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# GEOMETRICAL THEORY ON COMBINATORIAL MANIFOLDS

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## Abstract

For an integer $m \geq 1$, a combinatorial manifold $\widetilde{M}$ is defined to be a geometrical object $\widetilde{M}$ such that for $p \in \widetilde{M}$, there is a local chart $(U_p, \varphi_p)$ enable $\varphi_p : U_p \to B^{n_{i_1}} \cup B^{n_{i_2}} \cup \cdots \cup B^{n_{i_{s(p)}}}$ with $B^{n_{i_1}} \cap B^{n_{i_2}} \cap \cdots \cap B^{n_{i_{s(p)}}} \neq \emptyset$, where $B^{n_{i_j}}$ is an $n_{i_j}$-ball for integers $1 \leq j \leq s(p) \leq m$. Topological and differential structures such as those of $d$-pathwise connected, homotopy classes, fundamental $d$-groups in topology and tangent vector fields, tensor fields, connections, *Minkowski* norms in differential geometry on these finitely combinatorial manifolds are introduced. Some classical results are generalized to finitely combinatorial manifolds. Euler-Poincaré characteristic is discussed and geometrical inclusions in Smarandache geometries for various geometries are also presented by the geometrical theory on finitely combinatorial manifolds in this paper.

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## 1. Introduction

As a model of spacetimes in physics, various geometries such as those of Euclid, Riemannian and Finsler geometries are established by mathematicians. Today, more and more evidences have shown that our spacetime is not homogenous. Thereby models established on classical geometries are only unilateral. Then are there some kinds of overall geometries for spacetimes in physics? The answer is YES. Those are just Smarandache geometries established in last century but attract more one's attention now. According to the summary in [4], they are formally defined following.

**Definition 1.1** [4, 17]**.** A *Smarandache geometry* is a geometry which has at least one Smarandachely denied axiom (1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.

A Smarandache $n$-manifold is an $n$-manifold that support a Smarandache geometry.

For verifying the existence of Smarandache geometries, Kuciuk and Antholy gave a popular and easily understanding example on an Euclid plane in [4]. In [3], Iseri firstly presented a systematic construction for Smarandache geometries by equilateral triangular disks on Euclid planes, which are really Smarandache 2-dimensional geometries (see also [5]). In references [6, 7, 13], particularly in [7], a general constructing way for Smarandache 2-dimensional geometries on maps on surfaces, called *map geometries* was introduced, which generalized the construction of Iseri. For the case of dimensional number $\geq 3$, these pseudo-manifold geometries are proposed, which are approved to be Smarandache geometries and containing these Finsler and Kähler geometries as sub-geometries in [12].

In fact, by the Definition 1.1 a general but more natural way for constructing Smarandache geometries should be seeking for them on a union set of spaces with an axiom validated in one space but invalided in another, or invalided in a space in one way and another space in a different way. These unions are so-called Smarandache multi-spaces. This is the motivation for this paper. Notice that in [8], these multi-metric

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spaces have been introduced, which enables us to constructing Smarandache geometries on multi-metric spaces, particularly, on multi-metric spaces with a same metric.

**Definition 1.2.** A *multi-metric space* $\widetilde{A}$ is a union of spaces $A_1$, $A_2$, ..., $A_m$ for an integer $m \geq 2$ such that each $A_i$ is a space with metric $\rho_i$ for $i$, $1 \leq i \leq m$.

Now for any integer $n$, these $n$-manifolds $M^n$ are the main objects in modern geometry and mechanics, which are locally Euclidean spaces $\mathbf{R}^n$ satisfying the $T_2$ separation axiom in fact, i.e., for $p, q \in M^n$, there are local charts $(U_p, \varphi_p)$ and $(U_q, \varphi_q)$ such that $U_p \bigcap U_q = \varnothing$ and $\varphi_p : U_p \to \mathbf{B}^n$, $\varphi_q : U_q \to \mathbf{B}^n$, where

$$\mathbf{B}^n = \{(x_1, x_2, ..., x_n) | x_1^2 + x_2^2 + \cdots + x_n^2 < 1\}.$$

is an open ball.

These manifolds are locally Euclidean spaces. In fact, they are also homogenous spaces. But the world is not homogenous. Whence, a more important thing is considering these combinations of different dimensions, i.e., *combinatorial manifolds* defined following and finding their good behaviors for mathematical sciences besides just to research these manifolds. Two examples for these combinations of manifolds with different dimensions in $\mathbf{R}^3$ are shown in Figure 1.1, in where, (a) represents a combination of a 3-manifold, a torus and a 1-manifold, and (b) a torus with 4 bouquets of 1-manifolds.

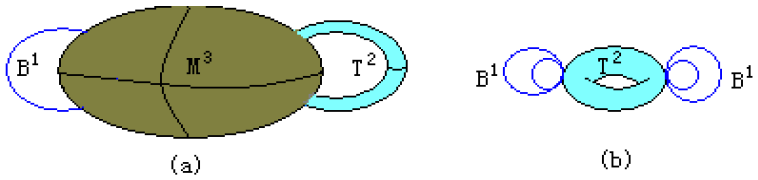


**Figure 1.1**

For an integer $s \geq 1$, let $n_1, n_2, ..., n_s$ be an integer sequence with $0 < n_1 < n_2 < \cdots < n_s$. Choose $s$ open unit balls $B_1^{n_1}, B_2^{n_2}, ..., B_s^{n_s}$, where

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$\bigcap_{i=1}^{s} B_i^{n_i} \neq \varnothing$ in $\mathbf{R}^{n_1+n_2+\cdots+n_s}$. Then a unit open combinatorial ball of degree $s$ is a union

$$\widetilde{B}(n_1, n_2, ..., n_s) = \bigcup_{i=1}^{s} B_i^{n_i}.$$

**Definition 1.3.** For a given integer sequence $n_1, n_2, ..., n_m, m \geq 1$ with $0 < n_1 < n_2 < \cdots < n_m$, a combinatorial manifold $\widetilde{M}$ is a *Hausdorff space* such that for any point $p \in \widetilde{M}$, there is a local chart $(U_p, \varphi_p)$ of $p$, i.e., an open neighborhood $U_p$ of $p$ in $\widetilde{M}$ and a homoeomorphism $\varphi_p : U_p \to \widetilde{B}(n_1(p), n_2(p), ..., n_{s(p)}(p))$ with $\{n_1(p), n_2(p), ..., n_{s(p)}(p)\} \subseteq \{n_1, n_2, ..., n_m\}$ and $\bigcup_{p\in\widetilde{M}} \{n_1(p), n_2(p), ..., n_{s(p)}(p)\} = \{n_1, n_2, ..., n_m\}$, denoted by $\widetilde{M}(n_1, n_2, ..., n_m)$ or $\widetilde{M}$ on the context and

$$\widetilde{\mathcal{A}} = \{(U_p, \varphi_p) | p \in \widetilde{M}(n_1, n_2, ..., n_m)\}$$

an atlas on $\widetilde{M}(n_1, n_2, ..., n_m)$. The maximum value of $s(p)$ and the dimension $\hat{s}(p)$ of $\bigcap_{i=1}^{s(p)} B_i^{n_i}$ are called the *dimension* and the *intersectional dimensional* of $\widetilde{M}(n_1, n_2, ..., n_m)$ at the point $p$, respectively.

A combinatorial manifold $\widetilde{M}$ is called *finite* if it is just combined by finite manifolds.

Notice that $\bigcap_{i=1}^{s} B_i^{n_i} \neq \varnothing$ by the definition of unit combinatorial balls of degree $s$. Thereby, for $p \in \widetilde{M}(n_1, n_2, ..., n_s)$, either it has a neighborhood $U_p$ with $\varphi_p : U_p \to \mathbf{R}^{\varsigma}, \varsigma \in \{n_1, n_2, ..., n_s\}$ or a combinatorial ball $\widetilde{B}(\tau_1, \tau_2, ..., \tau_l)$ with $\varphi_p : U_p \to \widetilde{B}(\tau_1, \tau_2, ..., \tau_l), l \leq s$ and $\{\tau_1, \tau_2, ..., \tau_l\} \subseteq \{n_1, n_2, ..., n_s\}$ hold.

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The main purpose of this paper is to characterize these finitely combinatorial manifolds, such as those of topological behaviors and differential structures on them by a combinatorial method. For these objectives, topological and differential structures such as those of $d$-pathwise connected, homotopy classes, fundamental $d$-groups in topology and tangent vector fields, tensor fields, connections, *Minkowski* norms in differential geometry on these combinatorial manifolds are introduced. Some results in classical differential geometry are generalized to finitely combinatorial manifolds. As an important invariant, Euler-Poincaré characteristic is discussed and geometrical inclusions in Smarandache geometries for various existent geometries are also presented by the geometrical theory on finitely combinatorial manifolds in this paper.

For terminologies and notations not mentioned in this section, we follow [1-2] for differential geometry, [5, 7] for graphs and [14, 18] for topology.

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## 2. Topological Structures on Combinatorial Manifolds

By a topological view, we introduce topological structures and characterize these finitely combinatorial manifolds in this section.

### 2.1. Pathwise connectedness

On the first, we define *d-dimensional pathwise connectedness* in a finitely combinatorial manifold for an integer $d$, $d \geq 1$, which is a natural generalization of pathwise connectedness in a topological space.

**Definition 2.1.** For two points $p$, $q$ in a *finitely combinatorial manifold* $\tilde{M}(n_1, n_2, ..., n_m)$, if there is a sequence $B_1, B_2, ..., B_s$ of $d$-dimensional open balls with two conditions following hold.

(1) $B_i \subset \tilde{M}(n_1, n_2, ..., n_m)$ for any integer $i$, $1 \leq i \leq s$ and $p \in B_1$, $q \in B_s$;

(2) The dimensional number $\dim(B_i \cap B_{i+1}) \geq d$ for $i, 1 \leq i \leq s-1$.

Then points $p$, $q$ are called *d-dimensional connected* in $\tilde{M}(n_1, n_2, ..., n_m)$ and the sequence $B_1, B_2, ..., B_e$ a $d$-dimensional path connecting $p$ and $q$, denoted by $P^d(p, q)$.

If each pair $p$, $q$ of points in the finitely combinatorial manifold $\tilde{M}(n_1, n_2, ..., n_m)$ is $d$-dimensional connected, then $\tilde{M}(n_1, n_2, ..., n_m)$ is called *d-pathwise connected* and say its connectivity $\geq d$.

Without loss of generality, we consider only finitely combinatorial manifolds with a connectivity $\geq 1$ in this paper. Let $\tilde{M}(n_1, n_2, ..., n_m)$ be a finitely combinatorial manifold and $d$, $d \geq 1$ an integer. We construct a labelled graph $G^d[\tilde{M}(n_1, n_2, ..., n_m)]$ by

$$V(G^d[\tilde{M}(n_1, n_2, ..., n_m)]) = V_1 \bigcup V_2,$$

where $V_1 = \{n_i$-manifolds $M^{n_i}$ in $\tilde{M}(n_1, n_2, ..., n_m) | 1 \leq i \leq m\}$ and $V_2 = \{$isolated intersection points $O_{M^{n_i}, M^{n_j}}$ of $M^{n_i}, M^{n_j}$ in $\tilde{M}(n_1, n_2, ..., n_m)$ for $1 \leq i, j \leq m\}$. Label $n_i$ for each $n_i$-manifold in $V_1$ and 0 for each vertex in $V_2$ and

$$E(G^d[\tilde{M}(n_1, n_2, ..., n_m)]) = E_1 \bigcup E_2,$$

where $E_1 = \{(M^{n_i}, M^{n_j}) | \dim(M^{n_i} \cap M^{n_j}) \geq d, 1 \leq i, j \leq m\}$ and $E_2 = \{(O_{M^{n_i}, M^{n_j}}, M^{n_i}), (O_{M^{n_i}, M^{n_j}}, M^{n_j}) | M^{n_i}$ tangent $M^{n_j}$ at the point $O_{M^{n_i}, M^{n_j}}$ for $1 \leq i, j \leq m\}$.

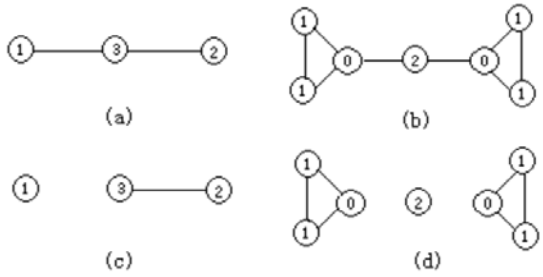


**Figure 2.1**

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For example, these correspondent labelled graphs gotten from finitely combinatorial manifolds in Figure 1.1 are shown in Figure 2.1, where $d = 1$ for (a) and (b), $d = 2$ for (c) and (d). By this construction, properties following can be easily gotten.

**Theorem 2.1.** *Let* $G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$ *be a labelled graph of a finitely combinatorial manifold* $\widetilde{M}(n_1, n_2, ..., n_m)$. *Then*

(1) $G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$ *is connected only if* $d \leq n_1$.

(2) *there exists an integer* $d$, $d \leq n_1$ *such that* $G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$ *is connected*.

**Proof.** By definition, there is an edge $(M^{n_i}, M^{n_j})$ in $G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$ for $1 \leq i, j \leq m$ if and only if there is a $d$-dimensional path $P^d(p, q)$ connecting two points $p \in M^{n_i}$ and $q \in M^{n_j}$. Notice that

$$(P^d(p, q) \backslash M^{n_i}) \subseteq M^{n_j} \text{ and } (P^d(p, q) \backslash M^{n_j}) \subseteq M^{n_i}.$$

Whence,

$$d \leq \min\{n_i, n_j\}. \tag{2.1}$$

Now if $G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$ is connected, then there is a $d$-path $P(M^{n_i}, M^{n_j})$ connecting vertices $M^{n_i}$ and $M^{n_j}$ for $M^{n_i}, M^{n_j} \in V(G^d[\widetilde{M}(n_1, n_2, ..., n_m)])$. Without loss of generality, assume

$$P(M^{n_i}, M^{n_j}) = M^{n_i} M^{s_1} M^{s_2} \cdots M^{s_{t-1}} M^{n_j}.$$

Then we get that

$$d \leq \min\{n_i, s_1, s_2, ..., s_{t-1}, n_j\} \tag{2.2}$$

by (2.1). However, according to Definition 1.4, we know that

$$\bigcup_{p \in \widetilde{M}} \{n_1(p), n_2(p), ..., n_{s(p)}(p)\} = \{n_1, n_2, ..., n_m\}. \tag{2.3}$$

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Therefore, we get that

$$d \le \min \bigcup_{p \in \tilde{M}} \{n_1(p), n_2(p), ..., n_{s(p)}(p)\} = \min\{n_1, n_2, ..., n_m\} = n_1$$

by combining (2.2) with (2.3). Notice that points labelled with 0 and 1 are always connected by a path. We get the conclusion (1).

