO, GENEVIÈVE BARRO and BLAISE SOME**

*Département de Mathématiques et Informatique
Faculté des Sciences Exactes et Appliquées
Université de Ndjaména, Tchad
e-mail: bakariabbo@yahoo.fr

Unité de Formation et de Recherche-Sciences Exactes et Appliquées
Université de Ouagadougou (LANIBIO)
01 BP: 7021 Ouagadougou 01
Burkina Faso UFR/SEA
e-mail: some@univ-ouaga.bf
barro_genevieve@yahoo.fr



**Abstract**

In this work, we present a new approach in the algorithm of the Adomian method for the resolution of nonlinear partial differential equations (PDE) with initial and boundary conditions.

This new method is based on a combination of Adomian decompositional method and the idea of the successive approximation method [3]. We have shown that this new algorithm is convergent with a few number of iterations.

2000 Mathematics Subject Classification: 65M55, 65N55.

Keywords and phrases: Adomian method, successive approximations, biomathematical modeling.

Received December 23, 2006

## 1. Introduction

We are interested to solve nonlinear PDE with initial and boundary conditions which model a big number of phenomena in biomathematics [9, 12, 13]. Several numerical methods of resolution of these PDEs have been developed. These classical methods of resolution, generally use discretisation or linearisation methods. The new approach of Adomian, that we propose, does not discretise, preserves the biological properties of the model and conditions imposed to the PDE, that is important for the applications [12-13].

The classical Adomian method [1, 2, 5] is not easy to compute and this algorithm often does not take into account the boundary conditions of PDE and it includes some special polynomials called *Adomian's polynomial* which are not easy to compute.

In this paper we introduce a new algorithm which takes into account the boundary conditions of nonlinear PDE and does not compute Adomian polynomials. This algorithm is numerically convergent with two or three iterations.

### 1.1. Description of the new algorithm proposed

Let us consider the following PDE with initial and boundary conditions in one space dimension:

$$\begin{cases} u_t = L(u(t, x)) + N(u(t, x)), \quad l_1 \leq x \leq l_2, \ t \geq 0 \\ u(0, x) = f(x), \\ u(t, l_1) = g(t), \\ u(t, l_2) = h(t), \end{cases} \tag{1}$$

where $L$ and $N$ are respectively linear and nonlinear operators with

$$L(u) = L_1(u) + L_2(u_x, u_{xx}, u_{xxx}, \ldots), \ L_2 \neq 0.$$

For a simple description, we can take the case where $L(u) = u_x$, then the equation (1) becomes

$$\begin{cases} u_t = u_x + N(u(t, x)), & l_1 \le x \le l_2, \; t \ge 0 \\ u(0, x) = f(x), \\ u(t, l_1) = g(t), \\ u(t, l_2) = h(t), \end{cases}$$

where

$$u = u(t, x), \; (t, x) \in [0, +\infty[ \times [l_1, l_2].$$

On the one hand, the equality $u_t = u_x + N(u)$ (that is $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + N(u)$) permits us to write boundary conditions while proceeding as follows:

$$u(t, l_2) = u(t, l_1) + \int_{l_1}^{l_2} \frac{\partial u(t, x)}{\partial t} dx - \int_{l_1}^{l_2} N(u(t, x))dx$$

$$\Rightarrow u(t, l_2) - u(t, l_1) - \int_{l_1}^{l_2} \frac{\partial u(t, x)}{\partial t} dx + \int_{l_1}^{l_2} N(u(t, x))dx = 0.$$

On the other hand, this equation gives

$$u(t, x) = u(0, x) + \int_0^t \frac{\partial u(\mu, x)}{\partial x} d\mu + \int_0^t N(u(\mu, x))d\mu,$$

that is equivalent to the following equation, where one added boundary conditions:

$$u(t, x) = u(0, x) + \int_0^t \frac{\partial u(\mu, x)}{\partial x} d\mu + \int_0^t N(u(\mu, x))d\mu + u(t, l_2) - u(t, l_1)$$

$$- \int_{l_1}^{l_2} \frac{\partial u(t, x)}{\partial t} dx + \int_{l_1}^{l_2} N(u(t, x))dx.$$

That give finally

$$\begin{cases} u(t, x) = \begin{pmatrix} u(0, x) + \int_0^t \frac{\partial u(\mu, x)}{\partial x} d\mu + \int_0^t N(u(\mu, x))d\mu + u(t, l_2) - u(t, l_1) \\ - \int_{l_1}^{l_2} \frac{\partial u(t, x)}{\partial t} dx + \int_{l_1}^{l_2} N(u(t, x))dx \end{pmatrix} \\ u(0, x) = f(x) \end{cases} \quad (2)$$

which is a Cauchy problem.

