JP Jour. Algebra, Number Theory & Appl. **7**(2) (2007), 233-240

SOME CONDITIONS FOR A FORM $u\alpha - v$ OF A TWO-GENERATOR EXTENSION $R[\alpha, \beta]$ TO BE A UNIT

KIYOSHI BABA

Department of Mathematics
Faculty of Education and Welfare Science
Oita University
Oita 870-1192, Japan
e-mail: baba@cc.oita-u.ac.jp

Abstract

Let α be an anti-integral element over an integral domain R and β be a linear fractional transform of α. Let u and v be elements of R. Then we give some conditions that $u\alpha - v$ is a unit of $R[\alpha, \beta]$.

Let R be an integral domain with quotient field K and $R[X]$ be a polynomial ring over R in an indeterminate X. Let α be an element of an algebraic field extension of K and $\pi : R[X] \to R[\alpha]$ be the *R-algebra homomorphism* defined by $\pi(X) = \alpha$. Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = t$, and write

$$\varphi_\alpha(X) = X^t + \eta_1 X^{t-1} + \cdots + \eta_t, \quad (n_1, \ldots, \eta_t \in K).$$

We define $I_{[\alpha]} := \bigcap_{i=1}^{t} (R :_R \eta_i)$, where $(R :_R \eta_i) = \{c \in R;\ c\eta_i \in R\}$. An element α is called an *anti-integral element* of degree t over R if $\operatorname{Ker} \pi = I_{[\alpha]} \varphi_\alpha(X) R[X]$.

2000 Mathematics Subject Classification: Primary 13A99; Secondary 13G05.

Keywords and phrases: unit, anti-integral element, denominator ideal.

Received November 1, 2006

Let β be a linear fractional transform of α, that is,

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R, ad - bc \in R^*, a\alpha - b \neq 0),$$

where R^* denotes the set of units of R.

Set $\varphi_\alpha(X, Y) = X^t\varphi_\alpha(Y/X)$. Since $\alpha = (b\beta - d)/(a\beta - c)$, it is easily verified that

$$\varphi_\beta(X) = \varphi_\alpha(a, b)^{-1}\varphi_\alpha(aX - c, bX - d).$$

Assume that α is in K. Ratliff [7] studied the conditions for $\mathrm{Ker}(\pi)$ to have a linear base, where it is said that $\mathrm{Ker}(\pi)$ has a linear base if $\mathrm{Ker}(\pi) = \sum(p_iX - q_i)R[X]$ with $\alpha = q_i/p_i (p_i \neq 0, q_i \in R)$, that is, $\mathrm{Ker}(\pi) = (R :_R \alpha)(X - \alpha)R[X]$, where $(R :_R \alpha) = \{c \in R; c\alpha \in R\}$. Subsequently, Mirbagheri and Ratliff [3] proved that $\mathrm{Ker}(\pi)$ has a linear base if and only if $R[\alpha] \cap R[\alpha^{-1}] = R$. In [5], an element $\alpha \in K$ is called an *anti-integral element* over R if $\mathrm{Ker}(\pi) = (R :_R \alpha)(X - \alpha)R[X]$. Furthermore, in [4], the notion of an anti-integral element over R was extended to the higher degree case, that is, the case that α is an element of an algebraic field extension of K. This notion is a generalization of linear base property.

Let α be an anti-integral element over R and u, v be elements of R. In [6], they gave a condition for $\alpha - v$ to be a unit of $R[\alpha]$. Moreover, [6] gave some conditions for $u\alpha - v$ to be a unit of $R[\alpha]$. In the case of Laurent extension $R[\alpha, \alpha^{-1}]$, [1] gave a condition for $\alpha - v$ to be a unit of $R[\alpha, \alpha^{-1}]$. Let β be a linear fractional transformation of α. Then the cases of $R[\alpha]$ and $R[\alpha, \alpha^{-1}]$ are special ones of $R[\alpha, \beta]$. So it will be worth considering the case $R[\alpha, \beta]$ and we generalize the results in [1] and [6] to the case $R[\alpha, \beta]$.