For the conclusion (2), notice that any finitely combinatorial manifold is always pathwise 1-connected by definition. Accordingly, $G^1[\tilde{M}(n_1, n_2, ..., n_m)]$ is connected. Thereby, there are at least one integer, for instance $d = 1$ enabling $G^d[\tilde{M}(n_1, n_2, ..., n_m)]$ to be connected. This completes the proof.

According to Theorem 2.1, we get immediately two following corollaries.

**Corollary 2.1.** *For a given finitely combinatorial manifold $\tilde{M}$, all connected graphs $G^d[\tilde{M}]$ are isomorphic if $d \le n_1$, denoted by $G[\tilde{M}]$.*

**Corollary 2.2.** *If there are $k$ 1-manifolds intersect at one point $p$ in a finitely combinatorial manifold $\tilde{M}$, then there is an induced subgraph $K^{k+1}$ in $G[\tilde{M}]$.*

Now we define an edge set $E^d(\tilde{M})$ in $G[\tilde{M}]$ by

$$E^d(\tilde{M}) = E(G^d[\tilde{M}]) \setminus E(G^{d+1}[\tilde{M}]).$$

Then we get a graphical recursion equation for graphs of a finitely combinatorial manifold $\tilde{M}$ as a by-product.

**Theorem 2.2.** *Let $\tilde{M}$ be a finitely combinatorial manifold. Then for any integer $d$, $d \ge 1$, there is a recursion equation*

$$G^{d+1}[\tilde{M}] = G^d[\tilde{M}] - E^d(\tilde{M})$$

*for graphs of $\tilde{M}$.*

**Proof.** It can be obtained immediately by definition.

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For a given integer sequence $1 \le n_1 < n_2 < \cdots < n_m$, $m \ge 1$, denote by $\mathcal{H}^d(n_1, n_2, ..., n_m)$ all these finitely combinatorial manifolds $\tilde{M}(n_1, n_2, ..., n_m)$ with connectivity $\ge d$, where $d \le n_1$ and $\mathcal{G}(n_1, n_2, ..., n_m)$ all these connected graphs $G[n_1, n_2, ..., n_m]$ with vertex labels $0, n_1, n_2, ..., n_m$ and conditions following hold.

(1) The induced subgraph by vertices labelled with 1 in $G$ is a union of complete graphs;

(2) All vertices labelled with 0 can only be adjacent to vertices labelled with 1.

Then we know a relation between sets $\mathcal{H}^d(n_1, n_2, ..., n_m)$ and $\mathcal{G}(n_1, n_2, ..., n_m)$.

**Theorem 2.3.** *Let $1 \le n_1 < n_2 < ... < n_m$, $m \ge 1$ be a given integer sequence. Then every finitely combinatorial manifold $\tilde{M} \in \mathcal{H}^d(n_1, n_2, ..., n_m)$ defines a labelled connected graph $G[n_1, n_2, ..., n_m] \in \mathcal{G}(n_1, n_2, ..., n_m)$. Conversely, every labelled connected graph $G[n_1, n_2, ..., n_m] \in \mathcal{G}(n_1, n_2, ..., n_m)$ defines a finitely combinatorial manifold $\tilde{M} \in \mathcal{H}^d(n_1, n_2, ..., n_m)$ for any integer $1 \le d \le n_1$.*

**Proof.** For $\tilde{M} \in \mathcal{H}^d(n_1, n_2, ..., n_m)$, there is a labelled graph $G[n_1, n_2, ..., n_m] \in \mathcal{G}(n_1, n_2, ..., n_m)$ correspondent to $\tilde{M}$ is already verified by Theorem 2.1. For completing the proof, we only need to construct a finitely combinatorial manifold $\tilde{M} \in \mathcal{H}^d(n_1, n_2, ..., n_m)$ for $G[n_1, n_2, ..., n_m] \in \mathcal{G}(n_1, n_2, ..., n_m)$. Denoted by $l(u) = s$ if the label of a vertex $u \in V(G[n_1, n_2, ..., n_m])$ is $s$. The construction is carried out by the following programming.

**Step 1.** Choose $|G[n_1, n_2, ..., n_m]| - |V_0|$ manifolds correspondent to each vertex $u$ with a dimensional $n_i$ if $l(u) = n_i$, where $V_0 = \{u \mid u \in V(G[n_1, n_2, ..., n_m])$ and $l(u) = 0\}$. Denoted by $V_{\ge 1}$ all these vertices in $G[n_1, n_2, ..., n_m]$ with label $\ge 1$.

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**Step 2.** For $u_1 \in V_{\geq 1}$ with $l(u_1) = n_{i_1}$, if its neighborhood set $N_{G[n_1, n_2, ..., n_m]}(u_1) \cap V_{\geq 1} = \{v_1^1, v_1^2, ..., v_1^{s(u_1)}\}$ with $l(v_1^1) = n_{11}$, $l(v_1^2) = n_{12}, ..., l(v_1^{s(u_1)}) = n_{1s(u_1)}$, then let the manifold correspondent to the vertex $u_1$ with an intersection dimension $\geq d$ with manifolds correspondent to vertices $v_1^1, v_1^2, ..., v_1^{s(u_1)}$ and define a vertex set $\Delta_1 = \{u_1\}$.

**Step 3.** If the vertex set $\Delta_l = \{u_1, u_2, ..., u_l\} \subseteq V_{\geq 1}$ has been defined and $V_{\geq 1} \setminus \Delta_l \neq \emptyset$, let $u_{l+1} \in V_{\geq 1} \setminus \Delta_l$ with a label $n_{i_{l+1}}$. Assume

$$(N_{G[n_1, n_2, ..., n_m]}(u_{l+1}) \cap V_{\geq 1}) \setminus \Delta_l = \{v_{l+1}^1, v_{l+1}^2, ..., v_{l+1}^{s(u_{l+1})}\}$$

with $l(v_{l+1}^1) = n_{l+1,1}$, $l(v_{l+1}^2) = n_{l+1,2}$, ..., $l(v_{l+1}^{s(u_{l+1})}) = n_{l+1,s(u_{l+1})}$. Then let the manifold correspondent to the vertex $u_{l+1}$ with an intersection dimension $\geq d$ with manifolds correspondent to these vertices $v_{l+1}^1, v_{l+1}^2$, ..., $v_{l+1}^{s(u_{l+1})}$ and define a vertex set $\Delta_{l+1} = \Delta_l \cup \{u_{l+1}\}$.

**Step 4.** Repeat steps 2 and 3 until a vertex set $\Delta_t = V_{\geq 1}$ has been constructed. This construction is ended if there are no vertices $w \in V(G)$ with $l(w) = 0$, i.e., $V_{\geq 1} = V(G)$. Otherwise, go to the next step.

**Step 5.** For $w \in V(G[n_1, n_2, ..., n_m]) \setminus V_{\geq 1}$, assume $N_{G[n_1, n_2, ..., n_m]}(w) = \{w_1, w_2, ..., w_e\}$. Let all these manifolds correspondent to vertices $w_1, w_2, ..., w_e$ intersects at one point simultaneously and define a vertex set $\Delta_{t+1}^* = \Delta_t \cup \{w\}$.

**Step 6.** Repeat Step 5 for vertices in $V(G[n_1, n_2, ..., n_m]) \setminus V_{\geq 1}$. This construction is finally ended until a vertex set $\Delta_{t+h}^* = V(G[n_1, n_2, ..., n_m])$ has been constructed.

As soon as the vertex set $\Delta_{t+h}^*$ has been constructed, we get a finitely combinatorial manifold $\widetilde{M}$. It can be easily verified that $\widetilde{M} \in \mathcal{H}^d(n_1, n_2, ..., n_m)$ by our construction way.

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**2.2. Combinatorial equivalence**

For a finitely combinatorial manifold $\widetilde{M}$ in $\mathcal{H}^d(n_1, n_2, ..., n_m)$, denoted by $G[\widetilde{M}(n_1, n_2, ..., n_m)]$ and $G[\widetilde{M}]$ the correspondent labelled graph in $\mathcal{G}(n_1, n_2, ..., n_m)$ and the graph deleted labels on $G[\widetilde{M}(n_1, n_2, ..., n_m)]$, $C(n_i)$ all these vertices with a label $n_i$ for $1 \le i \le m$, respectively.

**Definition 2.2.** Two finitely combinatorial manifolds $\widetilde{M}_1(n_1, n_2, ..., n_m)$, $\widetilde{M}_2(k_1, k_2, ..., k_l)$ are called *equivalent* if these correspondent labelled graphs

$$G[\widetilde{M}_1(n_1, n_2, ..., n_m)] \cong G[\widetilde{M}_2(k_1, k_2, ..., k_l)].$$

Notice that if $\widetilde{M}_1(n_1, n_2, ..., n_m)$, $\widetilde{M}_2(k_1, k_2, ..., k_l)$ are equivalent, then we can get that $\{n_1, n_2, ..., n_m\} = \{k_1, k_2, ..., k_l\}$ and $G[\widetilde{M}_1] \cong G[\widetilde{M}_2]$. Reversing this idea enables us classifying finitely combinatorial manifolds in $\mathcal{H}^d(n_1, n_2, ..., n_m)$ by the action of automorphism groups of these correspondent graphs without labels.

**Definition 2.3.** A labelled connected graph $G[\widetilde{M}(n_1, n_2, ..., n_m)]$ is combinatorially unique if all these correspondent finitely combinatorial manifolds $\widetilde{M}(n_1, n_2, ..., n_m)$ are equivalent.

A labelled graph $G[n_1, n_2, ..., n_m]$ is called *class-transitive* if the automorphism group Aut$G$ is transitive on $\{C(n_i), 1 \le i \le m\}$. We find a characteristic for combinatorially unique graphs.

**Theorem 2.4.** *A labelled connected graph $G[n_1, n_2, ..., n_m]$ is combinatorially unique if and only if it is class-transitive.*

**Proof.** For two integers $i, j$, $1 \le i, j \le m$, re-label vertices in $C(n_i)$ by $n_j$ and vertices in $C(n_j)$ by $n_i$ in $G[n_1, n_2, ..., n_m]$. Then we get a new labelled graph $G'[n_1, n_2, ..., n_m]$ in $\mathcal{G}[n_1, n_2, ..., n_m]$. According to Theorem 2.3, we can get two finitely combinatorial manifolds

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$\tilde{M}_1(n_1, n_2, ..., n_m)$ and $\tilde{M}_2(k_1, k_2, ..., k_l)$ correspondent to $G[n_1, n_2, ..., n_m]$ and $G'[n_1, n_2, ..., n_m]$.

Now if $G[n_1, n_2, ..., n_m]$ is combinatorially unique, we know $\tilde{M}_1(n_1, n_2, ..., n_m)$ is equivalent to $\tilde{M}_2(k_1, k_2, ..., k_l)$, i.e., there is an automorphism $\theta \in \text{Aut}\, G$ such that $C^{\theta}(n_i) = C(n_j)$ for $i, j, 1 \le i, j \le m$.

On the other hand, if $G[n_1, n_2, ..., n_m]$ is class-transitive, then for integers $i, j, 1 \le i, j \le m$, there is an automorphism $\tau \in \text{Aut}G$ such that $C^{\tau}(n_i) = C(n_j)$. Whence, for any re-labelled graph $G'[n_1, n_2, ..., n_m]$, we find that

$$G[n_1, n_2, ..., n_m] \cong G'[n_1, n_2, ..., n_m],$$

which implies that these finitely combinatorial manifolds correspondent to $G[n_1, n_2, ..., n_m]$ and $G'[n_1, n_2, ..., n_m]$ are combinatorially equivalent, i.e., $G[n_1, n_2, ..., n_m]$ is combinatorially unique.

Now assume that for parameters $t_{i1}, t_{i2}, ..., t_{is_i}$, we have known an enufunction

$$C_{M^{n_i}}[x_{i1}, x_{i2}, ...] = \sum_{t_{i1}, t_{i2}, ..., t_{is}} n_i(t_{i1}, t_{i2}, ..., t_{is}) x_{i1}^{t_{i1}} x_{i2}^{t_{i2}} \cdots x_{is}^{t_{is}}$$

for $n_i$-manifolds, where $n_i(t_{i1}, t_{i2}, ..., t_{is})$ denotes the number of non-homeomorphic $n_i$-manifolds with parameters $t_{i1}, t_{i2}, ..., t_{is}$. For instance the enufunction for compact 2-manifolds with parameter genera is

$$C_{\tilde{M}}[x](2) = 1 + \sum_{p \ge 1} 2x^p.$$

Consider the action of $\text{Aut}G[n_1, n_2, ..., n_m]$ on $G[n_1, n_2, ..., n_m]$. If the number of orbits of the automorphism group $\text{Aut}G[n_1, n_2, ..., n_m]$ action on $\{C(n_i), 1 \le i \le m\}$ is $\pi_0$, then we can only get $\pi_0!$ non-equivalent combinatorial manifolds correspondent to the labelled graph $G[n_1, n_2, ..., n_m]$ similar to Theorem 2.4. Calculation shows that there are $l!$ orbits action by its automorphism group for a complete $(s_1 + s_2 + \cdots + s_l)$-partite graph $K(k_1^{s_1}, k_2^{s_2}, ..., k_l^{s_l})$, where $k_i^{s_i}$ denotes

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that there are $s_i$ partite sets of order $k_i$ in this graph for any integer $i$, $1 \le i \le l$, particularly, for $K(n_1, n_2, ..., n_m)$ with $n_i \ne n_j$ for $i, j$, $1 \le i, j \le m$, the number of orbits action by its automorphism group is $m!$. Summarizing all these discussions, we get an enufunction for these finitely combinatorial manifolds $\tilde{M}(n_1, n_2, ..., n_m)$ correspondent to a labelled graph $G[n_1, n_2, ..., n_m]$ in $\mathcal{G}(n_1, n_2, ..., n_m)$ with each label $\ge 1$.

