

ORIENTABILITY OF REAL PARTS AND SPIN STRUCTURES

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Abstract

We establish the orientability and orientations of vector bundles that arise as the real parts of real structures by utilizing spin structures.

1. Introduction

Unlike complex algebraic varieties, real algebraic varieties are in general nonorientable, the simplest example being the real projective plane \mathbf{RP}^2 . Even if they are orientable, there may not be canonical orientations. It has been an important problem to resolve the orientability and orientation issues in real algebraic geometry. In 1974, Rokhlin introduced the complex orientation for dividing real algebraic curves in \mathbf{RP}^2 , which was then extended around 1982 by Viro to the so-called type-I real algebraic surfaces. A detailed historic count was presented in the lucid survey by Viro [10], where he also made some speculations on higher dimensional varieties.

In this short note, we investigate the following more general situation. We take $\sigma : X \rightarrow X$ to be a smooth involution on a smooth

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manifold of an arbitrary dimension. (It is possible to consider involutions on topological spaces with appropriate modifications.) Henceforth, we will assume that X is connected for certainty. In view of the motivation above, let us denote the fixed point set by $X_{\mathbf{R}}$, which in general is disconnected and will be assumed to be non-empty throughout the paper. Suppose $E \rightarrow X$ is a complex vector bundle and assume σ has an involutorial lifting σ_E on E that is *conjugate linear* fiberwise. We call σ_E a real structure on E and its fixed point set $E_{\mathbf{R}}$ a real part. Clearly the restricted projection $E_{\mathbf{R}} \rightarrow X_{\mathbf{R}}$ is a real vector bundle of a rank equal to the complex rank of E . The main results in the paper relate the orientability and orientations of the bundle $E_{\mathbf{R}}$ with spin structures on E . In the special cases of $\dim X = 2, 4$ and $E = TX$, we can already recover and extend results obtained by Rokhlin and Viro.

Aside from real algebraic varieties, we are mainly motivated by the real versions of Gauge Theory, where it is important to be able to determine the orientability and orientations of various real moduli spaces. In such a context, E will be the (virtual) index bundle of a family of Fredholm operators parameterized by a space X .

Edmonds [3] was the first to explore the orientability of the fixed point set of an involution with the help of spin structures. He showed that if the involution preserves a spin structure on the manifold, in particular it must preserve the orientation of the manifold, then the fixed point set is an orientable submanifold. Our result in the paper with $E = TX$, is complementary to Edmonds' in the sense that when $\dim X = 2(2k+1)$, our real structure reverses the orientation on X so Edmonds' theorem does not apply. In addition, we will have results about the orientation bundle of $E_{\mathbf{R}}$ when the spin structure is not preserved by σ_E . In terms of techniques, the approach taken in [3] is topological, while ours is more direct and geometrical. Bott and Taubes [2] gave a direct proof of Edmonds' result, and it seems that our approach is more comparable to theirs. Another direct proof of Edmonds' theorem appeared in Ono [7], which also contains a spin^c version. In turn this was extended in Ono and Stolz [8] to infinitely dimensional loop spaces and pin^{\pm} manifolds.

Moreover, Nagami [5] used branched covers to characterize those orientations on the fixed point sets that were obtained in [7]. Our paper differs from these papers mainly in that the dealing of real structures forces us to take a different (conjugate) lifting on the frame bundle of E , although there is a certain common thread of the methods used in all papers.

For a brief description of the main results, Theorem 1 shows that the real bundle $E_{\mathbf{R}}$ is orientable and an orientation is determined by a conjugate spin lifting, under the assumption that the real structure σ_E is compatible with a spin structure on E . Without assuming this compatibility, Theorem 4 characterizes the orientation bundle of $E_{\mathbf{R}}$ in terms of the deficiency of the compatibility. We also discuss a few examples as well as some remarks concerning the case when the bundle E is not spin.

2. The Compatible Spin Case

To set up the notations, consider a rank r Riemannian oriented vector bundle E on a smooth manifold X , with the $SO(r)$ -frame bundle of E denoted by P . By definition, a spin structure is a class $\xi \in H^1(P, \mathbf{Z}_2)$ that restricts fiberwise to the non-trivial element in $H^1(SO(r), \mathbf{Z}_2)$. Thus there associates a $Spin(r)$ -principal bundle $P_\xi \rightarrow X$ together with a fiber-preserving double covering $P_\xi \rightarrow P$. (To be precise, we should really call the covering $P_\xi \rightarrow P$ a spin structure on E .)