Our notation is standard and our general reference for unexplained terms is [2].

First we study a criterion for $\alpha - v$ to be a unit of $R[\alpha, \beta]$. We need some lemmas:

Lemma 1. *Let R be an integral domain with quotient field K. Let α be a non-zero element of an algebraic field extension of K and β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \, (a, b, c, d \in R, ad - bc \in R^*, a\alpha - b \neq 0).$$

Then $R[\alpha, \beta] = R\left[\alpha, \dfrac{1}{a\alpha - b}\right]$.

Proof. Since β is in $R\left[\alpha, \dfrac{1}{a\alpha - b}\right]$, we have $R(\alpha, \beta) \subset R\left[\alpha, \dfrac{1}{a\alpha - b}\right]$.

Conversely set $w = ad - bc$. Then w is a unit of R and $\dfrac{1}{a\alpha - b} = w^{-1}(c - a\beta)$. Hence $R\left[\alpha, \dfrac{1}{a\alpha - b}\right] \subset R[\alpha, \beta]$.

Lemma 2. *Let R be an integral domain and α be an anti-integral element over R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R, ad - bc \in R^*, a\alpha - b \neq 0).$$

Let u and v be elements of R such that $u \neq 0$. Assume that $u\alpha - v$ is a unit of $R[\alpha, \beta]$. Then $u(av - bu)$ is in $\sqrt{\varphi_\alpha(u, v) I_{[\alpha]}}$.

Proof. Since $u\alpha - v$ is a unit of $R[\alpha, \beta]$, there exists an element $g(X, Y)$ of $R[X, Y]$ such that $(u\alpha - v) g(\alpha, \beta) = 1$. Set $n = \deg g(X, Y)$ and

$$g(X) = (uX - v)(aX - b)^n g\left(X, \frac{c\alpha - d}{a\alpha - b}\right) - (aX - b)^n.$$

Then $g(X)$ is in $R[X]$ and $g(\alpha) = 0$. Since α is anti-integral over R, we see that $g(X)$ is in $I_{[\alpha]}\varphi_\alpha(X)R[X]$. Hence there exists an element $h(X)$ of $I_{[\alpha]}R[X]$ such that $g(X) = \varphi_\alpha(X)h(X)$. Substituting v/u for X, we get

$$-(av/u - b)^n = g(v/u) = \varphi_\alpha(v/u)h(v/u).$$

Set $t = \deg \varphi_\alpha(X), k = \deg h(X)$ and $h(X, Y) = X^k h(Y/X)$. Then $\varphi_\alpha(v/u) = u^{-t}\varphi_\alpha(u, v)$ and $h(v/u) = u^{-k}h(u, v)$. Moreover, $h(u, v)$ is in $I_{[\alpha]}$. Therefore, we have

$$-u^{t+k-n}(av - bu)^n = \varphi_\alpha(u, v)h(u, v).$$

Then, whether $t + k - n$ is non-negative or not, $u(av - bu)$ is in $\sqrt{\varphi_\alpha(u, v)I_{[\alpha]}}$.

Lemma 3. *Let R be an integral domain and α be an anti-integral element over R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R,\ ad - bc \in R^*,\ a\alpha - b \neq 0).$$

Let u and v be elements of R such that $u \neq 0$. Let P be a prime ideal of $R[\alpha]$ or $R[\alpha, \beta]$ such that $u\alpha - v \in P$. Assume that $u(av - bu) \in \sqrt{\varphi_\alpha(u, v)I_{[\alpha]}}$. Then $u^2(a\alpha - b) \in P$.