**Theorem 2.5.** *Let $G[n_1, n_2, ..., n_m]$ be a labelled graph in $\mathcal{G}(n_1, n_2, ..., n_m)$ with each label $\ge 1$. For an integer $i$, $1 \le i \le m$, let the enufunction of non-homeomorphic $n_i$-manifolds with given parameters $t_1, t_2, ...,$ be $C_{M^{n_i}}[x_{i1}, x_{i2}, ...]$ and $\pi_0$ the number of orbits of the automorphism group $\mathrm{Aut}G[n_1, n_2, ..., n_m]$ action on $\{C(n_i), 1 \le i \le m\}$, then the enufunction of combinatorial manifolds $\tilde{M}(n_1, n_2, ..., n_m)$ correspondent to a labelled graph $G[n_1, n_2, ..., n_m]$ is*

$$C_{\tilde{M}}(\bar{x}) = \pi_0! \prod_{i=1}^{m} C_{M^{n_i}}[x_{i1}, x_{i2}, ...],$$

*particularly, if $G[n_1, n_2, ..., n_m] = K(k_1^{s_1}, k_2^{s_2}, ..., k_m^{s_m})$ such that the number of partite sets labelled with $n_i$ is $s_i$ for any integer $i$, $1 \le i \le m$, then the enufunction correspondent to $K(k_1^{s_1}, k_2^{s_2}, ..., k_m^{s_m})$ is*

$$C_{\tilde{M}}(\bar{x}) = m! \prod_{i=1}^{m} C_{M^{n_i}}[x_{i1}, x_{i2}, ...]$$

*and the enufunction correspondent to a complete graph $K_m$ is*

$$C_{\tilde{M}}(\bar{x}) = \prod_{i=1}^{m} C_{M^{n_i}}[x_{i1}, x_{i2}, ...].$$

**Proof.** Notice that the number of non-equivalent finitely combinatorial manifolds correspondent to $G[n_1, n_2, ..., n_m]$ is

$$\pi_0 \prod_{i=1}^{m} n_i(t_{i1}, t_{i2}, ..., t_{is})$$

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for parameters $t_{i1}, t_{i2}, ..., t_{is}$, $1 \le i \le m$ by the product principle of enumeration. Whence, the enufunction of combinatorial manifolds $\widetilde{M}(n_1, n_2, ..., n_m)$ correspondent to a labelled graph $G[n_1, n_2, ..., n_m]$ is

$$C_{\widetilde{M}}(\overline{x}) = \sum_{t_{i1}, t_{i2}, ..., t_{is}} \left( \pi_0 \prod_{i=1}^{m} n_i(t_{i1}, t_{i2}, ..., t_{is}) \right) \prod_{i=1}^{m} x_{i1}^{t_{i1}} x_{i2}^{t_{i2}} \cdots x_{is}^{t_{is}}$$

$$= \pi_0! \prod_{i=1}^{m} C_{M^{n_i}}[x_{i1}, x_{i2}, ...].$$

### 2.3. Homotopy classes

Denote by $f \simeq g$ two homotopic mappings $f$ and $g$. Following the same pattern of homotopic spaces, we define homotopically combinatorial manifolds in the next.

**Definition 2.4.** Two finitely combinatorial manifolds $\widetilde{M}(k_1, k_2, ..., k_l)$ and $\widetilde{M}(n_1, n_2, ..., n_m)$ are said to be *homotopic* if there exist continuous maps

$$f : \widetilde{M}(k_1, k_2, ..., k_l) \to \widetilde{M}(n_1, n_2, ..., n_m),$$

$$g : \widetilde{M}(n_1, n_2, ..., n_m) \to \widetilde{M}(k_1, k_2, ..., k_l)$$

such that $gf \simeq identity : \widetilde{M}(k_1, k_2, ..., k_l) \to \widetilde{M}(k_1, k_2, .., k_l)$ and $fg \simeq identity : \widetilde{M}(n_1, n_2, ..., n_m) \to \widetilde{M}(n_1, n_2, .., n_m)$.

For equivalent homotopically combinatorial manifolds, we know the following result under these correspondent manifolds being homotopic. For this objective, we need an important lemma in algebraic topology.

**Lemma 2.1** (Gluing Lemma, [16])**.** *Assume that a space $X$ is a finite union of closed subsets: $X = \bigcup_{i=1}^{n} X_i$. If for some space $Y$, there are continuous maps $f_i : X_i \to Y$ that agree on overlaps, i.e., $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$ for all $i, j$, then there exists a unique continuous $f : X \to Y$ with $f|_{X_i} = f_i$ for all $i$.*

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**Theorem 2.6.** *Let* $\widetilde{M}(n_1, n_2, ..., n_m)$ *and* $\widetilde{M}(k_1, k_2, ..., k_l)$ *be finitely combinatorial manifolds with an equivalence* $\varpi : G[\widetilde{M}(n_1, n_2, ..., n_m)] \to G[\widetilde{M}(k_1, k_2, ..., k_l)]$. *If for* $M_1, M_2 \in V(G[\widetilde{M}(n_1, n_2, ..., n_m)])$, $M_i$ *is homotopic to* $\varpi(M_i)$ *with homotopic mappings* $f_{M_i} : M_i \to \varpi(M_i)$, $g_{M_i} : \varpi(M_i) \to M_i$ *such that* $f_{M_i}|_{M_i \cap M_j} = f_{M_j}|_{M_i \cap M_j}$, $g_{M_i}|_{M_i \cap M_j} = g_{M_j}|_{M_i \cap M_j}$ *providing* $(M_i, M_j) \in E(G[\widetilde{M}(n_1, n_2, ..., n_m)])$ *for* $1 \le i, j \le m$, *then* $\widetilde{M}(n_1, n_2, ..., n_m)$ *is homotopic to* $\widetilde{M}(k_1, k_2, ..., k_l)$.

**Proof.** By the Gluing Lemma, there are continuous mappings

$$f : \widetilde{M}(n_1, n_2, ..., n_m) \to \widetilde{M}(k_1, k_2, ..., k_l)$$

and

$$g : \widetilde{M}(k_1, k_2, ..., k_l) \to \widetilde{M}(n_1, n_2, ..., n_m)$$

such that

$$f|_M = f_M \text{ and } g|_{\varpi(M)} = g_{\varpi(M)}$$

for $M \in V(G[\widetilde{M}(n_1, n_2, ..., n_m)])$. Thereby, we also get that

$$gf \simeq identity : \widetilde{M}(k_1, k_2, ..., k_l) \to \widetilde{M}(k_1, k_2, ..., k_l)$$

and

$$fg \simeq identity : \widetilde{M}(n_1, n_2, ..., n_m) \to \widetilde{M}(n_1, n_2, ..., n_m)$$

as a result of $g_M f_M \simeq identity : M \to M$, $f_M g_M \simeq identity : \varpi(M) \to \varpi(M)$.

We have known that a finitely combinatorial manifold $\widetilde{M}(n_1, n_2, ..., n_m)$ is $d$-pathwise connected for some integers $1 \le d \le n_1$. This consequence enables us considering fundamental $d$-groups of finitely combinatorial manifolds.

**Definition 2.5.** Let $\widetilde{M}(n_1, n_2, ..., n_m)$ be a finitely combinatorial manifold. Then for an integer $d$, $1 \le d \le n_1$ and $x \in \widetilde{M}(n_1, n_2, ..., n_m)$, a

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*fundamental d-group* at the point $x$, denoted by $\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x)$ is defined to be a group generated by all homotopic classes of closed $d$-paths based at $x$.

If $d = 1$ and $\widetilde{M}(n_1, n_2, ..., n_m)$ is just a manifold $M$, we get that

$$\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x) = \pi(M, x).$$

Whence, fundamental $d$-groups are a generalization of fundamental groups in topology. We obtain the following characteristics for fundamental $d$-groups of finitely combinatorial manifolds.

**Theorem 2.7.** *Let* $\widetilde{M}(n_1, n_2, ..., n_m)$ *be a d-connected finitely combinatorial manifold with* $1 \le d \le n_1$. *Then*

(1) *for* $x \in \widetilde{M}(n_1, n_2, ..., n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x) \cong (\bigoplus_{M \in V(G^d)} \pi^d(M)) \oplus \pi(G^d),$$

*where* $G^d = G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$, $\pi^d(M)$, $\pi(G^d)$ *denote the fundamental d-groups of a manifold* $M$ *and the graph* $G^d$, *respectively and*

(2) *for* $x, y \in \widetilde{M}(n_1, n_2, ..., n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x) \cong \pi^d(\widetilde{M}(n_1, n_2, ..., n_m), y).$$

**Proof.** For proving the conclusion (1), we only need to prove that for any cycle $\widetilde{C}$ in $\widetilde{M}(n_1, n_2, ..., n_m)$, there are elements $C_1^M, C_2^M, ..., C_{l(M)}^M \in \pi^d(M)$, $\alpha_1, \alpha_2, ..., \alpha_{\beta(G^d)} \in \pi(G^d)$ and integers $a_i^M, b_j$ for $M \in V(G^d)$ and $1 \le i \le l(M)$, $1 \le j \le c(G^d) \le \beta(G^d)$ such that

$$\widetilde{C} \equiv \sum_{M \in V(G^d)} \sum_{i=1}^{l(M)} a_i^M C_i^M + \sum_{j=1}^{c(G^d)} b_j \alpha_j (\mathrm{mod}\ 2)$$

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and it is unique. Let $C_1^M, C_2^M, ..., C_{b(M)}^M$ be a base of $\pi^d(M)$ for $M \in V(G^d)$. Since $\widetilde{C}$ is a closed trail, there must exist integers $k_i^M$, $l_j$, $1 \le i \le b(M)$, $1 \le j \le \beta(G^d)$ and $h_P$ for an open $d$-path on $\widetilde{C}$ such that

$$\widetilde{C} = \sum_{M \in V(G^d)} \sum_{i=1}^{b(M)} k_i^M C_i^M + \sum_{j=1}^{\beta(G^d)} l_j \alpha_j + \sum_{P \in \Delta} h_P P,$$

where $h_P \equiv 0 (\mathrm{mod}\ 2)$ and $\Delta$ denotes all of these open $d$-paths on $\widetilde{C}$. Now let

$$\{a_i^M \,|\, 1 \le i \le l(M)\} = \{k_i^M \,|\, k_i^M \ne 0 \text{ and } 1 \le i \le b(M)\},$$

$$\{b_j \,|\, 1 \le j \le c(G^d)\} = \{l_j \,|\, l_j \ne 0, 1 \le j \le \beta(G^d)\}.$$

Then we get that

$$\widetilde{C} \equiv \sum_{M \in V(G^d)} \sum_{i=1}^{l(M)} a_i^M C_i^M + \sum_{j=1}^{c(G^d)} b_j \alpha_j (\mathrm{mod}\ 2). \tag{2.4}$$

If there is another decomposition

$$\widetilde{C} \equiv \sum_{M \in V(G^d)} \sum_{i=1}^{l'(M)} a_i'^M C_i^M + \sum_{j=1}^{c'(G^d)} b_j' \alpha_j (\mathrm{mod}\ 2),$$

without loss of generality, assume $l'(M) \le l(M)$ and $c'(M) \le c(M)$, then we know that

$$\sum_{M \in V(G^d)} \sum_{i=1}^{l(M)} (a_i^M - a_i'^M) C_i^M + \sum_{j=1}^{c(G^d)} (b_j - b_j') \alpha_{j'} = 0,$$

where $a_i'^M = 0$ if $i > l'(M)$, $b_j' = 0$ if $j' > c'(M)$. Since $C_i^M, 1 \le i \le b(M)$ and $\alpha_j, 1 \le j \le \beta(G^d)$ are bases of the fundamental group $\pi(M)$ and $\pi(G^d)$, respectively, we must have

$$a_i^M = a_i'^M, 1 \le i \le l(M) \text{ and } b_j = b_j', 1 \le j \le c(G^d).$$

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Whence, the decomposition (2.4) is unique.

For proving the conclusion (2), notice that $\widetilde{M}(n_1, n_2, ..., n_m)$ is pathwise $d$-connected. Let $P^d(x, y)$ be a $d$-path connecting points $x$ and $y$ in $\widetilde{M}(n_1, n_2, ..., n_m)$. Define

$$\omega_*(C) = P^d(x, y)C(P^d)^{-1}(x, y)$$

for $C \in \widetilde{M}(n_1, n_2, ..., n_m)$. Then it can be checked immediately that

$$\omega_* : \pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x) \to \pi^d(\widetilde{M}(n_1, n_2, ..., n_m), y)$$

is an isomorphism.

A $d$-connected finitely combinatorial manifold $\widetilde{M}(n_1, n_2, ..., n_m)$ is said to be *simply d-connected* if $\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x)$ is trivial. As a consequence, we get the following result by Theorem 2.7.

**Corollary 2.3.** *A d-connected finitely combinatorial manifold* $\widetilde{M}(n_1, n_2, ..., n_m)$ *is simply d-connected if and only if*

(1) *for* $M \in V(G^d[\widetilde{M}(n_1, n_2, ..., n_m)])$, *M is simply d-connected and*

(2) $G^d[\widetilde{M}(n_1, n_2, ..., n_m)]$ *is a tree.*

**Proof.** According to the decomposition for $\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x)$ in Theorem 2.7, it is trivial if and only if $\pi(M)$ and $\pi(G^d)$ both are trivial for $M \in V(G^d[\widetilde{M}(n_1, n_2, ..., n_m)])$, i.e, $M$ is simply $d$-connected and $G^d$ is a tree.

For equivalent homotopically combinatorial manifolds, we also get a criterion under a homotopically equivalent mapping in the next.

**Theorem 2.8.** *If* $f : \widetilde{M}(n_1, n_2, ..., n_m) \to \widetilde{M}(k_1, k_2, ..., k_l)$ *is a homotopic equivalence, then for any integer d,* $1 \le d \le n_1$ *and* $x \in \widetilde{M}(n_1, n_2, ..., n_m)$,

$$\pi^d(\widetilde{M}(n_1, n_2, ..., n_m), x) \cong \pi^d(\widetilde{M}(k_1, k_2, ..., k_l), f(x)).$$

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**Proof.** Notice that $f$ can naturally induce a homomorphism

$$f_\pi : \pi^d(\tilde{M}(n_1, n_2, ..., n_m), x) \to \pi^d(\tilde{M}(k_1, k_2, ..., k_l), f(x))$$

defined by $f_\pi\langle g\rangle = \langle f(g)\rangle$ for $g \in \pi^d(\tilde{M}(n_1, n_2, ..., n_m), x)$ since it can be easily checked that $f_\pi(gh) = f_\pi(g)f_\pi(h)$ for $g, h \in \pi^d(\tilde{M}(n_1, n_2, ..., n_m), x)$. We only need to prove that $f_\pi$ is an isomorphism.

By definition, there is also a homotopic equivalence $g : \tilde{M}(k_1, k_2, ..., k_l) \to \tilde{M}(n_1, n_2, ..., n_m)$ such that $gf \simeq identity : \tilde{M}(n_1, n_2, ..., n_m) \to \tilde{M}(n_1, n_2, ..., n_m)$. Thereby, $g_\pi f_\pi = (gf)_\pi = \mu(identity)_\pi$ :

$$\pi^d(\tilde{M}(n_1, n_2, ..., n_m), x) \to \pi^s(\tilde{M}(n_1, n_2, ..., n_m), x),$$

where $\mu$ is an isomorphism induced by a certain $d$-path from $x$ to $gf(x)$ in $\tilde{M}(n_1, n_2, ..., n_m)$. Therefore, $g_\pi f_\pi$ is an isomorphism. Whence, $f_\pi$ is a monomorphism and $g_\pi$ is an epimorphism.