So we bring back the resolution of problems (1) with initial and boundary conditions to Cauchy problem. This transformation permits us to solve this kind of problem with success. Afterwards we will illustrate that through examples. By using the idea of successive approximation (2), we give

$$u^k(t, x) = \underbrace{u^k(0, x) + u^k(t, l_2) - u^k(t, l_1) + \int_0^t \frac{\partial u^k(\mu, x)}{\partial x} d\mu}_{\widetilde{L}(u^k)} +$$

$$\underbrace{+ \int_0^t N(u^{k-1}(\mu, x)) d\mu - \int_{l_1}^{l_2} \frac{\partial u^{k-1}(t, s)}{\partial t} ds + \int_{l_1}^{l_2} N(u^{k-1}(t, s)) ds}_{\widetilde{N}(u^{k-1})}$$

with

$$\begin{cases} u^k(0, x) = f(x), \ \ k = 1, 2, \ldots \\ u^k(t, l_1) = g(t), \ \ k = 0, 1, 2, \ldots \\ u^k(t, l_2) = h(t), \ \ k = 0, 1, 2, \ldots \end{cases}$$

which is an Adomian canonical form.

So the Adomian algorithm is

$$\begin{cases} u_0^k = u^k(0, x) + u^k(t, l_2) - u^k(t, l_1) + \int_0^t N(u^{k-1}(\mu, x)) d\mu \\ - \int_{l_1}^{l_2} \frac{\partial u^{k-1}(t, x)}{\partial t} dx + \int_{l_1}^{l_2} N(u^{k-1}(t, x)) dx \\ u_n^k = \int_0^t \frac{\partial u_{n-1}^k(\mu, x)}{\partial x} d\mu, \quad k = 1, 2, 3, \ldots; \ n = 1, 2, \ldots. \end{cases}$$

At this level all happens as in [3].

Indeed, the resolution of the algorithm above by the successive approximation method, consists to determine at each iteration $(k = 1, 2, \ldots)$ the approached solutions

$$u^1, u^2, \ldots, u^n, \ldots,$$

where

$$u^k = \sum_{n=0}^{+\infty} u_n^k, \quad k = 1, 2, \ldots.$$

But it requires a choice of the initial condition $u^0$ beforehand.

By following the solution, $u$ of the problem (1) is

$$u = \lim_{k \to \infty} u^k,$$

if $(u^k)_{k \in \mathbb{N}}$ is convergent.

In summary the new algorithm is:

**Step 1.** Calcul of $u^1$

$$\begin{cases} u_0^1 & \text{1st} & \text{term of Adomian series at step 1;} \\ u_1^1 & \text{2nd} & \text{term of Adomian series at step 1;} \\ u_2^1 & \text{3rd} & \text{term of Adomian series at step 1;} \\ \vdots & \vdots & \vdots \\ u_n^1 & (n+1)\text{th} & \text{term of Adomian series at step 1.} \end{cases}$$

The approach solution of the problem is obtained by

$$u^1 = \sum_{n=0}^{\infty} u_n^1.$$

**Step 2.** Calcul of $u^2$

$$\begin{cases} u_0^2 & \text{1st} & \text{term of Adomian series at step 2;} \\ u_1^2 & \text{2nd} & \text{term of Adomian series at step 2;} \\ u_2^2 & \text{3rd} & \text{term of Adomian series at step 2;} \\ \vdots & \vdots & \vdots \\ u_n^2 & (n+1)\text{th} & \text{term of Adomian series at step 2.} \end{cases}$$