Let $\sigma : X \rightarrow X$ be a smooth, not necessarily orientation-preserving, involution. Denote its fixed point set by $X_{\mathbf{R}}$. Suppose from now on that E is a rank r Hermitian complex vector bundle and that σ can be lifted as a *real structure* σ_E on E . Namely σ_E is an involution and is fiberwise conjugate linear on E . We will also assume that σ_E preserves the Hermitian fiber metric so that it induces another lifting σ_U on the unitary frame bundle P_U . Note that this is a *conjugate morphism* in the sense that $\sigma_U(pg) = \sigma_U(p)\bar{g}$, where $p \in P_U$, $g \in U(r)$ and \bar{g} is the

complex conjugate. Using the standard inclusion $\rho : U(r) \hookrightarrow SO(2r)$, the lifting σ_U carries over to $P = P_U \times_{\rho} SO(2r)$, which we denote by σ_P . Note that σ_P is again a conjugate morphism since $\sigma_P(pg) = \sigma_P(p)\bar{g}$ for $p \in P$, $g \in SO(2r)$, where $\bar{g} = TgT^{-1}$ with $T = T^{-1}$ to be the $(2r) \times (2r)$ diagonal matrix

$$\text{diag}\{1, -1, 1, -1, \dots, 1, -1\}.$$

Remark. When the complex rank r is odd, the real structure σ_E on E is orientation reversing fiberwise, hence it does not induce a *bundle morphism* on the $SO(2r)$ -frame bundle P . When r is even, σ_E preserves the orientation and hence induces a bundle morphism on P which is however different from the σ_P defined above, as the latter is a conjugate morphism.

We now introduce the compatibility between a spin structure and a real structure.

Definition. We say a *real structure* $\sigma_E : E \rightarrow E$ is compatible with a spin structure $\xi \in H^1(P, \mathbf{Z}_2)$ on E if the induced conjugate lifting $\sigma_P : P \rightarrow P$ satisfies $\sigma_P^* \xi = \xi$, or equivalently, if there exists a conjugate lifting morphism $\sigma_{\xi} : P_{\xi} \rightarrow P_{\xi}$, namely $\sigma_{\xi}(pg) = \sigma_{\xi}(p)\bar{g}$, for $p \in P_{\xi}$, $g \in Spin(2r)$.

Here the conjugation on $Spin(2r)$ is the lifting of that on $SO(2r)$ via the double covering $Spin(2r) \rightarrow SO(2r)$. Alternatively, it is the restriction of the conjugation on the Clifford algebra $Cl(\mathbf{R}^{2r}) = Cl(\mathbf{C}^r)$, where it is induced by the complex conjugation on \mathbf{C} . Note there are always two conjugate liftings σ_{ξ} on P_{ξ} whenever there exists one. The equivalence in the definition is for the same topological reason as the usual case. In the definition above, we adopt the term “compatible” instead of the term “preserving”, in order to distinguish from the usual case where only non-conjugate lifting morphisms are involved.

Theorem 1. *Suppose a complex vector bundle E has a spin structure ξ and a real structure σ_E that is compatible with ξ . Then any conjugate*

lifting σ_ξ on the spin bundle P_ξ determines a unique orientation on the real vector bundle $E_{\mathbf{R}} \rightarrow X_{\mathbf{R}}$. The orientations resulting from the two different conjugate liftings are exactly opposite each other.

Proof. First we consider the special case that E is a complex line bundle. Here P is an $SO(2) = U(1)$ -bundle while the fixed point set $P^\sigma \rightarrow X_{\mathbf{R}}$ of the conjugate lifting $\sigma_P : P \rightarrow P$ is a \mathbf{Z}_2 -subbundle. In fact, if p, pg are fixed points on the same fiber of P^σ , then $pg = \sigma_P(p)\bar{g} = p\bar{g}$ forcing $\bar{g} = g$. It follows that $g = \pm 1 \in U(1)$ and P^σ is a \mathbf{Z}_2 -bundle. Clearly P^σ is the associated principal bundle of the real line bundle $E_{\mathbf{R}} \rightarrow X_{\mathbf{R}}$. Let σ_ξ be one of the two spin liftings on P_ξ . Since σ_ξ is also a conjugate morphism on the bundle P_ξ of the $Spin(2) = U(1)$ structure group, one shows similarly that the fixed point set $P_\xi^\sigma \rightarrow X_{\mathbf{R}}$ is again a principal \mathbf{Z}_2 -bundle.