Similarly, apply the same argument to the homotopy

$$fg \simeq identity : \tilde{M}(k_1, k_2, ..., k_l) \to \tilde{M}(k_1, k_2, ..., k_l),$$

we get that $f_\pi g_\pi = (fg)_\pi = \nu(identity)_\pi$ :

$$\pi^d(\tilde{M}(k_1, k_2, ..., k_l), x) \to \pi^s(\tilde{M}(k_1, k_2, ..., k_l), x),$$

where $\nu$ is an isomorphism induced by a $d$-path from $fg(x)$ to $x$ in $\tilde{M}(k_1, k_2, ..., k_l)$. So $g_\pi$ is a monomorphism and $f_\pi$ is an epimorphism. Combining these facts enables us to conclude that $f_\pi : \pi^d(\tilde{M}(n_1, n_2, ..., n_m), x) \to \pi^d(\tilde{M}(k_1, k_2, ..., k_l), f(x))$ is an isomorphism.

**Corollary 2.4.** *If* $f : \tilde{M}(n_1, n_2, ..., n_m) \to \tilde{M}(k_1, k_2, ..., k_l)$ *is a homeomorphism, then for any integer* $d$, $1 \le d \le n_1$ *and* $x \in \tilde{M}(n_1, n_2, ..., n_m)$

$$\pi^d(\tilde{M}(n_1, n_2, ..., n_m), x) \cong \pi^d(\tilde{M}(k_1, k_2, ..., k_l), f(x)).$$

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### 2.4. Euler-Poincaré characteristic

It is well-known that the integer

$$\chi(\mathfrak{M}) = \sum_{i=0}^{\infty} (-1)^i \alpha_i$$

with $\alpha_i$ the number of $i$-dimensional cells in a $CW$-complex $\mathfrak{M}$ is defined to be the Euler-Poincaré characteristic of this complex. In this subsection, we get the Euler-Poincaré characteristic for finitely combinatorial manifolds. For this objective, define a clique sequence $\{Cl(i)\}_{i\geq 1}$ in the graph $G[\tilde{M}]$ by the following programming.

**Step 1.** Let $Cl(G[\tilde{M}]) = l_0$. Construct

$$Cl(l_0) = \{K_1^{l_0}, K_2^{l_0}, ..., K_p^{i_0} \mid K_i^{l_0} \succ G[\tilde{M}] \text{ and } K_i^{l_0} \cap K_j^{l_0} = \varnothing,$$

$$\text{or a vertex} \in V(G[\tilde{M}]) \text{ for } i \neq j, 1 \leq i, j \leq p\}.$$

**Step 2.** Let $G_1 = \bigcup_{K^{l_0} \in Cl(l_0)} K^{l_0}$ and $Cl(G[\tilde{M}]\backslash G_1) = l_1$. Construct

$$Cl(l_1) = \{K_1^{l_1}, K_2^{l_1}, ..., K_q^{i_1} \mid K_i^{l_1} \succ G[\tilde{M}] \text{ and } K_i^{l_1} \cap K_j^{l_1} = \varnothing$$

$$\text{or a vertex} \in V(G[\tilde{M}]) \text{ for } i \neq j, 1 \leq i, j \leq q\}.$$

**Step 3.** Assume we have constructed $Cl(l_{k-1})$ for an integer $k \geq 1$. Let $G_k = \bigcup_{K^{l_{k-1}} \in Cl(l_{k-1})} K^{l_{k-1}}$ and $Cl(G[\tilde{M}]\backslash(G_1 \cup \cdots \cup G_k)) = l_k$. We construct

$$Cl(l_k) = \{K_1^{l_k}, K_2^{l_k}, ..., K_r^{l_k} \mid K_i^{l_k} \succ G[\tilde{M}] \text{ and } K_i^{l_k} \cap K_j^{l_k} = \varnothing,$$

$$\text{or a vertex} \in V(G[\tilde{M}]) \text{ for } i \neq j, 1 \leq i, j \leq r\}.$$

**Step 4.** Continue Step 3 until we find an integer $t$ such that there are no edges in $G[\tilde{M}]\backslash\bigcup_{i=1}^{t} G_i$.

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By this clique sequence $\{Cl(i)\}_{i\geq 1}$, we can calculate the Euler-Poincaré characteristic of finitely combinatorial manifolds.

**Theorem 2.9.** *Let $\tilde{M}$ be a finitely combinatorial manifold. Then*

$$\chi(\tilde{M}) = \sum_{K^k \in Cl(k),\, k\geq 2} \; \sum_{M_{i_j} \in V(K^k),\, 1\leq j\leq s\leq k} (-1)^{s+1} \chi(M_{i_1} \cap \cdots \cap M_{i_s}).$$

**Proof.** Denoted the numbers of all these $i$-dimensional cells in a combinatorial manifold $\tilde{M}$ or in a manifold $M$ by $\tilde{\alpha}_i$ and $\alpha_i(M)$. If $G[\tilde{M}]$ is nothing but a complete graph $K^k$ with $V(G[\tilde{M}]) = \{M_1, M_2, ..., M_k\}$, $k \geq 2$, by applying the inclusion-exclusion principle and the definition of Euler-Poincaré characteristic we get that

$$\begin{aligned}
\chi(\tilde{M}) &= \sum_{i=0}^{\infty} (-1)^i \tilde{\alpha}_i \\
&= \sum_{i=0}^{\infty} (-1)^i \sum_{M_{i_j} \in V(K^k),\, 1\leq j\leq s\leq k} (-1)^{s+1} \alpha_i(M_{i_1} \cap \cdots \cap M_{i_s}) \\
&= \sum_{M_{i_j} \in V(K^k),\, 1\leq j\leq s\leq k} (-1)^{s+1} \sum_{i=0}^{\infty} (-1)^i \alpha_i(M_{i_1} \cap \cdots \cap M_{i_s}) \\
&= \sum_{M_{i_j} \in V(K^k),\, 1\leq j\leq s\leq k} (-1)^{s+1} \chi(M_{i_1} \cap \cdots \cap M_{i_s}),
\end{aligned}$$

for instance, $\chi(\tilde{M}) = \chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2)$ if $G[\tilde{M}] = K^2$ and $V(G[\tilde{M}]) = \{M_1, M_2\}$. By the definition of clique sequence of $G[\tilde{M}]$, we finally obtain that

$$\chi(\tilde{M}) = \sum_{K^k \in Cl(k),\, k\geq 2} \; \sum_{M_{i_j} \in V(K^k),\, 1\leq j\leq s\leq k} (-1)^{i+1} \chi(M_{i_1} \cap \cdots \cap M_{i_s}).$$

If $G[\tilde{M}]$ is just one of some special graphs, we can get interesting consequences by Theorem 2.9.

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**Corollary 2.5.** *Let* $\widetilde{M}$ *be a finitely combinatorial manifold. If* $G[\widetilde{M}]$ *is* $K^3$ *-free, then*

$$\chi(\widetilde{M}) = \sum_{M \in V(G[\widetilde{M}])} \chi^2(M) - \sum_{(M_1, M_2) \in E(G[\widetilde{M}])} \chi(M_1 \cap M_2).$$

*Particularly, if* $\dim(M_1 \cap M_2)$ *is a constant for any* $(M_1, M_2) \in E(G[\widetilde{M}])$, *then*

$$\chi(\widetilde{M}) = \sum_{M \in V(G[\widetilde{M}])} \chi^2(M) - \chi(M_1 \cap M_2) | E(G[\widetilde{M}])|.$$

**Proof.** Notice that $G[\widetilde{M}]$ is $K^3$ -free, we get that

$$\begin{aligned}\chi(\widetilde{M}) &= \sum_{(M_1, M_2) \in E(G[\widetilde{M}])} (\chi(M_1) + \chi(M_2) - \chi(M_1 \cap M_2)) \\ &= \sum_{(M_1, M_2) \in E(G[\widetilde{M}])} (\chi(M_1) + \chi(M_2)) - \sum_{(M_1, M_2) \in E(G[\widetilde{M}])} \chi(M_1 \cap M_2) \\ &= \sum_{M \in V(G[\widetilde{M}])} \chi^2(M) - \sum_{(M_1, M_2) \in E(G[\widetilde{M}])} \chi(M_1 \cap M_2).\end{aligned}$$

Since the Euler-Poincaré characteristic of a manifold $M$ is 0 if $\dim M \equiv 1 (\mathrm{mod}\ 2)$, we get the following consequence.

**Corollary 2.6.** *Let* $\widetilde{M}$ *be a finitely combinatorial manifold with odd dimension number for any intersection of k manifolds with* $k \geq 2$. *Then*

$$\chi(\widetilde{M}) = \sum_{M \in V(G[\widetilde{M}])} \chi(M).$$

## 3. Differential Structures on Combinatorial Manifolds

We introduce differential structures on finitely combinatorial manifolds and characterize them in this section.

### 3.1. Tangent vector fields

**Definition 3.1.** For a given integer sequence $1 \leq n_1 < n_2 < \cdots < n_m$, a combinatorially $C^h$ differential manifold $(\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$ is a

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*finitely combinatorial manifold* $\widetilde{M}(n_1, n_2, ..., n_m)$, $\widetilde{M}(n_1, n_2, ..., n_m) = \bigcup_{i\in I} U_i$, endowed with an atlas $\widetilde{\mathcal{A}} = \{(U_\alpha; \varphi_\alpha) | \alpha \in I\}$ on $\widetilde{M}(n_1, n_2, ..., n_m)$ for an integer $h$, $h \geq 1$ with conditions following hold.

(1) $\{U_\alpha; \alpha \in I\}$ is an open covering of $\widetilde{M}(n_1, n_2, ..., n_m)$;

(2) For $\alpha, \beta \in I$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ are equivalent, i.e., $U_\alpha \cap U_\beta = \varnothing$ or $U_\alpha \cap U_\beta \neq \varnothing$ but the overlap maps

$$\varphi_\alpha\varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\beta) \text{ and } \varphi_\beta\varphi_\alpha^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha)$$

are $C^h$ mappings;

(3) $\widetilde{\mathcal{A}}$ is maximal, i.e., if $(U; \varphi)$ is a local chart of $\widetilde{M}(n_1, n_2, ..., n_m)$ equivalent with one of local charts in $\widetilde{\mathcal{A}}$, then $(U; \varphi) \in \widetilde{\mathcal{A}}$.

Denote by $(\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$ a combinatorially differential manifold. A finitely combinatorial manifold $\widetilde{M}(n_1, n_2, ..., n_m)$ is said to be *smooth* if it is endowed with a $C^\infty$ differential structure.

Let $\widetilde{\mathcal{A}}$ be an atlas on $\widetilde{M}(n_1, n_2, ..., n_m)$. Then choose a local chart $(U; \varpi)$ in $\widetilde{\mathcal{A}}$. For $p \in (U; \varphi)$, if $\varpi_p : U_p \to \bigcup_{i=1}^{s(p)} B^{n_i(p)}$ and $\hat{s}(p) = \dim\left(\bigcap_{i=1}^{s(p)} B^{n_i(p)}\right)$, the following $s(p) \times n_{s(p)}$ matrix $[\varpi(p)]$

$$[\varpi(p)] = \begin{bmatrix} \frac{x^{11}}{s(p)} & \cdots & \frac{x^{1\hat{s}(p)}}{s(p)} & x^{1(\hat{s}(p)+1)} & \cdots & x^{1n_1} & \cdots & 0 \\ \frac{x^{21}}{s(p)} & \cdots & \frac{x^{2\hat{s}(p)}}{s(p)} & x^{2(\hat{s}(p)+1)} & \cdots & x^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{x^{s(p)1}}{s(p)} & \cdots & \frac{x^{s(p)\hat{s}(p)}}{s(p)} & x^{s(p)(\hat{s}(p)+1)} & \cdots & \cdots & x^{s(p)n_{s(p)}-1} & x^{s(p)n_{s(p)}} \end{bmatrix}$$

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with $x^{is} = x^{js}$ for $1 \le i, j \le s(p), 1 \le s \le \hat{s}(p)$ is called the *coordinate matrix* of $p$. For emphasize $\varpi$ is a matrix, we often denote local charts in a combinatorially differential manifold by $(U; [\varpi])$. Using the coordinate matrix system of a combinatorially differential manifold $(\tilde{M}(n_1, n_2, ..., n_m); \tilde{\mathcal{A}})$, we introduce the conception of $C^h$ mappings and functions in the next.

**Definition 3.2.** Let $\tilde{M}_1(n_1, n_2, ..., n_m)$, $\tilde{M}_2(k_1, k_2, ..., k_l)$ be smoothly combinatorial manifolds and

$$f : \tilde{M}_1(n_1, n_2, ..., n_m) \to \tilde{M}_2(k_1, k_2, ..., k_l)$$

be a mapping, $p \in \tilde{M}_1(n_1, n_2, ..., n_m)$. If there are local charts $(U_p; [\varpi_p])$ of $p$ on $\tilde{M}_1(n_1, n_2, ..., n_m)$ and $(V_{f(p)}; [\omega_{f(p)}])$ of $f(p)$ with $f(U_p) \subset V_{f(p)}$ such that the composition mapping

$$\tilde{f} = [\omega_{f(p)}] \circ f \circ [\varpi_p]^{-1} : [\varpi_p](U_p) \to [\omega_{f(p)}](V_{f(p)})$$

is a $C^h$ mapping, then $f$ is called a $C^h$ *mapping* at the point $p$. If $f$ is $C^h$ at any point $p$ of $\tilde{M}_1(n_1, n_2, ..., n_m)$, then $f$ is called a $C^h$ *mapping*. Particularly, if $\tilde{M}_2(k_1, k_2, ..., k_l) = \mathbf{R}$, then $f$ is called a $C^h$ *function* on $\tilde{M}_1(n_1, n_2, ..., n_m)$. In the extreme $h = \infty$, these terminologies are called *smooth mappings* and *functions*, respectively. Denote by $\mathcal{C}_p^\infty$ all these $C^\infty$ functions at a point $p \in \tilde{M}(n_1, n_2, ..., n_m)$.

For the existence of combinatorially differential manifolds, we know the following result.

**Theorem 3.1.** *Let $\tilde{M}(n_1, n_2, ..., n_m)$ be a finitely combinatorial manifold and $d$, $1 \le d \le n_1$ an integer. If $\forall M \in V(G^d[\tilde{M}(n_1, n_2, ..., n_m)])$ is $C^h$ differential and $\forall (M_1, M_2) \in E(G^d[\tilde{M}(n_1, n_2, ..., n_m)])$ there exist atlas*

$$\mathcal{A}_1 = \{(V_x; \varphi_x) | \forall x \in M_1\}, \quad \mathcal{A}_2 = \{(W_y; \psi_y) | \forall y \in M_2\}$$

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*such that* $\varphi_x |_{V_x \cap W_y} = \psi_y |_{V_x \cap W_y}$ *for* $x \in M_1$, $y \in M_2$, *then there is a differential structures*

$$\widetilde{\mathcal{A}} = \{(U_p; [\varpi_p]) | \forall p \in \widetilde{M}(n_1, n_2, ..., n_m)\}$$

*such that* $(\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$ *is a combinatorially* $C^h$ *differential manifold.*

**Proof.** By definition, we only need to show that we can always choose a neighborhood $U_p$ and a homoeomorphism $[\varpi_p]$ for each $p \in \widetilde{M}(n_1, n_2, ..., n_m)$ satisfying these conditions (1)-(3) in Definition 3.1.