The approach solution of the problem is obtained by

$$u^2 = \sum_{n=0}^{\infty} u_n^2.$$

**Step** $k$**.** Calcul of $u^k$

$$\begin{cases} u_0^k & \text{1st} & \text{term of Adomian series at step } k; \\ u_1^k & \text{2nd} & \text{term of Adomian series at step } k; \\ u_2^k & \text{3rd} & \text{term of Adomian series at step } k; \\ \vdots & \vdots & \vdots \\ u_n^k & (n+1)\text{th} & \text{term of Adomian series at step } k. \end{cases}$$

The approach solution of the problem is obtained by

$$u^k = \sum_{n=0}^{\infty} u_n^k.$$

Finally the solution of the problem is

$$u = \lim_{k\to\infty} u^k = \lim_{k\to\infty}\left(\sum_{n=0}^{\infty} u_n^k\right).$$

### 1.2. Some examples of illustration

#### 1.2.1. Example 1

Let us consider the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + 2 + \dfrac{1}{4}\left(\dfrac{\partial^2 u}{\partial x^2}\right)^3, \quad 0 \le x \le 1, \ t \ge 0 \\ u(0, x) = x(1 - x), \\ u(t, 0) = t(1 - t), \\ u(t, 1) = -t(1 + t). \end{cases}$$

The equality

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + 2 + \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2}\right)^3$$

gives, on the one hand,

$$u(t,\ 1) = u(t,\ 0) + \int_0^1 \left(-\,2 + \frac{\partial u(t,\ x)}{\partial t}\right) dx - \frac{1}{4}\int_0^1 \left(\frac{\partial^2 u(t,\ x)}{\partial x^2}\right)^3 dx,$$

that is,

$$u(t,\ 1) - u(t,\ 0) + \int_0^1 \left(2 - \frac{\partial u(t,\ x)}{\partial t}\right) dx + \frac{1}{4}\int_0^1 \left(\frac{\partial^2 u(t,\ x)}{\partial x^2}\right)^3 dx = 0.$$

On the other hand, it gives

$$u(t,\ x) = u(0,\ x) + \int_0^t \left(2 + \frac{\partial u(\mu,\ x)}{\partial x}\right) d\mu + \frac{1}{4}\int_0^t \left(\frac{\partial^2 u(\mu,\ x)}{\partial x^2}\right)^3 d\mu$$

$$+\, u(t,\ 1) - u(t,\ 0) + \int_0^1 \left(2 - \frac{\partial u(t,\ x)}{\partial t}\right) dx + \frac{1}{4}\int_0^1 \left(\frac{\partial^2 u(t,\ x)}{\partial x^2}\right)^3 dx,$$

which can be written as:

$$u(t,\ x) = u(0,\ x) + u(t,\ 1) - u(t,\ 0) + \int_0^t \left(2 + \frac{\partial u}{\partial x}\right) d\mu$$

$$+\, \frac{1}{4}\int_0^t \left(\frac{\partial^2 u}{\partial x^2}\right)^3 d\mu + \int_0^1 \left(2 - \frac{\partial u}{\partial t}\right) ds - \frac{1}{4}\int_0^1 \left(\frac{\partial^2 u}{\partial s^2}\right)^3 ds.$$

The equation approached by the new technique can be written as:

$$u^k(t,\ x) = \underbrace{u^k(0,\ x) + u^k(t,\ 1) - u^k(t,\ 0) + \int_0^t \left(2 + \frac{\partial u^k}{\partial x}\right) d\mu}_{\widetilde{L}(u^k)} +$$

$$\underbrace{+\, \frac{1}{4}\int_0^t \left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^3 d\mu + \int_0^1 \left(2 - \frac{\partial u^{k-1}}{\partial t}\right) dx + \frac{1}{4}\int_0^1 \left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^3 dx}_{\widetilde{N}(u^{k-1})}.$$

For $k = 1$, with the choice of $u^0 = 2t \Rightarrow N(u^0) = 0$, $u^1$ is, therefore, the solution of

$$u^1(t,\ x) = u^1(0,\ x) + u^1(t,\ 1) - u^1(t,\ 0) + \int_0^t \left(2 + \frac{\partial u^1}{\partial x}\right) d\mu.$$