Fix a fiber of P_ξ^σ for a moment and let a, b be the two points on the fiber. Then $b = -a$ with $-1 \in \mathbf{Z}_2 \subset Spin(2)$, as we have shown above. Hence under the spin double covering $\pi : P_\xi \rightarrow P$, $\pi(b) = \pi(a)$, as -1 is mapped to 1 in the standard covering $Spin(2) \rightarrow SO(2)$. In other words, any fiber of P_ξ^σ is mapped to a single point of the corresponding fiber of P^σ . By varying the fibers on $X_{\mathbf{R}}$, the bundle P_ξ^σ is mapped onto a unique trivialization of $P^\sigma \rightarrow X_{\mathbf{R}}$. This in turn yields a unique orientation of the line bundle $E_{\mathbf{R}}$. Furthermore, if we take the other spin lifting $\sigma'_\xi := \sigma_\xi \circ \tau$, where $\tau : P_\xi \rightarrow P_\xi$ is the deck transformation of π , then any fiber of $P_\xi^{\sigma'}$ is mapped to the other point of the corresponding fiber of P^σ . Thus σ'_ξ yields exactly the opposite orientation of $E_{\mathbf{R}}$. This proves the theorem in the case of a complex line bundle E .

The higher rank case follows easily by applying the above argument to the complex determinant bundle $L = \det E = \wedge^r E$. Indeed, from E ,

the line bundle L inherits a spin structure as well as a compatible real structure. Furthermore any conjugate spin lifting for E induces one for L and consequently yields a unique orientation on $L_{\mathbf{R}}$. Finally note that $E = E_{\mathbf{R}} \otimes \mathbf{C}$ on $X_{\mathbf{R}}$, the real determinant $\det E_{\mathbf{R}}$ is exactly $L_{\mathbf{R}} = (\det E)_{\mathbf{R}}$. Hence $E_{\mathbf{R}}$ and $L_{\mathbf{R}}$ have the same orientations.

In the proof we have implicitly used the fact that on a complex vector bundle E , the spin structures are in one-to-one correspondence with the square roots of the canonical bundle (i.e., square roots of the determinant bundle). In this terminology, a real structure σ_E is compatible with a spin structure if and only if σ_E can be lifted to a conjugate linear homomorphism on the corresponding square root.

Remark. A pair of opposite orientations is sometimes referred to as a *semi-orientation*, which makes sense only if the underlying manifold is disconnected. Regardless of the two spin liftings on P_{ξ} , Theorem 1 can be put simply as that on a complex bundle E with a real structure, any spin structure on E that is compatible with the real structure determines a unique semi-orientation on $E_{\mathbf{R}}$. (The original Theorem 1 is a bit stronger in that it actually specifies the orientations, not just semi-orientations.) This is how we will state the next two corollaries for the sake of simplicity.

Corollary 2. *Assume X has a trivial $H^1(X, \mathbf{Z}_2)$ group and σ_E is a real structure on a complex vector bundle E . If E is spin, then there is a unique semi-orientation on $E_{\mathbf{R}}$.*

The proof is clear: there is only one spin structure on E and it must therefore be compatible with σ_E . The corollary is useful because without assuming $H^1(X, \mathbf{Z}_2) = 0$, it is often difficult to determine whether a real structure is compatible with a spin structure.

Suppose (X, J) is an almost complex manifold and $\sigma : X \rightarrow X$ is a real structure, that is, $\sigma_* : TX \rightarrow TX$ satisfies $\sigma_* \circ J = -J \circ \sigma_*$. Applying Theorem 1 to the complex vector bundle TX gives us

Corollary 3. *Any spin structure on X that is compatible with σ determines a unique semi-orientation on the real part $X_{\mathbf{R}}$.*

Example. Real algebraic curves. Here $X_{\mathbf{R}} \subset \mathbf{RP}^2$ is defined by a real homogeneous polynomial of three variables and $X \subset \mathbf{CP}^2$ is the complexification. The real structure σ on X is the restriction of the complex conjugation on \mathbf{CP}^2 . Of course $X_{\mathbf{R}}$ consists of circles topologically so is certainly orientable, although it does not inherit any obvious orientation. The complement $X \setminus X_{\mathbf{R}}$ can have at most two connected components. When there are two components, $X_{\mathbf{R}}$ is called a *dividing curve*. In this case, the orientations on both components yield the same semi-orientation on the boundary $X_{\mathbf{R}}$, which is then called the *complex orientation* of $X_{\mathbf{R}}$ by Rokhlin [10]. For non-dividing curves, no canonical semi-orientation exists on $X_{\mathbf{R}}$. Instead, semi-orientations depend on and are determined by compatible spin structures on X , a result first obtained by Natanzon [6], where he uses the Fuchsian group of hyperbolic automorphisms of the Riemann surface X . Applying Corollary 3 gives an alternative and easier proof of this. (The author has not checked how the two kinds of semi-orientations might be related.) Actually the first part of Theorem 1 states a bit stronger result that each conjugate lifting in the spin bundle determines a unique orientation, not just a semi-orientation, on $X_{\mathbf{R}}$. It can be shown that for any real algebraic curve with non-empty $X_{\mathbf{R}}$, there is always a spin structure on X , that is, compatible with the conjugation. (This follows from [6; Theorem 5.1], where compatible spin structures are shown in one-to-one correspondence with compatible Arf functions on X . The existence of the latter is given in [6; Theorem 3.4]. In [6], non-singular Arf functions mean $X_{\mathbf{R}} \neq \emptyset$ and spinor bundles on real curves mean compatible spin structures in our sense, cf. Section 3 and Section 5 of the paper.) As alluded in the Introduction, Edmonds' result [3] is not applicable to $X_{\mathbf{R}}$ here, since σ is orientation reversing on X .