By assumption, each manifold $\forall M \in V(G^d[\widetilde{M}(n_1, n_2, ..., n_m)])$ is $C^h$ differential, accordingly there is an index set $I_M$ such that $\{U_\alpha; \alpha \in I_M\}$ is an open covering of $M$, local charts $(U_\alpha; \varphi_\alpha)$ and $(U_\beta; \varphi_\beta)$ of $M$ are equivalent and $\mathcal{A} = \{(U; \varphi)\}$ is maximal. Since for $p \in \widetilde{M}(n_1, n_2, ..., n_m)$, there is a local chart $(U_p; [\varpi_p])$ of $p$ such that $[\varpi_p] : U_p \to \bigcup_{i=1}^{s(p)} B^{n_i(p)}$, i.e., $p$ is an intersection point of manifolds $M^{n_i(p)}$, $1 \le i \le s(p)$. By assumption each manifold $M^{n_i(p)}$ is $C^h$ differential, there exists a local chart $(V_p^i; \varphi_p^i)$ while the point $p \in M^{n_i(p)}$ such that $\varphi_p^i \to B^{n_i(p)}$. Now we define

$$U_p = \bigcup_{i=1}^{s(p)} V_p^i.$$

Then applying the Gluing Lemma again, we know that there is a homoeomorphism $[\varpi_p]$ on $U_p$ such that

$$[\varpi_p]|_{M^{n_i(p)}} = \varphi_p^i$$

for any integer $i$, $1 \le i \le s(p)$. Thereafter,

$$\widetilde{\mathcal{A}} = \{(U_p; [\varpi_p]) | \forall p \in \widetilde{M}(n_1, n_2, ..., n_m)\}$$

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is a $C^h$ differential structure on $\widetilde{M}(n_1, n_2, ..., n_m)$ satisfying conditions (1)-(3). Thereby $(\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$ is a combinatorially $C^h$ differential manifold.

**Definition 3.3.** Let $(\widetilde{M}(n_1, n_2, ..., n_m), \widetilde{\mathcal{A}})$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, ..., n_m)$. A tangent vector $v$ at $p$ is a mapping $v : \mathcal{C}_p^\infty \to \mathbf{R}$ with conditions following hold.

(1) $\forall g, h \in \mathcal{C}_p^\infty, \forall \lambda \in \mathbf{R}, v(h + \lambda h) = v(g) + \lambda v(h)$;

(2) $\forall g, h \in \mathcal{C}_p^\infty, v(gh) = v(g)h(p) + g(p)v(h)$.

Denoted all tangent vectors at $p \in \widetilde{M}(n_1, n_2, ..., n_m)$ by $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ and define addition $+$ and scalar multiplication $\cdot$ for $u, v \in T_p\widetilde{M}(n_1, n_2, ..., n_m)$, $\lambda \in \mathbf{R}$ and $f \in \mathcal{C}_p^\infty$ by

$$(u + v)(f) = u(f) + v(f), (\lambda u)(f) = \lambda \cdot u(f).$$

Then it can be shown immediately that $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ is a vector space under these two operations $+$ and $\cdot$.

**Theorem 3.2.** *For any point $p \in \widetilde{M}(n_1, n_2, ..., n_m)$ with a local chart $(U_p; [\varphi_p])$, the dimension of $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ is*

$$\dim T_p\widetilde{M}(n_1, n_2, ..., n_m) = \hat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \hat{s}(p))$$

*with a basis matrix*

$$\left[\frac{\partial}{\partial \bar{x}}\right]_{s(p)\times n_{s(p)}} =$$

$$\begin{bmatrix}
\frac{1}{s(p)}\frac{\partial}{\partial x^{11}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{1\hat{s}(p)}} & \frac{\partial}{\partial x^{1(\hat{s}(p)+1)}} & \cdots \frac{\partial}{\partial x^{1n_1}} & \cdots & 0 \\
\frac{1}{s(p)}\frac{\partial}{\partial x^{21}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{2\hat{s}(p)}} & \frac{\partial}{\partial x^{2(\hat{s}(p)+1)}} & \cdots \frac{\partial}{\partial x^{2n_2}} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \cdots & & \\
\frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)1}} & \cdots & \frac{1}{s(p)}\frac{\partial}{\partial x^{s(p)\hat{s}(p)}} & \frac{\partial}{\partial x^{s(p)(\hat{s}(p)+1)}} & \cdots \cdots & \frac{\partial}{\partial x^{s(p)(n_{s(p)}-1)}} & \frac{\partial}{\partial x^{s(p)n_{s(p)}}}
\end{bmatrix}$$

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*where* $x^{il} = x^{jl}$ *for* $1 \le i, j \le s(p), 1 \le l \le \hat{s}(p)$, *namely there is a smoothly functional matrix* $[v_{ij}]_{s(p)\times n_{s(p)}}$ *such that for any tangent vector* $\bar{v}$ *at a point* $p$ *of* $\widetilde{M}(n_1, n_2, ..., n_m)$,

$$\bar{v} = [v_{ij}]_{s(p)\times n_{s(p)}} \odot \left[\frac{\partial}{\partial \bar{x}}\right]_{s(p)\times n_{s(p)}},$$

*where* $[a_{ij}]_{k\times l} \odot [b_{ts}]_{k\times l} = \sum_{i=1}^{k}\sum_{j=1}^{l} a_{ij}b_{ij}$.

**Proof.** For $f \in \mathscr{C}_p^{\sim}$, let $\widetilde{f} = f \cdot [\varphi_p]^{-1} \in \mathscr{C}_{[\varphi_p](p)}^{\sim}$. We only need to prove that $f$ can be spanned by elements in

$$\left\{\frac{\partial}{\partial x^{hj}}|_p \,|\, 1 \le j \le \hat{s}(p)\right\} \bigcup \left( \bigcup_{i=1}^{s(p)} \bigcup_{j=\hat{s}(p)+1}^{n_i} \left\{\frac{\partial}{\partial x^{ij}}|_p \,|\, 1 \le j \le s\right\}\right), \tag{3.1}$$

for a given integer $h$, $1 \le h \le s(p)$, namely (3.1) is a basis of $T_p\widetilde{M}(n_1, n_2, ..., n_m)$. In fact, for $\bar{x} \in [\varphi_p](U_p)$, since $\widetilde{f}$ is smooth, we know that

$$\begin{aligned}\widetilde{f}(\bar{x}) - \widetilde{f}(\bar{x}_0) &= \int_0^1 \frac{d}{dt}\widetilde{f}(\bar{x}_0 + t(\bar{x} - \bar{x}_0))dt \\ &= \sum_{i=1}^{s(p)}\sum_{j=1}^{n_i} \eta^j_{\hat{s}(p)}(x^{ij} - x_0^{ij})\int_0^1 \frac{\partial \widetilde{f}}{\partial x^{ij}}(\bar{x}_0 + t(\bar{x} - \bar{x}_0))dt\end{aligned}$$

in a spherical neighborhood of the point $p$ in

$$[\varphi_p](U_p) \subset \mathbf{R}^{\hat{s}(p)-s(p)\hat{s}(p)+n_1+n_2+\cdots+n_{s(p)}}$$

with $[\varphi_p](p) = \bar{x}_0$, where

$$\eta^j_{\hat{s}(p)} = \begin{cases}\dfrac{1}{\hat{s}(p)}, & \text{if } 1 \le j \le \hat{s}(p), \\ 1, & \text{otherwise.}\end{cases}$$

Define

$$\widetilde{g}_{ij}(\bar{x}) = \int_0^1 \frac{\partial \widetilde{f}}{\partial x^{ij}}(\bar{x}_0 + t(\bar{x} - \bar{x}_0))dt$$

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and $g_{ij} = \widetilde{g}_{ij} \cdot [\varphi_p]$. Then we find that

$$g_{ij}(p) = \widetilde{g}_{ij}(\overline{x}_0) = \frac{\partial \widetilde{f}}{\partial x^{ij}}(\overline{x}_0)$$

$$= \frac{\partial (f \cdot [\varphi_p]^{-1})}{\partial x^{ij}}([\varphi_p](p)) = \frac{\partial f}{\partial x^{ij}}(p).$$

Therefore, for $q \in U_p$, there are $g_{ij}$, $1 \le i \le s(p)$, $1 \le j \le n_i$ such that

$$f(q) = f(p) + \sum_{i=1}^{s(p)} \sum_{j=1}^{n_i} \eta^j_{\hat{s}(p)} (x^{ij} - x^{ij}_0) g_{ij}(p).$$

Now let $\overline{v} \in T_p \widetilde{M}(n_1, n_2, ..., n_m)$. Application of the condition (2) in Definition 3.1 shows that

$$v(f(p)) = 0, \text{ and } v(\eta^j_{\hat{s}(p)} x^{ij}_0) = 0.$$

Accordingly, we obtain that

$$\overline{v}(f) = \overline{v}\left( f(p) + \sum_{i=1}^{s(p)} \sum_{j=1}^{n_i} \eta^j_{\hat{s}(p)} (x^{ij} - x^{ij}_0) g_{ij}(p) \right)$$

$$= \overline{v}\left( f(p) + \sum_{i=1}^{s(p)} \sum_{j=1}^{n_i} \overline{v}(\eta^j_{\hat{s}(p)} (x^{ij} - x^{ij}_0) g_{ij}(p)) \right)$$

$$= \sum_{i=1}^{s(p)} \sum_{j=1}^{n_i} (\eta^j_{\hat{s}(p)} g_{ij}(p) \overline{v}(x^{ij} - x^{ij}_0) + (x^{ij}(p) - x^{ij}_0) \overline{v}(\eta^j_{\hat{s}(p)} g_{ij}(p)))$$

$$= \sum_{i=1}^{s(p)} \sum_{j=1}^{n_i} \eta^j_{\hat{s}(p)} \frac{\partial f}{\partial x^{ij}}(p) \overline{v}(x^{ij})$$

$$= \sum_{i=1}^{s(p)} \sum_{j=1}^{n_i} \overline{v}(x^{ij}) \eta^j_{\hat{s}(p)} \frac{\partial}{\partial x^{ij}}|_p (f) = [v_{ij}]_{s(p) \times n_{s(p)}} \odot \left[ \frac{\partial}{\partial \overline{x}} \right]_{s(p) \times n_{s(p)}} |_p (f).$$

Therefore, we get that

$$\overline{v} = [v_{ij}]_{s(p) \times n_{s(p)}} \odot \left[ \frac{\partial}{\partial \overline{x}} \right]_{s(p) \times n_{s(p)}}. \tag{3.2}$$

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The formula (3.2) shows that any tangent vector $\overline{v}$ in $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ can be spanned by elements in (3.1).

Notice that all elements in (3.1) are also linearly independent. Otherwise, if there are numbers $a^{ij}$, $1 \le i \le s(p)$, $1 \le j \le n_i$ such that

$$\left(\sum_{j=1}^{\hat{s}(p)} a^{hj} \frac{\partial}{\partial x^{hj}} + \sum_{i=1}^{s(p)} \sum_{j=\hat{s}(p)+1}^{n_i} a^{ij} \frac{\partial}{\partial x^{ij}}\right)|_p = 0,$$

then we get that

$$a^{ij} = \left(\sum_{j=1}^{\hat{s}(p)} a^{hj} \frac{\partial}{\partial x^{hj}} + \sum_{i=1}^{s(p)} \sum_{j=\hat{s}(p)+1}^{n_i} a^{ij} \frac{\partial}{\partial x^{ij}}\right)(x^{ij}) = 0$$

for $1 \le i \le s(p)$, $1 \le j \le n_i$. Therefore, (3.1) is a basis of the tangent vector space $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ at the point $p \in (\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$.

By Theorem 3.2, if $s(p) = 1$ for any point $p \in (\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$, then $\dim T_p\widetilde{M}(n_1, n_2, ..., n_m) = n_1$. This can only happens while $\widetilde{M}(n_1, n_2, ..., n_m)$ is combined by one manifold. As a consequence, we get a well-known result in classical differential geometry again.

**Corollary 3.1** [2]. *Let* $(M^n; \mathcal{A})$ *be a smooth manifold and* $p \in M^n$. *Then*

$$\dim T_pM^n = n$$

*with a basis*

$$\left\{\frac{\partial}{\partial x^i}|_p \,|\, 1 \le i \le n\right\}.$$

**Definition 3.4.** For $p \in (\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$, the dual space $T_p^*\widetilde{M}(n_1, n_2, ..., n_m)$ is called a *co-tangent vector space* at $p$.

**Definition 3.5.** For $f \in \mathcal{C}_p^\infty$, $d \in T_p^*\widetilde{M}(n_1, n_2, ..., n_m)$ and $\overline{v} \in T_p\widetilde{M}(n_1, n_2, ..., n_m)$, the action of $d$ on $f$, called a *differential operator* $d: \mathcal{C}_p^\infty \to \mathbf{R}$, is defined by

$$df = \overline{v}(f).$$

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Then we immediately obtain the result following.

**Theorem 3.3.** *For* $p \in (\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{\mathcal{A}})$ *with a local chart* $(U_p; [\varphi_p])$, *the dimension of* $T_p^*\widetilde{M}(n_1, n_2, ..., n_m)$ *is*

$$\dim T_p^*\widetilde{M}(n_1, n_2, ..., n_m) = \hat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \hat{s}(p))$$

*with a basis matrix*

$$[d\bar{x}]_{s(p)\times n_{s(p)}}$$

$$= \begin{bmatrix} \frac{dx^{11}}{s(p)} & \cdots & \frac{dx^{1\hat{s}(p)}}{s(p)} & dx^{1(\hat{s}(p)+1)} & \cdots dx^{1n_1} & \cdots & 0 \\ \frac{dx^{21}}{s(p)} & \cdots & \frac{dx^{2\hat{s}(p)}}{s(p)} & dx^{2(\hat{s}(p)+1)} & \cdots dx^{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \frac{dx^{s(p)1}}{s(p)} & \cdots & \frac{dx^{s(p)\hat{s}(p)}}{s(p)} & dx^{s(p)(\hat{s}(p)+1)} & \cdots \quad dx^{s(p)n_{s(p)}-1} & & dx^{s(p)n_{s(p)}} \end{bmatrix},$$

*where* $x^{il} = x^{jl}$ *for* $1 \le i, j \le s(p), 1 \le l \le \hat{s}(p)$, *namely for any co-tangent vector* $d$ *at a point* $p$ *of* $\widetilde{M}(n_1, n_2, ..., n_m)$, *there is a smoothly functional matrix* $[u_{ij}]_{s(p)\times s(p)}$ *such that*

$$d = [u_{ij}]_{s(p)\times n_{s(p)}} \odot [d\bar{x}]_{s(p)\times n_{s(p)}}.$$

### 3.2. Tensor fields

**Definition 3.6.** Let $\widetilde{M}(n_1, n_2, ..., n_m)$ be a smoothly combinatorial manifold and $p \in \widetilde{M}(n_1, n_2, ..., n_m)$. A tensor of type $(r, s)$ at the point $p$ on $\widetilde{M}(n_1, n_2, ..., n_m)$ is an $(r + s)$-*multilinear function* $\tau$,

$$\tau : \underbrace{T_p^*\widetilde{M} \times \cdots \times T_p^*\widetilde{M}}_{r} \times \underbrace{T_p\widetilde{M} \times \cdots \times T_p\widetilde{M}}_{s} \to \mathbf{R},$$

where $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, ..., n_m)$ and $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, ..., n_m)$.