With initial and boundary conditions, we have

$$u^1(t, x) = x(1-x) + \int_0^t \left(\frac{\partial u^1}{\partial x}\right) d\mu.$$

At this level the classic algorithm of Adomian is

$$\begin{cases} u_0^1 = x(1-x) \\ u_n^1 = \int_0^t \frac{\partial u_{n-1}^1}{\partial x} d\mu, \quad n = 1, 2, \dots . \end{cases}$$

The solution of this stage is

$$u^1 = \sum_{n=0}^{\infty} (u_n^1) = x(1-x) + (1-2x)t - t^2.$$

For $k \geq 2$, we get at each stage the same solution

$$u^1 = u^2 = \cdots u^k = x(1-x) + (1-2x)t - t^2.$$

So solution of the problem is obtained by

$$u = \lim_{k\to\infty} u^k = \sum_{n=0}^{\infty} (u_n^k) = x(1-x) + (1-2x)t - t^2.$$



**1.2.2. Example 2**

Let us consider the following nonlinear PDE:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - e^t(1-u)^2 - e^{-t}\sin^2 x + e^{-t}, \ 0 \leq x \leq \frac{\pi}{2}, t \geq 0 \\ u(0, x) = 2\sin^2\left(\frac{x}{2}\right), \\ u(t, 0) = 1 - e^{-t}, \\ u\left(t, \frac{\pi}{2}\right) = 1. \end{cases}$$

While considering the operator $\frac{\partial u}{\partial t}$, on the one hand,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - e^t(1-u)^2 - e^{-t}\sin^2 x + e^{-t},$$

gives

$$u(t, x) = u(0, x) + \int_0^t \frac{\partial^2 u(\mu, x)}{\partial x^2} d\mu$$

$$- \int_0^t (e^{\mu}(1-u)^2 + e^{-\mu} \sin^2 x) d\mu - e^{-t} + 1. \tag{3}$$

On the other hand, the choice of the operator $\frac{\partial^2 u}{\partial x^2}$ gives

$$u\left(t, \frac{\pi}{2}\right) = u(t, 0) + \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\partial u(t, x)}{\partial t} dx dx$$

$$+ \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (-e^t (1-u)^2 - e^{-t} \sin^2 x + e^{-t}) dx dx,$$

that is,

$$u\left(t, \frac{\pi}{2}\right) - u(t, 0) - \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\partial u(t, x)}{\partial t} dx dx$$

$$- \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (e^t (1-u)^2 + e^{-t} \sin^2 x - e^{-t}) dx dx = 0. \tag{4}$$

(3) and (4) $\Rightarrow$

$$u(t, x) = u(0, x) + \int_0^t \frac{\partial^2 u(\mu, x)}{\partial x^2} d\mu$$

$$- \int_0^t (e^{\mu}(1-u)^2 - e^{-\mu} \sin^2 x) d\mu - e^{-t} + 1 + u\left(t, \frac{\pi}{2}\right) - u(t, 0)$$

$$- \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\partial u(t, x)}{\partial t} dx dx - \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (e^t (1-u)^2 + e^{-t} \sin^2 x - e^{-t}) dx dx.$$

The approached canonical form associated to the above Adomian canonical form is

$$u^k(t, x) = \underbrace{u^k(0, x) + u^k\left(t, \frac{\pi}{2}\right) - u^k(t, 0) + \int_0^t \frac{\partial^2 u^k(\mu, x)}{\partial x^2} d\mu - e^{-t}}_{L(u^k)} +$$

$$\underbrace{\begin{aligned} &- \int_0^t (e^{\mu}(1 - u^{k-1})^2 + e^{-\mu} \sin^2 x) d\mu \\ &- \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (e^t (1 - u)^2 - e^{-t} \sin^2 x + e^{-t}) dx dx \ . \\ &- \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\partial u^{k-1}(t, x)}{\partial t} dx dx + 1 \end{aligned}}_{N(u^{k-1})}$$