Example. Real algebraic surfaces. Here $X_{\mathbf{R}} \subset \mathbf{RP}^3$ and $X \subset \mathbf{CP}^3$, given by a real polynomial of four variables. The Lefschetz hyperplane

theorem tells us that X is simply connected and is spin if further the defining polynomial has an even degree. Thus Corollaries 2 and 3 imply that $X_{\mathbf{R}}$ carries a canonical semi-orientation, as long as the degree of $X_{\mathbf{R}}$ is even. In the special case of type-I real surfaces, that is, for those with $[X_{\mathbf{R}}] = 0 \in H_2(X, \mathbf{Z}_2)$, Viro [10] has constructed semi-orientations on $X_{\mathbf{R}}$ by using the double cover of X branched along $X_{\mathbf{R}}$. Our corollaries here apply to both type-I and non type-I surfaces. Note Edmonds' result [3] can also be applied to show that $X_{\mathbf{R}}$ is orientable when X is spin, although it does not tell us anything about the semi-orientation.

Theorem 1 and its corollaries apply equally well to higher dimensional real algebraic varieties. In Viro [10], only some speculated results were possible to make for higher dimensions, which moreover require some unsettled assumptions such as $[X_{\mathbf{R}}] = 0 \in H_r(X, \mathbf{Z}_2)$, (where $\dim_{\mathbf{R}} X = 2r$) and $H^1(X, \mathbf{Z}_2) = 0$.

The set \mathcal{S}^{σ} of spin structures compatible with a given real structure σ_E on E is an affine space modelled on $H^1(X, \mathbf{Z}_2)^{\sigma}$. The case of real algebraic curves already shows that $E_{\mathbf{R}}$ gets different semi-orientations from different spin structures in \mathcal{S}^{σ} . In general it remains to be seen how semi-orientations depend on classes in $H^1(X, \mathbf{Z}_2)^{\sigma}$. This will be answered readily in the next section.

3. The Incompatible Spin Case

Next we move on to study the case where the real structure is not compatible with the spin structure.

Let $E, P, \xi, \sigma_E, \sigma_P$ be defined as in the previous section, but no longer assume $\sigma_P^* \xi - \xi = 0 \in H^1(P, \mathbf{Z}_2)$. Nonetheless it is easy to check that $\sigma_P^* \xi - \xi$ is the pull-back of a class $\alpha \in H^1(X, \mathbf{Z}_2)^{\sigma}$, i.e., α is invariant under σ . So we can write $\sigma_P^* \xi = \xi + \alpha$ using the injection $H^1(X, \mathbf{Z}_2) \rightarrow H^1(P, \mathbf{Z}_2)$.

Fix a spin bundle P_ξ for the spin structure ξ and choose a real line bundle $l \rightarrow X$ representing $\alpha: w_1(l) = \alpha$. Put any metric on l and let P_l denote the principal \mathbf{Z}_2 -bundle. Then we have the twisted bundle $P_{\xi, l}$ of P_ξ by P_l and $P_{\xi, l}$ is a spin bundle of the spin structure $\sigma^*\xi$. Here $P_{\xi, l}$ is defined by the transition functions $g_{ij} \cdot h_{ij}$, if g_{ij}, h_{ij} are respectively the transition functions of P_ξ and P_l . We need to set up bundles carefully, not just their isomorphism types, so that we can state the following result.

Theorem 4. *Suppose σ_E is a real structure on a complex bundle E that admits a spin structure ξ . Then for the real part $E_{\mathbf{R}} \rightarrow X_{\mathbf{R}}$, we have $w_1(\det E_{\mathbf{R}}) = \alpha'$, where $\alpha' = \alpha|_{X_{\mathbf{R}}}$. Furthermore, up to a sign, there is a canonical isomorphism*

$$\det E_{\mathbf{R}} \rightarrow l',$$

where l' is the restriction of l to $X_{\mathbf{R}}$. In other words, the line bundle $\det E_{\mathbf{R}} \otimes (l')^{-1}$ carries a canonical semi-orientation.