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Denoted by $T_s^r(p, \widetilde{M})$ all tensors of type $(r, s)$ at a point $p$ of $\widetilde{M}(n_1, n_2, ..., n_m)$. Then we know its structure by Theorems 3.2 and 3.3.

**Theorem 3.4.** *Let* $\widetilde{M}(n_1, n_2, ..., n_m)$ *be a smoothly combinatorial manifold and* $p \in \widetilde{M}(n_1, n_2, ..., n_m)$. *Then*

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s},$$

*where* $T_p\widetilde{M} = T_p\widetilde{M}(n_1, n_2, ..., n_m)$ *and* $T_p^*\widetilde{M} = T_p^*\widetilde{M}(n_1, n_2, ..., n_m)$, *particularly*,

$$\dim T_s^r(p, \widetilde{M}) = \left(\hat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \hat{s}(p))\right)^{r+s}.$$

**Proof.** By definition and multilinear algebra, any tensor $t$ of type $(r, s)$ at the point $p$ can be uniquely written as

$$t = \sum t_{j_1 \cdots j_s}^{i_1 \cdots i_r} \frac{\partial}{\partial x^{i_1 j_1}}|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r j_r}}|_p \otimes dx^{k_1 l_1} \otimes \cdots \otimes dx^{k_s l_s}$$

for smooth components $t_{j_1 \cdots j_s}^{i_1 \cdots i_r}$ on a neighborhood $U_p$ according to Theorems 3.2 and 3.3, where $1 \le i_h, k_h \le s(p)$ and $1 \le j_h \le i_h, 1 \le l_h \le k_h$ for $1 \le h \le r$. As a consequence, we obtain that

$$T_s^r(p, \widetilde{M}) = \underbrace{T_p\widetilde{M} \otimes \cdots \otimes T_p\widetilde{M}}_{r} \otimes \underbrace{T_p^*\widetilde{M} \otimes \cdots \otimes T_p^*\widetilde{M}}_{s}.$$

Since $\dim T_p\widetilde{M} = \dim T_p^*\widetilde{M} = \hat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \hat{s}(p))$ by Theorems 3.2 and 3.3, we also know that

$$\dim T_s^r(p, \widetilde{M}) = \left(\hat{s}(p) + \sum_{i=1}^{s(p)} (n_i - \hat{s}(p))\right)^{r+s}.$$

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**Definition 3.7.** Let $T_s^r(\tilde{M}) = \bigcup_{p\in\tilde{M}} T_s^r(p, \tilde{M})$ for a smoothly combinatorial manifold $\tilde{M} = \tilde{M}(n_1, n_2, ..., n_m)$. A tensor field of type $(r, s)$ on $\tilde{M}(n_1, n_2, ..., n_m)$ is a *mapping* $\tau : \tilde{M}(n_1, n_2, ..., n_m) \to T_s^r(\tilde{M})$ such that $\tau(p) \in T_s^r(p, \tilde{M})$ for $p \in \tilde{M}(n_1, n_2, ..., n_m)$.

A $k$-form on $\tilde{M}(n_1, n_2, ..., n_m)$ is a tensor field $\omega \in T_0^k(\tilde{M})$. Denoted all $k$-form of $\tilde{M}(n_1, n_2, ..., n_m)$ by $\Lambda^k(\tilde{M})$ and

$$\Lambda(\tilde{M}) = \bigoplus_{k=0}^{\hat{s}(p)-s(p)\hat{s}(p)+\sum_{i=1}^{s(p)} n_i} \Lambda^k(\tilde{M}),\ \mathcal{X}(\tilde{M}) = \bigcup_{p\in\tilde{M}} \mathcal{X}_p.$$

Similar to the classical differential geometry, we can also define operations $\varphi \wedge \psi$ for $\varphi, \psi \in T_s^r(\tilde{M})$, $[X, Y]$ for $X, Y \in \mathcal{X}(\tilde{M})$ and obtain a *Lie algebra* under the commutator. For the exterior differentiations on combinatorial manifolds, we find results following.

**Theorem 3.5.** *Let $\tilde{M}$ be a smoothly combinatorial manifold. Then there is a unique exterior differentiation $\tilde{d} : \Lambda(\tilde{M}) \to \Lambda(\tilde{M})$ such that for any integer $k \geq 1$, $\tilde{d}(\Lambda^k) \subset \Lambda^{k+1}(\tilde{M})$ with conditions following hold.*

(1) *$\tilde{d}$ is linear, i.e., for $\varphi, \psi \in \Lambda(\tilde{M})$, $\lambda \in \mathbf{R}$,*

$$\tilde{d}(\varphi + \lambda\psi) = \tilde{d}\varphi \wedge \psi + \lambda\tilde{d}\psi$$

*and for $\varphi \in \Lambda^k(\tilde{M})$, $\psi \in \Lambda(\tilde{M})$,*

$$\tilde{d}(\varphi \wedge \psi) = \tilde{d}\varphi + (-1)^k \varphi \wedge \tilde{d}\psi.$$

(2) *For $f \in \Lambda^0(\tilde{M})$, $\tilde{d}f$ is the differentiation of $f$.*

(3) $\tilde{d}^2 = \tilde{d} \cdot \tilde{d} = 0$.

(4) *$\tilde{d}$ is a local operator, i.e., if $U \subset V \subset \tilde{M}$ are open sets and $\alpha \in \Lambda^k(V)$, then $\tilde{d}(\alpha|_U) = (\tilde{d}\alpha)|_U$.*

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**Proof.** Let $(U; [\varphi])$, where $[\varphi] : p \to \bigcup_{i=1}^{s(p)} [\varphi](p) = [\varphi(p)]$, be a local chart for a point $p \in \widetilde{M}$ and $\alpha = \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)} dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k}$ with $1 \le \nu_j \le n_{\mu_i}$ for $1 \le \mu_i \le s(p)$, $1 \le i \le k$. We first establish the uniqueness. If $k = 0$, the local formula $\widetilde{d}\alpha = \dfrac{\partial \alpha}{\partial x^{\mu\nu}} dx^{\mu\nu}$ applied to the coordinates $x^{\mu\nu}$ with $1 \le \nu_j \le n_{\mu_i}$ for $1 \le \mu_i \le s(p)$, $1 \le i \le k$ shows that the differential of $x^{\mu\nu}$ is 1-form $dx^{\mu\nu}$. From (3), $\widetilde{d}(x^{\mu\nu}) = 0$, which combining with (1) shows that $\widetilde{d}(dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k}) = 0$. This, again by (1),

$$\widetilde{d}\alpha = \frac{\partial \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)}}{\partial x^{\mu\nu}} dx^{\mu\nu} \wedge dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k} \tag{3.3}$$

and $\widetilde{d}$ is uniquely determined on $U$ by properties (1)-(3) and by (4) on any open subset of $\widetilde{M}$.

For existence, define on every local chart $(U; [\varphi])$ the operator $\widetilde{d}$ by (3.3). Then (2) is trivially verified as is **R**-linearity. If $\beta = \beta_{(\sigma_1\varsigma_1)\cdots(\sigma_l\varsigma_l)} dx^{\sigma_1\varsigma_1} \wedge \cdots \wedge dx^{\sigma_l\varsigma_l} \in \Lambda^l(U)$, then

$$\begin{aligned}
&\widetilde{d}(\alpha \wedge \beta) \\
&= \widetilde{d}(\alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)} \beta_{(\sigma_1\varsigma_1)\cdots(\sigma_l\varsigma_l)} dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k} \wedge dx^{\sigma_1\varsigma_1} \wedge \cdots \wedge dx^{\sigma_l\varsigma_l}) \\
&= \left( \frac{\partial \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)}}{\partial x^{\mu\nu}} \beta_{(\sigma_1\varsigma_1)\cdots(\sigma_l\varsigma_l)} + \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)} \right. \\
&\quad \left. \times \frac{\partial \beta_{(\sigma_1\varsigma_1)\cdots(\sigma_l\varsigma_l)}}{\partial x^{\mu\nu}} \right) dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k} \wedge dx^{\sigma_1\varsigma_1} \wedge \cdots \wedge dx^{\sigma_l\varsigma_l} \\
&= \frac{\partial \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)}}{\partial x^{\mu\nu}} dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k} \wedge \beta_{(\sigma_1\varsigma_1)\cdots(\sigma_l\varsigma_l)} ds^{\sigma_1\varsigma_1} \wedge \cdots \wedge dx^{\sigma_l\varsigma_l}
\end{aligned}$$

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$$+(-1)^k \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)} dx^{\mu_1\nu_1} \cdots \wedge dx^{\mu_k\nu_k} \wedge \frac{\partial \beta_{(\sigma_1\varsigma_1)\cdots(\sigma_l\varsigma_l)}}{\partial x^{\mu\nu}} dx^{\sigma_1\varsigma_1} \cdots \wedge dx^{\sigma_l\varsigma_l}$$

$$= \tilde{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \tilde{d}\beta$$

and (1) is verified. For (3), symmetry of the second partial derivatives shows that

$$\tilde{d}(\tilde{d}\alpha) = \frac{\partial^2 \alpha_{(\mu_1\nu_1)\cdots(\mu_k\psi_k)}}{\partial x^{\mu\nu} \partial x^{\sigma\varsigma}} dx^{\mu_1\nu_1} \wedge \cdots \wedge dx^{\mu_k\nu_k} \wedge dx^{\sigma_1\varsigma_1} \wedge \cdots \wedge dx^{\sigma_l\varsigma_l} = 0.$$

Thus, in every local chart $(U; [\varphi])$, (3.3) defines an operator $\tilde{d}$ satisfying (1)-(3). It remains to be shown that $\tilde{d}$ really defines an operator $\tilde{d}$ on any open set and (4) holds. To do so, it suffices to show that this definition is chart independent. Let $\tilde{d}'$ be the operator given by (3.3) on a local chart $(U'; [\varphi'])$, where $U \cap U' \neq \varnothing$. Since $\tilde{d}'$ also satisfies (1)-(3) and the local uniqueness has already been proved, $\tilde{d}'\alpha = \tilde{d}\alpha$ on $U \cap U'$. Whence, (4) thus follows.

**Corollary 3.2.** *Let* $\tilde{M} = \tilde{M}(n_1, n_2, ..., n_m)$ *be a smoothly combinatorial manifold and* $d_M : \Lambda^k(M) \to \Lambda^{k+1}(M)$ *the unique exterior differentiation on* $M$ *with conditions following hold for* $M \in V(G^l[\tilde{M}(n_1, n_2, ..., n_m)])$, *where* $1 \leq l \leq \min\{n_1, n_2, ..., n_m\}$.

(1) $d_M$ *is linear, i.e., for* $\varphi, \psi \in \Lambda(M)$, $\lambda \in \mathbf{R}$,

$$d_M(\varphi + \lambda\psi) = d_M\varphi + \lambda d_M\psi.$$

(2) *For* $\varphi \in \Lambda^r(M)$, $\psi \in \Lambda(M)$,

$$d_M(\varphi \wedge \psi) = d_M\varphi + (-1)^r \varphi \wedge d_M\psi.$$

(3) *For* $f \in \Lambda^0(M)$, $d_M f$ *is the differentiation of* $f$.

(4) $d_M^2 = d_M \cdot d_M = 0$.

*Then*

$$\tilde{d}|_M = d_M.$$

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**Proof.** By Theorem 2.4.5 in [1], $d_M$ exists uniquely for any smoothly manifold $M$. Now since $\tilde{d}$ is a local operator on $\tilde{M}$, i.e., for any open subset $U_\mu \subset \tilde{M}$, $\tilde{d}(\alpha|_{U_\mu}) = (\tilde{d}\alpha)|_{U_\mu}$ and there is an index set $J$ such that $M = \bigcup_{\mu\in J} U_\mu$, we finally get that

$$\tilde{d}|_M = d_M$$

by the uniqueness of $\tilde{d}$ and $d_M$.

**Theorem 3.6.** *Let* $\omega \in \Lambda^1(\tilde{M})$. *Then for* $X, Y \in \mathscr{X}(\tilde{M})$,

$$\tilde{d}\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

**Proof.** Denote by $\alpha(X, Y)$ the right hand side of the formula. We know that $\alpha : \tilde{M} \times \tilde{M} \to C^\infty(\tilde{M})$. It can be checked immediately that $\alpha$ is bilinear and for $X, Y \in \mathscr{X}(\tilde{M})$, $f \in C^\infty(\tilde{M})$,

$$\begin{aligned}\alpha(fX, Y) &= fX(\omega(Y)) - Y(\omega(fX)) - \omega([fX, Y]) \\ &= fX(\omega(Y)) - Y(f\omega(X)) - \omega(f[X, Y] - Y(f)X) \\ &= f\alpha(X, Y)\end{aligned}$$

and

$$\alpha(X, fY) = -\alpha(fY, X) = -f\alpha(Y, X) = f\alpha(X, Y)$$

by definition. Accordingly, $\alpha$ is a differential 2-form. We only need to prove that for a local chart $(U, [\varphi])$,

$$\alpha|_U = \tilde{d}\omega|_U.$$

In fact, assume $\omega|_U = \omega_{\mu\nu}dx^{\mu\nu}$. Then

$$\begin{aligned}(\tilde{d}\omega)|_U = \tilde{d}(\omega|_U) &= \frac{\partial\omega_{\mu\nu}}{\partial x^{\sigma\varsigma}}\, dx^{\sigma\varsigma} \wedge dx^{\mu\nu} \\ &= \frac{1}{2}\left(\frac{\partial\omega_{\mu\nu}}{\partial x^{\sigma\varsigma}} - \frac{\partial\omega_{\varsigma\tau}}{\partial x^{\mu\nu}}\right) dx^{\sigma\varsigma} \wedge dx^{\mu\nu}.\end{aligned}$$

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On the other hand, $\alpha|_U = \frac{1}{2}\alpha\left(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\sigma\varsigma}}\right) dx^{\sigma\varsigma} \wedge dx^{\mu\nu}$, where

$$\alpha\left(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\sigma\varsigma}}\right) = \frac{\partial}{\partial x^{\sigma\varsigma}}\left(\omega\left(\frac{\partial}{\partial x^{\mu\nu}}\right)\right) - \frac{\partial}{\partial x^{\mu\nu}}\left(\omega\left(\frac{\partial}{\partial x^{\sigma\varsigma}}\right)\right) - \omega\left(\left[\frac{\partial}{\partial x^{\mu\nu}} - \frac{\partial}{\partial x^{\sigma\varsigma}}\right]\right)$$

$$= \frac{\partial \omega_{\mu\nu}}{\partial x^{\sigma\varsigma}} - \frac{\partial \omega_{\sigma\varsigma}}{\partial x^{\mu\nu}}.$$

Therefore, $\tilde{d}\omega|_U = \alpha|_U$.