For $k = 1$, with the choice of $u^0 = 1$ (that is, $x = \frac{\pi}{2}$) $\Rightarrow N(u^0) = 0$, $u^1$ is, therefore, the solution of

$$u^1(t, x) = u^1(0, x) + u^1\left(t, \frac{\pi}{2}\right) - u^1(t, 0) - e^{-t} + \int_0^t \frac{\partial^2 u^1(\mu, x)}{\partial x^2} d\mu.$$

To this level the classic algorithm of Adomian is

$$\begin{cases} u_0^1 = u^1(0, x) = 2 \sin^2\left(\frac{x}{2}\right) \\ u_n^1 = \int_0^t \frac{\partial^2 u_{n-1}^1}{\partial x^2} d\mu, \quad n = 1, 2, \ldots. \end{cases}$$

The solution of this stage is

$$u^1 = \sum_{n=0}^{\infty} (u_n^1) = 1 - e^{-t} \cos x.$$

For $k \geq 2$, we get at every stage the same solution

$$u^1 = u^2 = \cdots = u^k = 1 - e^{-t} \cos x.$$

The solution of the problem is obtained by

$$u = \lim_{k \to \infty} u^k = \sum_{n=0}^{\infty} (u_n^k) = 1 - e^{-t} \cos x.$$

### 1.2.3. Example 3

Let us consider the following nonlinear PDE:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + 2\sqrt{\left(\dfrac{\partial^2 u}{\partial x^2}\right)u^2 - \left(\dfrac{\partial^2 u}{\partial x^2}\right)^3 + \dfrac{1}{2}(1 + \cos 2t)} - 2\cos t, \\ u(0,\ x) = \sin x, \\ u\left(t,\ -\dfrac{\eth}{2}\right) = -\cos t, \\ u\left(t,\ \dfrac{\eth}{2}\right) = \cos t, \end{cases}$$

where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $t \geq 0$.

While considering respectively the operators $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$, we get like above, on the one hand,

$$u(t,\ x) = u(0,\ x) + \int_0^t \int_0^t \frac{\partial^2 u(\mu,\ x)}{\partial x^2} d\mu d\mu + 2\cos t - 2$$

$$+ \int_0^t \int_0^t 2\sqrt{\left(\frac{\partial^2 u}{\partial x^2}\right)u^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \frac{1}{2}(1 + \cos 2\mu)}\, d\mu d\mu,$$



and on the other hand,

$$u\left(t,\ \frac{\pi}{2}\right) = u\left(t,\ -\frac{\pi}{2}\right) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial^2 u(t,\ x)}{\partial t^2} + 2\cos t\right) dx dx$$

$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{\left(\frac{\partial^2 u}{\partial x^2}\right)u^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \frac{1}{2}(1 + \cos 2t)}\, dx dx,$$

that is,

$$u\left(t,\ \frac{\pi}{2}\right) - u\left(t,\ -\frac{\pi}{2}\right) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial^2 u(t,\ x)}{\partial t^2} + 2\cos t\right) dx dx$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{\left(\frac{\partial^2 u}{\partial x^2}\right)u^2 - \left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \frac{1}{2}(1 + \cos 2t)}\, dx dx = 0.$$

The above equalities give the following approached canonical form:

$$u^k(t,\ x) = u^k(0,\ x) + \int_0^t\int_0^t \frac{\partial^2 u^k(\mu,\ x)}{\partial x^2} d\mu d\mu + 2\cos t + u^k\left(t,\ \frac{\pi}{2}\right)$$

$$- u^k\left(t,\ -\frac{\pi}{2}\right) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\partial^2 u^{k-1}(t,\ x)}{\partial t^2} + 2\cos t\right)dxdx$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{\left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)(u^{k-1})^2 - \left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^3} + \frac{1}{2}(1+\cos 2t)dxdx - 2$$

$$+ \int_0^t\int_0^t 2\sqrt{\left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)(u^{k-1})^2 - \left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^3} + \frac{1}{2}(1+\cos 2\mu)d\mu d\mu.$$