Proof. As in the proof of Theorem 1, we can assume E has rank 1 by taking determinant if necessary. The idea is then to consider the double cover of X corresponding to α and apply Theorem 1 to the pull backs on the cover.

Rename $\tilde{X} = P_l$ so we have a double covering $\pi: \tilde{X} \rightarrow X$. By definition, $\tilde{X} \rightarrow X$ consists of local fiberwise orientations of l and is endowed with the natural topology. Thus the pull back bundle $\tilde{l} = \pi^*l \rightarrow \tilde{X}$ is trivial with a canonical orientation and the deck transformation $\tau: \tilde{X} \rightarrow \tilde{X}$ of π lifts to -1 under the trivialization, so we have the quotient bundle $\tilde{l}/\tau = l$.

Since $\sigma^*\alpha = \alpha \in H^1(X, \mathbf{Z}_2)$, there is a lifting homomorphism $\sigma_l: l \rightarrow l$ (not unique, $-\sigma_l$ is the other). This in turn gives a lifting involution $\tilde{\sigma}: \tilde{X} \rightarrow \tilde{X}$ on the associated principal bundle. Note on the

real parts, the restriction $\pi : \tilde{X}_{\mathbf{R}} \rightarrow X_{\mathbf{R}}$ is not necessarily surjective. To identify the image, note the lifting homomorphism σ_l must restrict to ± 1 on the fibers of l over $X_{\mathbf{R}}$, as it preserves a fiber metric. Let $X_{\mathbf{R}}^{\pm}$ denote the sets of those components of $X_{\mathbf{R}}$, where $\sigma_l = \pm 1$, respectively over the fibers. Then $X_{\mathbf{R}} = X_{\mathbf{R}}^+ \cup X_{\mathbf{R}}^-$ and $\pi : \tilde{X}_{\mathbf{R}} \rightarrow X_{\mathbf{R}}^+$ is surjective, hence is a double cover. Of course, the double cover is the principal bundle of the line bundle

$$l^+ := l'|_{X_{\mathbf{R}}^+} \rightarrow X_{\mathbf{R}}^+.$$

Our first goal is to show that the orientation bundle of $E_{\mathbf{R}}$ restricted to $X_{\mathbf{R}}^+$ is naturally isomorphic to l^+ up to a sign.

Let $\tilde{P} \rightarrow \tilde{X}$ be the pull back of P via π . So we have also the homomorphism $\pi : \tilde{P} \rightarrow P$ and the pull back $\tilde{\xi} \in H^1(\tilde{P}, \mathbf{Z}_2)$ of ξ , which is a spin structure on \tilde{P} . Clearly the pull back bundle $\tilde{P}_{\tilde{\xi}}$ of P_{ξ} is the spin bundle of $\tilde{\xi}$.

Now the conjugate lifting $\sigma_P : P \rightarrow P$ pulls back a conjugate lifting $\tilde{\sigma}_P : \tilde{P} \rightarrow \tilde{P}$ over the involution $\tilde{\sigma} : \tilde{X} \rightarrow \tilde{X}$. Since $\pi^* \alpha = 0 \in H^1(\tilde{X}, \mathbf{Z}_2)$, the last lifting preserves the spin structure $\tilde{\xi}$ on \tilde{P} :

$$\tilde{\sigma}_P^* \tilde{\xi} - \tilde{\xi} = \pi^*(\sigma_P^* \xi - \xi) = \pi^* \alpha = 0.$$

Thus Theorem 1 applies to show that the fixed point set $\tilde{P}^{\tilde{\sigma}}$ of $\tilde{\sigma}_P$ is a trivial bundle on $\tilde{X}_{\mathbf{R}}$. To actually obtain a specific trivialization, we need to examine the spin bundle of $\tilde{\xi}$ and the spin liftings of $\tilde{\sigma}_P$ as in the proof of Theorem 1. So consider the spin bundles $P_{\xi}, P_{\xi, l}$ of ξ and $\sigma^* \xi$ that were set up early. By definition of l , we have a conjugate lifting $\check{\sigma} : P_{\xi} \rightarrow P_{\xi, l}$ (not unique, $-\check{\sigma}$ is the other) of $\sigma : X \rightarrow X$. Since the pull back of l via π is a trivial bundle, we have a pull back conjugate morphism of $\check{\sigma}$

$$\tilde{\sigma}_\xi : \tilde{P}_\xi \rightarrow \tilde{P}_\xi \quad (1)$$

on the double cover \tilde{X} , where \tilde{P}_ξ is the pull back of P_ξ . It is not hard to see that \tilde{P}_ξ is the spin bundle of $\tilde{\xi}$ and the conjugate morphism $\tilde{\sigma}_\xi$ is a spin lifting of $\tilde{\sigma}_P : \tilde{P} \rightarrow \tilde{P}$. Let \tilde{P}_ξ^σ be the fixed point set of $\tilde{\sigma}_\xi$. From the proof of Theorem 1, the image set $\tilde{P}_\xi^\sigma \rightarrow \tilde{P}^\sigma$ gives rise to our desired trivialization of $\tilde{P}^\sigma \rightarrow \tilde{X}_\mathbf{R}$.