Proof. Since $u(av - bu) \in \sqrt{\varphi_\alpha(u, v)I_{[\alpha]}}$, there exist a positive integer m and an element r of $I_{[\alpha]}$ such that $u^m(av - bu)^m = r\varphi_\alpha(u, v)$. Set

$$\varphi_\alpha(X) = X^t + \eta_1 X^{t-1} + \cdots + \eta_t, \ (\eta_1, \ldots, \eta_t \in K),$$

where K is the quotient field of R. Then there exist elements $\lambda_1, \ldots, \lambda_t \in K$ such that

$$\begin{aligned} &X^t + \eta_1 uX^{t-1} + \cdots + \eta_{t-1}u^{t-1}X + \eta_t u^t \\ &= (X - v)^t + \lambda_1(X - v)^{t-1} + \cdots + \lambda_{t-1}(X - v) + \lambda_t. \end{aligned} \tag{1}$$

Note that $\lambda_1, \ldots, \lambda_t$ are in $(\eta_1, \ldots, \eta_t)$ and

$$\lambda_t = v^t + \eta_1 uv^{t-1} + \cdots + \eta_{t-1}u^{t-1}v + \eta_t u^t = \varphi_\alpha(u, v),$$

where $(\eta_1, \ldots, \eta_t)$ is the R-module generated by $\eta_1, \ldots, \eta_t$. In equation (1), substituting $u\alpha$ for X and multiplying r by the both sides of equation (1), we have

$$r(u\alpha - v)^t + r\lambda_1(u\alpha - v)^{t-1} + \cdots + r\lambda_{t-1}(u\alpha - v) + r\lambda_t = 0.$$

Therefore,

$$r(u\alpha - v)^t + r\lambda_1(u\alpha - v)^{t-1} + \cdots + r\lambda_{t-1}(u\alpha - v)$$

$$= -r\lambda_t = -r\varphi_\alpha(u, v) = -u^m(av - bu)^m.$$

Because $u\alpha - v \in P$ and $r\lambda_1, \ldots, r\lambda_{t-1} \in R$, we obtain $u^m(av - bu)^m \in P$. Hence $u(av - bu) \in P$. Moreover, $u\alpha - v \in P$. Therefore, $u^2(a\alpha - b) = ua(u\alpha - v) + u(av - bu) \in P$.

Theorem 4. *Let R be an integral domain and α be an anti-integral element over R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R,\ ad - bc \in R^*,\ a\alpha - b \neq 0).$$

Let v be an element of R. Then the following three conditions are equivalent:

(i) $\alpha - v$ *is a unit of* $R[\alpha, \beta]$.

(ii) $av - b \in \sqrt{\varphi_\alpha(v) I_{[\alpha]}}$.

(iii) $a\alpha - b \in \sqrt{(\alpha - v)R[\alpha]}$.

Proof. (i) $\Rightarrow$ (ii) In Lemma 2, set $u = 1$. Then we see that $av - b$ is in $\sqrt{\varphi_\alpha(1, v) I_{[\alpha]}} = \sqrt{\varphi_\alpha(v) I_{[\alpha]}}$.

(ii) $\Rightarrow$ (iii) Set $A = R[\alpha]$ and let P be an arbitrary prime ideal of A such that $\sqrt{(\alpha - v)A} \subset P$. Lemma 3 asserts that $a\alpha - b$ is in P. Therefore, $\sqrt{(a\alpha - b)A} \subset P$. Since $\bigcap_{\alpha - v \in P} P = \sqrt{(\alpha - v)A}$, this asserts that $a\alpha - b \in \sqrt{(a\alpha - b)A}$.

(iii) $\Rightarrow$ (i) By the condition (iii), there exist a positive integer k and an element $p(X)$ of $R[X]$ such that $(a\alpha - b)^k = (\alpha - v)p(\alpha)$. Then Lemma 1 implies that $p(\alpha)/(a\alpha - b)^k$ is in $R[\alpha, \beta]$. Hence $(\alpha - v)p(\alpha)/(a\alpha - b)^k = 1$ and $\alpha - v$ is a unit of $R[\alpha, \beta]$.

The following generalizes the results of [6, Theorem 6] and [1, Proposition 9].