So the exact solution is

$$u(t,\ x) = \cos t \sin x.$$

### 1.2.4. Example 4

Let us consider the following PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + 1 - \left(\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2}\right)^2, & 0 \le x \le l,\ t \ge 0 \\ u(0,\ x) = x(l-x), & \\ u(t,\ 0) = t, & \\ u(t,\ l) = t. & \end{cases}$$

The equation

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + 1 - \left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)^2$$

can be written as (while considering the operator $\frac{\partial u}{\partial t}$), on the one hand

$$u(t,\ x) = u(0,\ x) + t - \frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x^2} d\mu - \int_0^t\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)^2 d\mu.$$

On the other hand, the operator $\dfrac{\partial^2 u}{\partial x^2}$ gives

$$u(t,\ l) = u(t,\ 0) + l\,\frac{\partial u(t,\ 0)}{\partial t} - 2\int_0^l\int_0^l\left(\frac{\partial u}{\partial t} - 2\right)dsds - \frac{1}{2}\int_0^l\int_0^l\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dxdx,$$

that is,

$$u(t,\ l) - u(t,\ 0) - l\,\frac{\partial u(t,\ 0)}{\partial x} + 2\int_0^l\int_0^l\left(\frac{\partial u}{\partial t} - 2\right)dxdx + \frac{1}{2}\int_0^l\int_0^l\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dxdx = 0.$$

That gives

$$u(t,\ x) = u(0,\ x) + t - \frac{1}{2}\int_0^t \frac{\partial^2 u}{\partial x^2}\,d\mu + u(t,\ l) - u(t,\ 0) - l\,\frac{\partial u(t,\ 0)}{\partial x}$$

$$+\,2\int_0^l\int_0^l\left(\frac{\partial u}{\partial t} - 2\right)dxdx + \frac{1}{2}\int_0^l\int_0^l \frac{\partial^2 u^2}{\partial x^2}\,dxdx - \int_0^t\left(\frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\right)^2 d\mu.$$

The approached canonical form, according to the method of successive approximations can be written as:

$$u^k(t,\ x) = u^k(0,\ x) + t - \frac{1}{2}\int_0^t \frac{\partial^2 u^k}{\partial x^2}\,d\mu + u^{k-1}(t,\ l) - u^k(t,\ 0) - l\,\frac{\partial u^k(t,\ 0)}{\partial x}$$

$$+\,2\int_0^l\int_0^l\left(\frac{\partial u^{k-1}}{\partial t} - 2\right)dxdx + \frac{1}{2}\int_0^l\int_0^l\left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^2 dxdx$$

$$-\int_0^t\left(\frac{1}{2}\,\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^2 d\mu, \quad k = 1,\ 2,\ \ldots,$$

that is,

$$u^k(t,\ x) = \underbrace{u^k(0,\ x) - u^k(t,\ 0) + t - \frac{1}{2}\int_0^t \frac{\partial^2 u^k}{\partial x^2}\,d\mu}_{L(u^k)} +$$

$$\underbrace{\begin{aligned}&+u^{k-1}(t,\ l) + 2\int_0^l\int_0^l\left(\frac{\partial u^{k-1}}{\partial t} + 2\right)dxdx\\ &+\frac{1}{2}\int_0^l\int_0^l\left(\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^2 dxdx - \int_0^t\left(\frac{1}{2}\,\frac{\partial^2 u^{k-1}}{\partial x^2}\right)^2 d\mu\end{aligned}}_{N(u^{k-1})}.$$

For $k = 1$, with the choice of $u^0 = 0 \Rightarrow N(u^0) = 0$, $u^1$ is therefore, solution of

$$u^1(t, x) = u^1(0, x) - u^1(t, 0) + t - \frac{1}{2}\int_0^t \frac{\partial^2 u^1}{\partial x^2}\, d\mu.$$

The algorithm of Adomian gives

$$\begin{cases} u_0^1 = u^1(0, x) = x(l - x) \\ u_n^1 = -\dfrac{1}{2}\displaystyle\int_0^t \frac{\partial^2 u_{n-1}^1}{\partial x^2}\, d\mu, \quad n = 1, 2, \dots . \end{cases}$$