Next check the lifting $\tilde{\tau}$ of the deck transformation τ on \tilde{P}^σ . Since the construction of the conjugate morphism $\tilde{\sigma}_\xi$ in (1) uses the lifting -1 on the trivial bundle \tilde{l} (namely, the lifting switches the orientations of the associated trivial principal bundle), the lifting of τ on \tilde{P}_ξ will switch the fixed points in \tilde{P}_ξ^σ into the non-fixed points. Therefore, the push-forward lifting $\tilde{\tau}$ on \tilde{P}^σ will switch the trivialization obtained from the image of \tilde{P}_ξ^σ into the opposite. In other words, the quotient bundle $\tilde{P}^\sigma / \tilde{\tau} \rightarrow X_\mathbf{R}^+$ is the principal bundle associated to the double cover $\pi : \tilde{X}_\mathbf{R} \rightarrow X_\mathbf{R}^+$. Since $\det E_\mathbf{R} |_{X_\mathbf{R}^+}$ is the line bundle of $P^\sigma = \tilde{P}^\sigma / \tilde{\tau}$ and l^+ is that of the principal bundle $\pi : \tilde{X}_\mathbf{R} \rightarrow X_\mathbf{R}^+$, this means that $\det E_\mathbf{R} |_{X_\mathbf{R}^+}$ is isomorphic to l^+ .

So far, the only choice we have made is for the lifting $\check{\sigma} : P_\xi \rightarrow P_\xi, l$. If we use $-\check{\sigma}$ instead, we would get the opposite trivialization on \tilde{P}^σ , which descends to the negative of the isomorphism on $P^\sigma = \tilde{P}^\sigma / \tilde{\tau}$. That is to say, the above isomorphism $\det E_\mathbf{R} |_{X_\mathbf{R}^+} \rightarrow l^+$ is well-defined up to a sign.

Finally we consider the other components $X_\mathbf{R}^- \subset X_\mathbf{R}$, where the lift $\sigma_l = -1$ over the fibers of l . However for the lifting $-\sigma_l : l \rightarrow l$, the argument above is suitable and can be repeated to show that $\det E_\mathbf{R} |_{X_\mathbf{R}^-}$

is naturally isomorphic to $l^- = l'|_{X_{\mathbf{R}}}$ up to a sign. Combining the two parts together, we have the required isomorphism $\det E_{\mathbf{R}} \rightarrow l'$. This surely implies $w_1(E_{\mathbf{R}}) = \alpha'$ as stated in the theorem.

Note that Theorem 4 is consistent with Theorem 1: In the compatible spin case, $l \rightarrow X$ is a trivial bundle. Fix a trivialization on l . Each spin lifting in Theorem 1 picks up the lifting $\check{\sigma}$ as in the proof of Theorem 4, hence determines a unique isomorphism $\det E_{\mathbf{R}} \rightarrow l'$ and an orientation on $E_{\mathbf{R}}$ as well. Switching between the two spin liftings corresponds to switching between $\check{\sigma}$ and $-\check{\sigma}$.

For a simple illustrative example, consider the Hopf surface $X = (\mathbf{C}^2 \setminus \{0\})/G$, where G is the infinite cyclic group generated by $(z_1, z_2) \mapsto (2z_2, 2z_1)$. The complex conjugation on \mathbf{C}^2 descends to a real structure σ on X . Note that σ switches the two spin structures on $X \approx S^1 \times S^3$ and the real part is a Klein bottle.

Remark. In [3], Edmonds showed that for a smooth involution on a spin manifold X , the orientation bundle of the fixed point set F lies in the image set of the restriction map $H^1(X, \mathbf{Z}_2) \rightarrow H^1(F, \mathbf{Z}_2)$. It is not clear to us how the proof of Theorem 4 can be adapted here since we have used essentially that E is a complex vector bundle, although we speculate that Edmonds' result can be strengthened in a way similar to Theorem 4.