### 3.3. Connections on tensors

We introduce connections on tensors of smoothly combinatorial manifolds by the next definition.

**Definition 3.8.** Let $\tilde{M}$ be a smoothly combinatorial manifold. A connection on tensors of $\tilde{M}$ is a *mapping* $\tilde{D} : \mathscr{X}(\tilde{M}) \times T_s^r\tilde{M} \to T_s^r\tilde{M}$ with $\tilde{D}_X\tau = \tilde{D}(X, \tau)$ such that for $X, Y \in \mathscr{X}\tilde{M}$, $\tau, \pi \in T_s^r(\tilde{M})$, $\lambda \in \mathbf{R}$ and $f \in C^\infty(\tilde{M})$,

(1) $\tilde{D}_{X+fY}\tau = \tilde{D}_X\tau + f\tilde{D}_Y\tau$; and $\tilde{D}_X(\tau + \lambda\pi) = \tilde{D}_X\tau + \lambda\tilde{D}_X\pi$;

(2) $\tilde{D}_X(\tau \otimes \pi) = \tilde{D}_X\tau \otimes \pi + \sigma \otimes \tilde{D}_X\pi$;

(3) for any contraction $C$ on $T_s^r(\tilde{M})$,

$$\tilde{D}_X(C(\tau)) = C(\tilde{D}_X\tau).$$

We get results following for these connections on tensors of smoothly combinatorial manifolds.

**Theorem 3.7.** *Let $\tilde{M}$ be a smoothly combinatorial manifold. Then there exists a connection $\tilde{D}$ locally on $\tilde{M}$ with a form*

$$(\tilde{D}_X\tau)|_U$$

$$= X^{\sigma\varsigma}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)} \frac{\partial}{\partial x^{\mu_1\nu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_r\nu_r}} \otimes dx^{\kappa_1\lambda_1} \otimes \cdots \otimes dx^{\kappa_s\lambda_s}$$

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*for* $Y \in \mathscr{X}(\widetilde{M})$ *and* $\tau \in T_s^r(\widetilde{M})$, *where*

$$\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)} = \frac{\partial \tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)}}{\partial x^{\mu\nu}} + \sum_{a=1}^{r} \tau^{(\mu_1\nu_1)\cdots(\mu_{a-1}\nu_{a-1})(\sigma\varsigma)(\mu_{a+1}\nu_{a+1})\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)} \Gamma^{\mu_a\nu_a}_{(\sigma\varsigma)(\mu\nu)} - \sum_{b=1}^{s} \tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)\cdots(\kappa_{b-1}\lambda_{b-1})(\mu\nu)(\sigma_{b+1}\varsigma_{b+1})\cdots(\kappa_s\lambda_s)} \Gamma^{\sigma\varsigma}_{(\sigma_b\varsigma_b)(\mu\nu)}$$

*and* $\Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$ *is a function determined by*

$$\widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} \frac{\partial}{\partial x^{\sigma\varsigma}} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)} \frac{\partial}{\partial x^{\sigma\varsigma}}$$

*on* $(U_p; [\varphi_p]) = (U_p; x^{\mu\nu})$ *of a point* $p \in \widetilde{M}$, *also called the coefficient on a connection.*

**Proof.** We first prove that any connection $\widetilde{D}$ on smoothly combinatorial manifolds $\widetilde{M}$ is local by definition, namely for $X_1, X_2 \in \mathscr{X}(\widetilde{M})$ and $\tau_1, \tau_2 \in T_s^r(\widetilde{M})$, if $X_1|_U = X_2|_U$ and $\tau_1|_U = \tau_2|_U$, then $(\widetilde{D}_{X_1}\tau_1)_U = (\widetilde{D}_{X_2}\tau_2)_U$. For this objective, we need to prove that $(\widetilde{D}_{X_1}\tau_1)_U = (\widetilde{D}_{X_1}\tau_2)_U$ and $(\widetilde{D}_{X_1}\tau_1)_U = (\widetilde{D}_{X_2}\tau_1)_U$. Since their proofs are similar, we check the first only.

In fact, if $\tau = 0$, then $\tau = \tau - \tau$. By the definition of connection,

$$\widetilde{D}_X\tau = \widetilde{D}_X(\tau - \tau) = \widetilde{D}_X\tau - \widetilde{D}_X\tau = 0.$$

Now let $p \in U$. Then there is a neighborhood $V_p$ of $p$ such that $\overline{V}$ is compact and $\overline{V} \subset U$. By a result in topology, i.e., for two open sets $V_p$, $U$ of $\mathbf{R}^{\hat{s}(p)-s(p)\hat{s}(p)+n_1+\cdots+n_{s(p)}}$ with compact $\overline{V}_p$ and $\overline{V}_p \subset U$, there exists a function $f \in C^\infty(\mathbf{R}^{\hat{s}(p)-s(p)\hat{s}(p)+n_1+\cdots+n_{s(p)}})$ such that $0 \le f \le 1$ and

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$f|_{V_p} \equiv 1$, $f|_{\mathbf{R}^{\hat{s}(p)-s(p)\hat{s}(p)+n_1+\cdots+n_{s(p)}} \setminus U} \equiv 0$, we find that $f \cdot (\tau_2 - \tau_1) = 0$. Whence, we know that

$$0 = \tilde{D}_{X_1}((f \cdot (\tau_2 - \tau_1))) = X_1(f)(\tau_2 - \tau_1) + f(\tilde{D}_{X_1}\tau_2 - \tilde{D}_{X_1}\tau_1).$$

As a consequence, we get that $(\tilde{D}_{X_1}\tau_1)_V = (\tilde{D}_{X_1}\tau_2)_V$, particularly, $(\tilde{D}_{X_1}\tau_1)_p = (\tilde{D}_{X_1}\tau_2)_p$. For the arbitrary choice of $p$, we get that $(\tilde{D}_{X_1}\tau_1)_U = (\tilde{D}_{X_1}\tau_2)_U$ finally.

The local property of $\tilde{D}$ enables us to find an induced connection $\tilde{D}^U : \mathscr{X}(U) \times T_s^r(U) \to T_s^r(U)$ such that $\tilde{D}^U_{X|_U}(\tau|_U) = (\tilde{D}_X\tau)|_U$ for $X \in \mathscr{X}(\tilde{M})$ and $\tau \in T_s^r\tilde{M}$. Now for $X_1, X_2 \in \mathscr{X}(\tilde{M})$, $\forall\tau_1, \tau_2 \in T_s^r(\tilde{M})$ with $X_1|_{V_p} = X_2|_{V_p}$ and $\tau_1|_{V_p} = \tau_2|_{V_p}$, define a mapping $\tilde{D}^U : \mathscr{X}(U) \times T_s^r(U) \to T_s^r(U)$ by

$$(\tilde{D}_{X_1}\tau_1)|_{V_p} = (\tilde{D}_{X_1}\tau_2)|_{V_p}$$

for any point $p \in U$. Then since $\tilde{D}$ is a connection on $\tilde{M}$, it can be checked easily that $\tilde{D}^U$ satisfies all conditions in Definition 3.8. Whence, $\tilde{D}^U$ is indeed a connection on $U$.

Now we calculate the local form on a chart $(U_p, [\varphi_p])$ of $p$. Since

$$\tilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)} \frac{\partial}{\partial x^{\sigma\varsigma}},$$

it can find immediately that

$$\tilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} dx^{\kappa\lambda} = -\Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)} dx^{\sigma\varsigma}$$

by Definition 3.8. Therefore, we find that

$$(\tilde{D}_X\tau)|_U$$

$$= X^{\sigma\varsigma}\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)} \frac{\partial}{\partial x^{\mu_1\nu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_r\nu_r}} \otimes dx^{\kappa_1\lambda_1} \otimes \cdots \otimes dx^{\kappa_s\lambda_s}$$

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with

$$\tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s),(\mu\nu)}$$

$$= \frac{\partial \tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)}}{\partial x^{\mu\nu}}$$

$$+ \sum_{a=1}^{r} \tau^{(\mu_1\nu_1)\cdots(\mu_{a-1}\nu_{a-1})(\sigma\varsigma)(\mu_{a+1}\nu_{a+1})\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)(\kappa_2\lambda_2)\cdots(\kappa_s\lambda_s)} \Gamma^{\mu_a\nu_a}_{(\sigma\varsigma)(\mu\nu)}$$

$$- \sum_{b=1}^{s} \tau^{(\mu_1\nu_1)(\mu_2\nu_2)\cdots(\mu_r\nu_r)}_{(\kappa_1\lambda_1)\cdots(\kappa_{b-1}\lambda_{b-1})(\mu\nu)(\sigma_{b+1}\varsigma_{b+1})\cdots(\kappa_s\lambda_s)} \Gamma^{\sigma\varsigma}_{(\sigma_b\varsigma_b)(\mu\nu)}.$$

This completes the proof.

**Theorem 3.8.** *Let* $\widetilde{M}$ *be a smoothly combinatorial manifold with a connection* $\widetilde{D}$. *Then for* $X, Y \in \mathscr{X}(\widetilde{M})$,

$$\widetilde{T}(X, Y) = \widetilde{D}_X Y - \widetilde{D}_Y X - [X, Y]$$

*is a tensor of type* $(1, 2)$ *on* $\widetilde{M}$.

**Proof.** By definition, it is clear that $\widetilde{T} : \mathscr{X}(\widetilde{M}) \times \mathscr{X}(\widetilde{M}) \to \mathscr{X}(\widetilde{M})$ is antisymmetrical and bilinear. We only need to check it is also linear on each element in $C^\infty(\widetilde{M})$ for variables $X$ or $Y$. In fact, for $f \in C^\infty(\widetilde{M})$,

$$\widetilde{T}(fX, Y) = \widetilde{D}_{fX} Y - \widetilde{D}_Y(fX) - [fX, Y]$$

$$= f\widetilde{D}_X Y - (Y(f)X + f\widetilde{D}_Y X)$$

$$- (f[X, Y] - Y(f)X) = f\widetilde{T}(X, Y)$$

and

$$\widetilde{T}(X, fY) = -\widetilde{T}(fY, X) = -f\widetilde{T}(Y, X) = f\widetilde{T}(X, Y).$$

Notice that

$$T\left(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\sigma\varsigma}}\right) = \widetilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} \frac{\partial}{\partial x^{\sigma\varsigma}} - \widetilde{D}_{\frac{\partial}{\partial x^{\sigma\varsigma}}} \frac{\partial}{\partial x^{\mu\nu}} = (\Gamma^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} - \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}) \frac{\partial}{\partial x^{\kappa\lambda}}$$

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under a local chart $(U_p; [\varphi_p])$ of a point $p \in \tilde{M}$. If $T\left(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\sigma\varsigma}}\right) \equiv 0$, we call $T$ *torsion-free*. This enables us getting the next useful result.

**Theorem 3.9.** *A connection $\tilde{D}$ on tensors of a smoothly combinatorial manifold $\tilde{M}$ is torsion-free if and only if $\Gamma^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$.*

Now we turn our attention to the case of $s = r = 1$. Similarly, a *combinatorially Riemannian geometry* is defined in the next definition.

**Definition 3.9.** Let $\tilde{M}$ be a smoothly combinatorial manifold and $g \in A^2(\tilde{M}) = \bigcup\limits_{p\in\tilde{M}} T_2^0(p, \tilde{M})$. If $g$ is symmetrical and positive, then $\tilde{M}$ is called a *combinatorially Riemannian manifold*, denoted by $(\tilde{M}, g)$. In this case, if there is a connection $\tilde{D}$ on $(\tilde{M}, g)$ with equality following holds

$$Z(g(X, Y)) = g(\tilde{D}_Z X, Y) + g(X, \tilde{D}_Z Y), \tag{3.4}$$

then $\tilde{M}$ is called a *combinatorially Riemannian geometry*, denoted by $(\tilde{M}, g, \tilde{D})$.

We get a result for connections on smoothly combinatorial manifolds similar to that of Riemannian geometry.

**Theorem 3.10.** *Let $(\tilde{M}, g)$ be a combinatorially Riemannian manifold. Then there exists a unique connection $\tilde{D}$ on $(\tilde{M}, g)$ such that $(\tilde{M}, g, \tilde{D})$ is a combinatorially Riemannian geometry.*

**Proof.** By definition, we know that

$$\tilde{D}_Z g(X, Y) = Z(g(X, Y)) - g(\tilde{D}_Z X, Y) - g(X, \tilde{D}_Z Y)$$

for a connection $\tilde{D}$ on tensors of $\tilde{M}$ and $\forall Z \in \mathscr{X}(\tilde{M})$. Thereby, the equality (3.4) is equivalent to that of $\tilde{D}_Z g = 0$ for $Z \in \mathscr{X}(\tilde{M})$, namely $\tilde{D}$ is torsion-free.

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Without loss of generality, assume $g = g_{(\mu\nu)(\sigma\varsigma)}dx^{\mu\nu}dx^{\sigma\varsigma}$ in a local chart $(U_p; [\varphi_p])$ of a point $p$, where $g_{(\mu\nu)(\sigma\varsigma)} = g\left(\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\sigma\varsigma}}\right)$. Then we find that

$$\tilde{D}g = \left(\frac{\partial g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}} - g_{(\zeta\eta)(\sigma\varsigma)}\Gamma^{\zeta\eta}_{(\mu\nu)(\sigma\varsigma)} - g_{(\mu\nu)(\zeta\eta)}\Gamma^{\zeta\eta}_{(\sigma\varsigma)(\kappa\lambda)}\right)dx^{\mu\nu}\otimes dx^{\sigma\varsigma}\otimes dx^{\kappa\lambda}.$$

Therefore, we get that

$$\frac{\partial g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\kappa\lambda}} = g_{(\zeta\eta)(\sigma\varsigma)}\Gamma^{\zeta\eta}_{(\mu\nu)(\sigma\varsigma)} + g_{(\mu\nu)(\zeta\eta)}\Gamma^{\zeta\eta}_{(\sigma\varsigma)(\kappa\lambda)} \tag{3.5}$$

if $\tilde{D}_Z g = 0$ for $Z \in \mathcal{X}(\tilde{M})$. The formula (3.5) enables us to get that

$$\Gamma^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} = \frac{1}{2}g^{(\kappa\lambda)(\zeta\eta)}\left(\frac{\partial g_{(\mu\nu)(\zeta\eta)}}{\partial x^{\sigma\varsigma}} + \frac{\partial g_{(\zeta\eta)(\sigma\varsigma)}}{\partial x^{\mu\nu}} - \frac{\partial g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\zeta\eta}}\right),$$

where $g^{(\kappa\lambda)(\zeta\eta)}$ is an element in the matrix inverse of $[g_{(\mu\nu)(\sigma\varsigma)}]$.