The solution at this stage is:

$$u^1 = \sum_{n=0}^{\infty} (u_n^1) = x(l - x) + t.$$

For $k \geq 2$, we get at every stage the same solution

$$u^1 = u^2 = \cdots = u^k = x(l - x) + t.$$

So the solution of the problem is obtained by

$$u = \lim_{k \to \infty} u^k = \sum_{n=0}^{\infty} (u_n^k) = x(l - x) + t.$$

## 2. Discussions

As for nonlinear PDE of Cauchy type, examples below show that the new algorithm is very efficient to solve nonlinear PDE with initial and boundary conditions. In fact the exact solution of the equation is obtained very often at the first iteration; this algorithm contrary to the initial algorithm of Adomian takes into account all conditions imposed to the PDE. This second point is very revolutionary because the original algorithm of Adomian cannot take into account all conditions imposed to a PDE. The first point is very important for the numerical application because the exact solution is obtained very quickly and explicitly.

But some problems stay opened. In the present paper, we have not studied the global theoretical convergence of this algorithm and the case of system of PDE. We also think that the generalization of this method to more than two or three space dimensions is easy and we hope to resolve more complicated PDE problems. These points will be the object of our future articles.

## References

[1] K. Abbaoui and Y. Cherruault, Convergence of Adomian method applied to differential equation, Math. Comput. Modelling 28(5) (1994), 103-109.

[2] K. Abbaoui, Les fondements mathématiques de la méthode décomposition-nelle d'Adomian et application à la résolution des équations issues de la biologie et de la médecine, Thèse de Doctorat de l'université de Paris, VI, Octobre 1995.

[3] B. Abbo, N. Ngarasta, B. Mampassi, B. Some and L. Some, A new approach of the Adomian algorithm for solving nonlinear ordinary or partial differential equations, Far East J. Appl. Math. 23(3) (2006), 299-312.

[4] B. Abbo, Nouvel algorithme numérique de résolution des équations différentielles ordinaires (EDO) et des équations aux dérivées partielles (EDP) non linéaires, Thèse de Doctorat unique, de l' Université de Ouagadougou, Burkina Faso, Janvier 2007.

[5] G. Adomian, Nonlinear Stochastic Systems Theory and Application to Physics, Kluwer Academic Publishers, 1989.

[6] E. Babolian and Sh. Javadi, New method for calculating Adomian polynomials, Appl. Math. Comput. 153 (2004), 253-259.

[7] G. Barro, O. So, B. O. Konfe and B. Some, Solving the Cauchy problem for quasilinear equation with power law nonlinearities by the Adomian decomposition method (ADM), Far East J. Appl. Math. 17(3) (2004), 277-285.

[8] W. Chen and Z. Lu, An algorithm for Adomian decomposition method, Appl. Math. Comput. 159 (2004), 221-235.

[9] Y. Cherruault, Modèles et Méthodes Mathématiques pour les Sciences du Vivant, Presses Universitaires de France, Paris, 1998.

[10] Hassan K. Khalil, Nonlinear Systems, Second edition, Prentice-Hall, Inc., 1996.

[11] D. Lesnic, Blow-up solutions obtained using the decomposition method, Chaos, Solitons Fractals 28(3) (2006), 776-787.

[12] J. D. Murray, Nonlinear Differential Equation Models in Biology, Clarendon Press, Oxford, 1977.

[13] J. D. Murray, Mathematical Biology, Springer, Berlin, 1989.

[14] Shepley L. Ross, Introduction to Ordinary Differential Equations, Second edition, University of New Hampshire, 1974.

[15] Andreas Ruffing et al., Comparing algebraic and numerical solutions of classical diffusion process equations in computational financial mathematics, Discrete Dynamics Nature Society 6 (2001), 157-169.

[16] B. Some, Convergence of the Adomian method applied to Fredholm integro-differential equations, Africa Mat. 14(3) (2001), 71-88.

[17] A. Wazwaz, The numerical solution of sixth-order boundary value problems by the modified decomposition method, Appl. Math. Comput. 118 (2001), 311-325.

[18] Y. Zhu, Q. Chang and S. Wu, A new algorithm for calculating Adomian polynomials, Appl. Math. Comput. 169 (2005), 402-416.

■