Corollary 5. *Let R be an integral domain and α be an anti-integral element over R. Let v be an element of R. Then the following two assertions hold*:

(1) $\alpha - v$ *is a unit of* $R[\alpha]$ *if and only if* $\varphi_\alpha(v)I_{[\alpha]} = R$.

(2) *Suppose that* $\alpha \neq 0$. *Then* $\alpha - v$ *is a unit of* $R[\alpha, \alpha^{-1}]$ *if and only if* $v \in \sqrt{\varphi_\alpha(v)I_{[\alpha]}}$.

Proof. It is immediate from Theorem 4 by setting $a = 0, b = -1, c = 1, d = 0$ in (1), and setting $a = 1, b = 0, c = 0, d = -1$ in (2).

Our main theorem is the following:

Theorem 6. *Let R be an integral domain and α be an anti-integral element over R. Let β be a linear fractional transform of α, that is,*

$$\beta = \frac{c\alpha - d}{a\alpha - b} \quad (a, b, c, d \in R, ad - bc \in R^*, a\alpha - b \neq 0).$$

Let u and v be elements of R such that $u \neq 0$. Then the following three conditions are equivalent:

(i) $u\alpha - v$ *is a unit of* $R[\alpha, \beta]$.

(ii) $(u, v)R[\alpha, \beta] = R[\alpha, \beta]$ *and* $u(av - bu) \in \sqrt{\varphi_\alpha(u, v)I_{[\alpha]}}$.

(iii) $a\alpha - b \in \sqrt{(u\alpha - v)R[\alpha, \beta]}$.

Proof. (i) $\Rightarrow$ (ii) Assume that $(u, v)R[\alpha, \beta] \neq R[\alpha, \beta]$. Then there exists a prime ideal P of $R[\alpha, \beta]$ such that $(u, v)R[\alpha, \beta] \subset P$. Thus $u\alpha - v$ is in P. This contradicts the condition (i). The latter half is proved by Lemma 2.

(ii) $\Rightarrow$ (iii) Let P be an arbitrary prime ideal of $R[\alpha, \beta]$ such that $u\alpha - v \in P$. Then $u^2(a\alpha - b) \in P$ by Lemma 3. We shall show that u is not in P. Assume that $u \in P$. Then u and v are in P because $u\alpha - v \in P$. This contradicts the fact $(u, v)R[\alpha, \beta] = R[\alpha, \beta]$. Hence $a\alpha - b \in P$. Note that $\sqrt{(u\alpha - v)R[\alpha, \beta]} = \bigcap P$, where the intersection is taken over P such that $u\alpha - v \in P$ and $P \in \operatorname{Spec} R[\alpha, \beta]$. This implies that $a\alpha - b \in \sqrt{(u\alpha - v)R[\alpha, \beta]}$.

(iii) $\Rightarrow$ (i) By the condition (3), there exist a positive integer k and an element $p(X, Y)$ of $R[X, Y]$ such that $(a\alpha - b)^k = (u\alpha - v)p(\alpha, \beta)$. Lemma 1 implies that $p(\alpha, \beta)/(a\alpha - b)^k \in R[\alpha, \beta]$. Hence $(u\alpha - v)(p(\alpha, \beta)/(a\alpha - b)^k) = 1$, and $u\alpha - v$ is a unit of $R[\alpha, \beta]$.

Corollary 7 (cf. [6, Theorem 11])**.** *Let R be an integral domain and α be an anti-integral element over R. Let u and v be elements of R such that $u \neq 0$. Then the following conditions are equivalent*:

(i) $u\alpha - v$ *is a unit of* $R[\alpha]$.

(ii) $(u, v)R[\alpha] = R[\alpha]$ *and* $u \in \sqrt{\varphi_\alpha(u, v)I_{[\alpha]}}$.

Proof. By setting $a = 0$, $b = -1$, $c = 1$, $d = 0$ in Theorem 6, we can prove Corollary 7.