If $\xi_1 \in H^1(P, \mathbf{Z}_2)$ is another spin structure on E and $\alpha_1 = \sigma_P^* \xi_1 - \xi_1 \in H^1(X, \mathbf{Z}_2)$, Theorem 4 would imply $\alpha'_1 = \alpha_1|_{X_{\mathbf{R}}}$ must be α' . This indeed is true, following from the simple calculation

$$\alpha_1 - \alpha = \sigma_P^* \gamma - \gamma,$$

where $\gamma = \xi_1 - \xi \in H^1(P, \mathbf{Z}_2)$, and σ_P^* of course restricts to the identity on $X_{\mathbf{R}}$, so $\alpha'_1 - \alpha' = 0$.

As a kind of by-product, we now find out how the orientations on $E_{\mathbf{R}}$ depend on the compatible spin structures, which is an extension of

Theorem 1. So continue using E, σ_E, ξ, P_ξ as in Theorem 1. Moreover, let η be a second spin structure on E that is compatible with σ_E and P_η be the spin principal bundle. Assume $\eta = \xi + \alpha$. We have seen that $\alpha \in H^1(X, \mathbf{Z}_2)^\sigma$ at the end of Section 2. Choose a line bundle $l \rightarrow X$ representing α . Then as in the proof of Theorem 4, σ lifts to an involution σ_l on l and the restriction $l' = \pm 1$ on fibers over $X_{\mathbf{R}}$. Recall from Theorem 1, each spin conjugate lifting on P_ξ or P_η determines a unique orientation on $E_{\mathbf{R}}$.

Proposition 5. *Upon suitable choices of spin conjugate liftings on P_ξ, P_η , the corresponding orientations on $E_{\mathbf{R}}$ differ by a factor of value $l' = \pm 1$ on $X_{\mathbf{R}}$.*

Proof. Adopting a similar strategy as in Theorem 4, but here for a slight advantage, we will work with vector bundles instead of principal bundles.

So let the complex line bundle K be a square root of $\det E$, representing the spin structure ξ . Then by definition of $l, K \otimes_{\mathbf{R}} l$ is a square root of $\det E$, representing the spin structure η . Since σ_E is compatible with ξ , $\det \sigma_E$, lifts to a conjugate linear morphism $\hat{\sigma}_\xi$ on K via the squaring map

$$K \rightarrow \det E, v \mapsto v \otimes v.$$

Moreover, similar to the proof of Theorem 1, the fixed points K^σ of $\hat{\sigma}_\xi$ is a real line bundle on $X_{\mathbf{R}}$, and it determines a unique orientation on $E_{\mathbf{R}}$ through the squaring map $K^\sigma \rightarrow \det E_{\mathbf{R}}$. (Surely this is another proof of Theorem 1 using vector bundles alone.)

Now the lifting $\hat{\sigma}_\xi$ coupled with σ_l yields a conjugate linear lifting $\hat{\sigma}_{\xi, l}$ on $K \otimes_{\mathbf{R}} l$. Then the fixed points of $K \otimes_{\mathbf{R}} l$ gives another orientation on $E_{\mathbf{R}}$, as in the previous paragraph. Clearly, the two orientations on $E_{\mathbf{R}}$ constructed here differ by a factor of l' . The theorem

follows if we choose the spin conjugate liftings on P_ξ, P_η that are transferred from $\hat{\sigma}_\xi$ and $\hat{\sigma}_{\xi,l}$.

Proposition 5 can be generalized to incompatible spin structures in the situation of Theorem 4. The details are left to the reader.

4. Remarks on the Non-spin Case

A complex vector bundle E has a canonical spin^c bundle $P_c = P_U \times_{\rho} \text{Spin}^c(2r)$ from the complex frame bundle P_U and the natural homomorphism $\rho : U(r) \rightarrow \text{Spin}^c(2r)$. Note, unlike the spin case, that there is always a canonical conjugate lifting morphism σ_c on P_c , where $\text{Spin}^c(2r) = \text{Spin}(2r) \times_{\pm} U(1)$ inherits a conjugation from $\text{Spin}(2r)$ and $U(1)$.