Now if there exists another torsion-free connection $\tilde{D}^*$ on $(\tilde{M}, g)$ with

$$\tilde{D}^*_{\frac{\partial}{\partial x^{\mu\nu}}} = \Gamma^{*\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial}{\partial x^{\kappa\lambda}},$$

then we must get that

$$\Gamma^{*\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} = \frac{1}{2}g^{(\kappa\lambda)(\zeta\eta)}\left(\frac{\partial g_{(\mu\nu)(\zeta\eta)}}{\partial x^{\sigma\varsigma}} + \frac{\partial g_{(\zeta\eta)(\sigma\varsigma)}}{\partial x^{\mu\nu}} - \frac{\partial g_{(\mu\nu)(\sigma\varsigma)}}{\partial x^{\zeta\eta}}\right).$$

Accordingly, $\tilde{D} = \tilde{D}^*$. Whence, there are at most one torsion-free connection $\tilde{D}$ on a combinatorially Riemannian manifold $(\tilde{M}, g)$.

For the existence of torsion-free connection $\tilde{D}$ on $(\tilde{M}, g)$, let $\Gamma^{\kappa\lambda}_{(\mu\nu)(\sigma\varsigma)} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}$ and define a connection $\tilde{D}$ on $(\tilde{M}, g)$ such that

$$\tilde{D}_{\frac{\partial}{\partial x^{\mu\nu}}} = \Gamma^{\kappa\lambda}_{(\sigma\varsigma)(\mu\nu)}\frac{\partial}{\partial x^{\kappa\lambda}},$$

then $\tilde{D}$ is torsion-free by Theorem 3.9. This completes the proof.

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**Corollary 3.3** [2]. *For a Riemannian manifold* $(M, g)$, *there exists only one torsion-free connection* $D$, *i.e.*,

$$D_Z g(X, Y) = Z(g(X, Y)) - g(D_Z X, Y) - g(X, D_Z Y) \equiv 0$$

*for* $X, Y, Z \in \mathscr{X}(M)$.

### 3.4. Minkowski norms

These Minkowski norms are the fundamental in Finsler geometry. Certainly, they can be also generalized on smoothly combinatorial manifolds.

**Definition 3.10.** A *Minkowski norm* on a vector space $V$ is a function $F : V \to \mathbf{R}$ such that

(1) $F$ is smooth on $V \setminus \{0\}$ and $F(v) \geq 0$ for $v \in V$;

(2) $F$ is 1-homogenous, i.e., $F(\lambda v) = \lambda F(v)$ for $\lambda > 0$;

(3) for all $y \in V \setminus \{0\}$, the symmetric bilinear form $g_y : V \times V \to \mathbf{R}$ with

$$g_y(u, v) = \sum_{i,j} \frac{\partial^2 F(y)}{\partial y^i \partial y^j}$$

is positive definite for $u, v \in V$.

Denoted by $T\widetilde{M} = \bigcup_{p \in \widetilde{M}} T_p \widetilde{M}$. Similar to Finsler geometry, we introduce combinatorially Finsler geometries on a Minkowski norm defined on $T\widetilde{M}$.

**Definition 3.11.** A combinatorially Finsler geometry is a *smoothly combinatorial manifold* $\widetilde{M}$ endowed with a Minkowski norm $\widetilde{F}$ on $T\widetilde{M}$, denoted by $(\widetilde{M}; \widetilde{F})$.

Then we get the following result.

**Theorem 3.11.** *There are combinatorially Finsler geometries.*

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**Proof.** Let $\widetilde{M}(n_1, n_2, ..., n_m)$ be a smoothly combinatorial manifold. We construct Minkowski norms on $T\widetilde{M}(n_1, n_2, ..., n_m)$. Let $\mathbf{R}^{n_1+n_2+\cdots+n_m}$ be a Euclidean space. Then there exists a Minkowski norm $F(\bar{x}) = |\bar{x}|$ in $\mathbf{R}^{n_1+n_2+\cdots+n_m}$ at least, in here $|\bar{x}|$ denotes the Euclidean norm on $\mathbf{R}^{n_1+n_2+\cdots+n_m}$. According to Theorem 3.2, $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ is homeomorphic to $\mathbf{R}^{\hat{s}(p)-s(p)\hat{s}(p)+n_{i_1}+\cdots+n_{i_{s(p)}}}$. Whence there are Minkowski norms on $T_p\widetilde{M}(n_1, n_2, ..., n_m)$ for $p \in U_p$, where $(U_p; [\varphi_p])$ is a local chart.

Notice that the number of manifolds is finite in a smoothly combinatorial manifold $\widetilde{M}(n_1, n_2, ..., n_m)$ and each manifold has a finite cover $\{(U_\alpha; \varphi_\alpha)|\alpha \in I\}$, where $I$ is a finite index set. We know that there is a finite cover

$$\bigcup_{M\in V(G[\widetilde{M}(n_1, n_2, ..., n_m)])} \{(U_{M\alpha}; \varphi_{M\alpha})|\alpha \in I_M\}.$$

By the decomposition theorem for unit, we know that there are smooth functions $h_{M\alpha}$, $\alpha \in I_M$ such that

$$\sum_{M\in V(G[\widetilde{M}(n_1, n_2, ..., n_m)])} \sum_{\alpha\in I_M} h_{M\alpha} = 1 \text{ with } 0 \le h_{M\alpha} \le 1.$$

Now we choose a Minkowski norm $\widetilde{F}^{M\alpha}$ on $T_pM_\alpha$ for $p \in U_{M\alpha}$. Define

$$\widetilde{F}_{M\alpha} = \begin{cases} h^{M\alpha}\widetilde{F}^{M\alpha}, & \text{if } p \in U_{M\alpha}, \\ 0, & \text{if } p \notin U_{M\alpha} \end{cases}$$

for $p \in \widetilde{M}$. Now let

$$\widetilde{F} = \sum_{M\in V(G[\widetilde{M}(n_1, n_2, .., n_m)])} \sum_{\alpha\in I} \widetilde{F}_{M\alpha}.$$

Then $\widetilde{F}$ is a Minkowski norm on $T\widetilde{M}(n_1, n_2, ..., n_m)$ since it can be checked immediately that all conditions (1)-(3) in Definition 3.10 hold.

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For the relation of combinatorially Finsler geometries with these Smarandache geometries, we obtain the next consequence.

**Theorem 3.12.** *A combinatorially Finsler geometry* $(\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{F})$ *is a Smarandache geometry if* $m \geq 2$.

**Proof.** Notice that if $m \geq 2$, then $\widetilde{M}(n_1, n_2, ..., n_m)$ is combined by at least two manifolds $M^{n_1}$ and $M^{n_2}$ with $n_1 \neq n_2$. By definition, we know that

$$M^{n_1} \setminus M^{n_2} \neq \varnothing \text{ and } M^{n_2} \setminus M^{n_1} \neq \varnothing.$$

Now the axiom there is an integer $n$ such that there exists a neighborhood homeomorphic to an open ball $\mathbf{B}^n$ for any point in this space is Smarandachely denied, since for points in $M^{n_1} \setminus M^{n_2}$, each has a neighborhood homeomorphic to $B^{n_1}$, but each point in $M^{n_2} \setminus M^{n_1}$ has a neighborhood homeomorphic to $B^{n_2}$.

Theorems 3.11 and 3.12 imply inclusions in Smarandache geometries for classical geometries in the following.

**Corollary 3.4.** *There are inclusions among Smarandache geometries, Finsler geometry, Riemannian geometry and Weyl geometry*:

$$\{\textit{Smarandache geometries}\} \supset \{\textit{combinatorially Finsler geometries}\}$$

$$\supset \{\textit{Finsler geometry}\} \textit{ and } \{\textit{combinatorially Riemannian geometries}\}$$

$$\supset \{\textit{Riemannian geometry}\} \supset \{\textit{Weyl geometry}\}.$$

**Proof.** Let $m = 1$. Then a combinatorially Finsler geometry $(\widetilde{M}(n_1, n_2, ..., n_m); \widetilde{F})$ is nothing but just a Finsler geometry. Applying Theorems 3.11 and 3.12 to this special case, we get these inclusions as expected.

**Corollary 3.5.** *There are inclusions among Smarandache geometries, combinatorially Riemannian geometries and Kähler geometry*:

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$$\{\textit{Smarandache geometries}\} \supset \{\textit{combinatorially Riemannian geometries}\}$$
$$\supset \{\textit{Riemannian geometry}\}$$
$$\supset \{\textit{Kähler geometry}\}.$$

**Proof.** Let $m = 1$ in a combinatorial manifold $\tilde{M}(n_1, n_2, ..., n_m)$ and applies Theorems 3.10 and 3.12, we get inclusions

$$\{\text{Smarandache geometries}\} \supset \{\text{combinatorially Riemannian geometries}\}$$
$$\supset \{\text{Riemannian geometry}\}.$$

For the Kähler geometry, notice that any complex manifold $M_c^n$ is equal to a smoothly real manifold $M^{2n}$ with a natural base $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}$ for $T_p M_c^n$ at each point $p \in M_c^n$. Whence, we get

$$\{\text{Riemannian geometry}\} \supset \{\text{Kähler geometry}\}.$$

## 4. Further Discussions

### 4.1. Embedding problem

Whitney had shown that any smooth manifold $M^d$ can be embedded as a closed submanifold of $\mathbf{R}^{2d+1}$ in 1936 [1]. The same embedding problem for finitely combinatorial manifold in a Euclidean space is also interesting. Since $\tilde{M}$ is finite, by applying Whitney theorem, we know that there is an integer $n(\tilde{M})$, $n(\tilde{M}) < +\infty$ such that $\tilde{M}$ can be embedded as a closed submanifold in $\mathbf{R}^{n(\tilde{M})}$. Then what is the minimum dimension of Euclidean spaces embeddable a given finitely combinatorial manifold $\tilde{M}$? Whether can we determine it for some combinatorial manifolds with a given graph structure, such as those of complete graphs $K^n$, circuits $P^n$ or cubic graphs $Q^n$?

**Conjecture 4.1.** The minimum dimension of Euclidean spaces embeddable a finitely combinatorial manifold $\tilde{M}$ is

$$2 \min_{p \in \tilde{M}} \{\hat{s}(p) - s(p)\hat{s}(p) + n_{i_1} + n_{i_2} + \cdots + n_{i_{s(p)}}\} + 1.$$

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### 4.2. *D*-dimensional holes

For these closed 2-manifolds $S$, it is well known that

$$\chi(S) = \begin{cases} 2 - 2p(S), & \text{if } S \text{ is orientable,} \\ 2 - q(S), & \text{if } S \text{ is non-orientable,} \end{cases}$$

with $p(S)$ or $q(S)$ the orientable genus or non-orientable genus of $S$, namely 2-dimensional holes adjacent to $S$. For general case of $n$-manifolds $M$, we know that

$$\chi(M) = \sum_{k=0}^{\infty} (-1)^k \dim H_k(M),$$

where $\dim H_k(M)$ is the rank of these $k$-dimensional homology groups $H_k(M)$ in $M$, namely the number of $k$-dimensional holes adjacent to the manifold $M$. By the definition of combinatorial manifolds, some $k$-dimensional holes adjacent to a combinatorial manifold are increased. Then what is the relation between the Euler-Poincaré characteristic of a combinatorial manifold $\tilde{M}$ and the $i$-dimensional holes adjacent to $\tilde{M}$? Whether can we find a formula likewise the Euler-Poincaré formula? Calculation shows that even for the case of $n = 2$, the situation is complex. For example, choose $n$ different orientable 2-manifolds $S_1$, $S_2$, ..., $S_n$ and let them intersect one after another at $n$ different points in $\mathbf{R}^3$. We get a combinatorial manifold $\tilde{M}$. Calculation shows that

$$\chi(\tilde{M}) = (\chi(S_1) + \chi(S_2) + \cdots + \chi(S_n)) - n$$

by Theorem 2.9. But it only increases one 2-holes. What is the relation of 2-dimensional holes adjacent to $\tilde{M}$?

### 4.3. Local properties

Although a finitely combinatorial manifold $\tilde{M}$ is not homogenous in general, namely the dimension of local charts of two points in $\tilde{M}$ may be different, we have still constructed global operators such as those of exterior differentiation $\tilde{d}$ and connection $\tilde{D}$ on $T_s^r \tilde{M}$.

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An operator $\widetilde{\mathfrak{D}}$ is said to be *local* on a subset $W \subset T_s^r \widetilde{M}$ if for any local chart $(U_p, [\varphi_p])$ of a point $p \in W$,

$$\widetilde{\mathfrak{D}}|_{U_p}(W) = \widetilde{\mathfrak{D}}(W)_{U_p}.$$

Of course, nearly all existent operators with local properties on $T_s^r \widetilde{M}$ in Finsler or Riemannian geometries can be reconstructed in these combinatorially Finsler or Riemannian geometries and find the local forms similar to those in Finsler or Riemannian geometries.

**Global properties**

To find global properties on manifolds is a central task in classical differential geometry. The same is true for combinatorial manifolds. In classical geometry on manifolds, some global results, such as those of de Rham theorem and Atiyah-Singer index theorem, etc. are well known. Remember that the $p$th de Rham cohomology group on a manifold $M$ and the index $\mathrm{Ind}\mathcal{D}$ of a Fredholm operator $\mathcal{D} : H^k(M, E) \to L^2(M, F)$ are defined to be a *quotient space*

$$H^p(M) = \frac{Ker(d : \Lambda^p(M) \to \Lambda^{p+1}(M))}{\mathrm{Im}(d : \Lambda^{p-1}(M) \to \Lambda^p(M))}$$

and an integer

$$\mathrm{Ind}\mathcal{D} = \dim Ker(\mathcal{D}) - \dim\left(\frac{L^2(M, F)}{\mathrm{Im}\,\mathcal{D}}\right),$$

respectively. The de Rham theorem and the Atiyah-Singer index theorem respectively conclude that for any manifold $M$, a mapping $\varphi : \Lambda^p(M) \to Hom(\Pi_p(M), \mathbf{R})$ induces a natural isomorphism $\varphi^* : H^p(M) \to H^n(M; \mathbf{R})$ of cohomology groups, where $\Pi_p(M)$ is the free Abelian group generated by the set of all $p$-simplexes in $M$ and

$$\mathrm{Ind}\mathcal{D} = \mathrm{Ind}_T(\sigma(D)),$$

where $\sigma(\mathcal{D}) : T^*M \to Hom(E, F)$ and $\mathrm{Ind}_T(\sigma(\mathcal{D}))$ is the topological index of $\sigma(\mathcal{D})$. Now the questions for these finitely combinatorial manifolds are given in the following.

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(1) Is the de Rham theorem and Atiyah-Singer index theorem still true for finitely combinatorial manifolds? If not, what are its modified forms?

(2) Check other global results for manifolds whether true or get their new modified forms for finitely combinatorial manifolds.

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