Corollary 8. *Let R be an integral domain and α be non-zero anti-integral element over R. Let u and v be elements of R such that $u \neq 0$. Then the following conditions are equivalent*:

(i) $u\alpha - v$ *is a unit of* $R[\alpha, \alpha^{-1}]$.

(ii) $(u, v)R[\alpha, \alpha^{-1}] = R[\alpha, \alpha^{-1}]$ *and* $uv \in \sqrt{\varphi_\alpha(u, v)I_{[\alpha]}}$.

(iii) $\alpha \in \sqrt{(u\alpha - v)R[\alpha, \alpha^{-1}]}$.

Proof. It is clear by Theorem 6 by setting $a = 1$, $b = 0$, $c = 0$, $d = -1$.

Let R be an integral domain and α be an anti-integral element over R. Let β be a linear fractional transform of α and u, v be elements of R such that $u \neq 0$. We cannot use the condition $(u, v)R[\alpha] = R[\alpha]$ instead of $(u, v)R[\alpha, \beta] = R[\alpha, \beta]$ in the condition (ii) of Theorem 6 as the following example shows. Also we cannot use the condition $a\alpha - b \in \sqrt{(u\alpha - v)R[\alpha]}$ instead of $a\alpha - b \in \sqrt{(u\alpha - v)R[\alpha, \beta]}$ in the condition (ii) of Theorem 6.

Example 9. Set $R = \mathbf{Z}$ and $\alpha = \sqrt{3}$. Then α is an anti-integral element over $\mathbf{Z}$. Set $u = 3$, $v = 0$ and $\beta = (\sqrt{3})^{-1}$. Then the following are easily verified:

(1) $u\alpha - v = 3\sqrt{3}$ is a unit of $\mathbf{Z}[\sqrt{3}, (\sqrt{3})^{-1}]$.

(2) $(u, v)\mathbf{Z}[\sqrt{3}] = 3\mathbf{Z}[\sqrt{3}] \neq \mathbf{Z}[\sqrt{3}]$.

(3) $a\alpha - b = \sqrt{3} \notin 3\sqrt{3}\mathbf{Z}[\sqrt{3}] = \sqrt{(u\alpha - v)R[\alpha]}$.

(4) Set $P = \sqrt{3}\mathbf{Z}[\sqrt{3}]$. Then P is a prime ideal of $\mathbf{Z}[\sqrt{3}]$, $u\alpha - v = 3\sqrt{3} \in P$ and $(u, v)\mathbf{Z}[\sqrt{3}] = 3\mathbf{Z}[\sqrt{3}] \subset P$. Hence $u\alpha - v$ is not a unit of $\mathbf{Z}[\sqrt{3}]$.

References

[1] M. Kanemitsu, K. Baba and K. Yoshida, The invertibility of an element $\alpha^2 - a$ of a super-primitive extension $R[\alpha]/R$ and a linear form of a Laurent extension $R[\alpha, \alpha^{-1}]$, Scientiae Mathematicae 3(3) (2000), 371-376.

[2] H. Matsumura, Commutative Algebra, 2nd ed., Benjamin, New York, 1980.

[3] A. Mirbagheri and L. J. Ratliff, Jr., On the intersection of two overrings, Houston J. Math. 8 (1982), 525-535.

[4] S. Oda, J. Sato and K. Yoshida, High degree anti-integral extensions of Noetherian domains, Osaka J. Math. 30(1) (1993), 119-135.

[5] S. Oda and K. Yoshida, Anti-integral extensions of Noetherian domains, Kobe J. Math. 5 (1988), 43-56.

[6] S. Oda and K. Yoshida, On units of an anti-integral extension $R[\alpha]$ and conditions for a linear fractional transformation β of α to satisfy $R[\beta] = R[\alpha]$, submitted to Algebra Colloquium.

[7] L. J. Ratliff, Jr., Conditions for $\mathrm{Ker}(R[X] \to R[c/b])$ to have a linear base, Proc. Amer. Math. Soc. 39 (1973), 509-514.

■