Squaring the $U(1)$ factor of $\text{Spin}^c(2r)$ yields a bundle morphism $P_c \rightarrow P_L$, where P_L is the principal bundle of $L = \det E$. However, unlike the spin case, the fixed point set P_c^σ of σ_c no longer gives any trivialization of P_L^σ . To see this, consider the essential case that $E = L$ is a complex line bundle. Then P_c is a $\text{Spin}^c(2)$ -bundle and P_c^σ is a $\mathbf{Z}_2 \times \mathbf{Z}_2$ -bundle. It is not hard to check that $P_c^\sigma \rightarrow P_L^\sigma$ is a surjective homomorphism, hence does not single out a trivialization of the \mathbf{Z}_2 -bundle P_L^σ . (Essentially, this is due to that the real element $\{\mathbf{i}, \mathbf{i}\}$ of $\text{Spin}^c(2) = U(1) \times_{\pm} U(1)$ is mapped to $\mathbf{i}^2 = -1 \in \mathbf{Z}_2$.) Of course we know already that P_L^σ namely $E_{\mathbf{R}}$ may not be orientable, as shown by the real part \mathbf{RP}^2 of \mathbf{CP}^2 with the standard complex conjugation. (As pointed out in the previous paragraph, the canonical spin^c structure on a complex manifold is always compatible with any real structures.)

Borel and Haefliger [1, Proposition 5.18] showed that for an algebraic complex vector bundle E defined over \mathbf{R} , the Chern class $c_1(E)$ determines $w_1(E_{\mathbf{R}})$, both classes as algebraic cycles, via a map that is

constructed using their \mathbf{Z}_2 intersection theory and fundamental class. This suggests that in general, $w_2(E)$ could relate to $w_1(E_{\mathbf{R}})$ in some way. One way to clarify this sentence seems to be the following “virtual bundle” version of our early results in Section 2.

Consider two complex vector bundles $E, F \rightarrow X$ both of which are equipped with real structures. Let $S = E - F$ denote the virtual bundle in the K -theory. Then S has an induced real structure in a proper sense. Suppose S is spin, i.e., the Stiefel-Whitney class

$$w_2(S) = w_2(E) - w_2(F) = 0$$

(E, F are orientable!), and assume there is a spin structure on S that is compatible with the real structure. Then $S_{\mathbf{R}}$ has a unique semi-orientation, namely, the orientation bundles of $E_{\mathbf{R}}, F_{\mathbf{R}}$ are isomorphic and the isomorphisms are unique up to constant multiples. All of these amount to applying Theorem 1 to the determinant $\det S = \det E \otimes (\det F)^*$, which is a usual complex line bundle.

The previous remark is meant to reflect that roughly speaking, complex vector bundles with the same w_2 have the same kind of orientability for their real parts. Besides the virtual bundle version, there is also a “relative” version, where we consider complex vector bundles $E \rightarrow X, F \rightarrow Y$ with real structures and a smooth map $f : X \rightarrow Y$ that is compatible with the involutions. Then assuming f has a spin structure that is compatible with the real structure in a proper sense, one concludes that the map f is orientable with a well-defined semi-orientation. In essence this relative version is simply to translate everything from the virtual bundle $E - f^*F$. The orientability of f is used in order to define the degree of f , which will be studied elsewhere [4].

Without having $w_1(E_{\mathbf{R}}) = 0$, one might turn to a characteristic submanifold W of $w_1(E_{\mathbf{R}})$, i.e., a codimension 1 submanifold of $X_{\mathbf{R}}$ such that $PD[W] = w_1(E_{\mathbf{R}})$. It is perhaps possible to have a natural candidate of W in some situations, and it would be interesting to figure out the structure of “chambers” in the complement of W .

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References

- [1] A. Borel and A. Haefliger, La classe d'homologie fondamentale d'un espace analytique, *Bull. Soc. Math. France* 89 (1961), 461-513.
- [2] R. Bott and C. Taubes, On the rigidity theorems of Witten, *J. Amer. Math. Soc.* 2 (1989), 137-186.
- [3] A. L. Edmonds, Orientability of fixed point sets, *Proc. Amer. Math. Soc.* 82 (1981), 120-124.
- [4] J. Lee, in preparation.
- [5] S. Nagami, A note on orientations of fixed point sets of spin structure preserving involutions, *Kobe J. Math.* 20 (2003), 39-51.
- [6] S. M. Natanzon, Moduli of real algebraic surfaces and their superanalogues. Differentials, spinors and Jacobians of real curves, *Russian Math. Surveys* 54 (1999), 1091-1147.
- [7] K. Ono, On a theorem of Edmonds, *Progress in Differential Geometry*, Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 1993.
- [8] K. Ono and S. Stolz, An analogue of Edmonds' theorem for loop spaces, *Symplectic Geometry and Quantization* (Sanda and Yokohama, 1993), *Contemp. Math.* 179, Amer. Math. Soc, 1994, pp. 241-260.
- [9] G. Tian and S. Wang, Real Seiberg-Witten invariants, in preparation.
- [10] O. Viro, Complex orientations of real algebraic surfaces, *Topology of Manifolds and Varieties*, *Advances in Soviet Math.* 18, Amer. Math. Soc., 1994, pp. 261-284.

