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ISOMORPHISM OF MODULAR GROUP ALGEBRAS OF p-SPLITTING ABELIAN W-GROUPS

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Abstract

Let R be a commutative unital ring of prime characteristic p and G be a p-splitting abelian μ-elementary W-group for some ordinal μ. A complete set of invariants for the R-group algebra RG is found. The main result is a supplement to our recent publications in [8] and [7].

1. Preliminary Facts

Throughout this short article, suppose that RG is the group algebra of an abelian group G over a commutative ring R with identity of prime characteristic p. For such a group G, the letter G_p is reserved for the p-component of torsion in G and G_t for the torsion part (= maximal torsion subgroup) of G.

In [8] we have proved that if the *Isomorphism Problem* for p-mixed groups holds true, that is G being p-mixed and $RG \cong RH$ being R-isomorphic for some arbitrary group H imply $G \cong H$, then one can find a suitable complete set of invariants for RG provided that G is p-splitting. As application, we have considered the concrete class of Warfield groups utilizing our recently obtained results from [5] and [6] (for more details see also [2] and [3]). Moreover, in [7] we have shown that the

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Isomorphism Problem for p-mixed μ-elementary W-groups for some ordinal μ, which class of global mixed groups properly contains the Warfield ones, holds in the affirmative thus extending earlier results from [5] and [6].

The purpose of the present brief note is to combine these two achievements in order to establish a full system of invariants for RG whenever G is a p-splitting μ-elementary W-group for some ordinal μ, thus strengthening the chief statement in [8].

2. The Main Assertion

Before proceed by proving our affirmation, we need the following technicalities. We shall verify only the first one which is necessary for the convenience of the readers.

Proposition 1 [8]. *Let G be an abelian group. Then G p-splits $\Leftrightarrow G/\coprod_{q\neq p} G_q$ splits.*

Proof. First of all, note that

$$\left(G/\coprod_{q\neq p} G_q\right)_t = G_t/\coprod_{q\neq p} G_q = \left(G_p \times \coprod_{q\neq p} G_q\right)/\coprod_{q\neq p} G_q \cong G_p.$$

(1) "$\Rightarrow$" Write $G = G_p \times M$ for some $M \leq G$. Therefore, it is self-evident that

$$G/\coprod_{q\neq p} G_q = \left(G_t/\coprod_{q\neq p} G_q\right)\left(M\left(\coprod_{q\neq p} G_q\right)/\coprod_{q\neq p} G_q\right)$$

$$= \left(G/\coprod_{q\neq p} G_q\right)_t\left(M\left(\coprod_{q\neq p} G_q\right)/\coprod_{q\neq p} G_q\right).$$

What remains to demonstrate is that the intersection between the two factors is equal to one. This, certainly, is accomplished by showing that $G_t \cap \left(M\coprod_{q\neq p} G_q\right) = \coprod_{q\neq p} G_q$. Indeed, with the aid of the modular law, we calculate that

$$G_t \cap \left(M \coprod_{q \neq p} G_q \right)$$

$$= \left(\coprod_{q \neq p} G_q \right)(G_t \cap M) = \left(\coprod_{q \neq p} G_q \right)((G_p \times M_t) \cap M)$$

$$= \left(\coprod_{q \neq p} G_q \right)\left(\left(G_p \times \coprod_{q \neq p} M_q \right) \cap M \right)$$

$$= \left(\coprod_{q \neq p} G_q \right)(G_p \cap M) = \coprod_{q \neq p} G_q,$$

so completing this part-half.

(2) "$\Leftarrow$" Write

$$G \Big/ \coprod_{q \neq p} G_q = \left(G \Big/ \coprod_{q \neq p} G_q \right)_t \times \left(N \Big/ \coprod_{q \neq p} G_q \right)$$

or equivalently

$$G \Big/ \coprod_{q \neq p} G_q = \left(G_t \Big/ \coprod_{q \neq p} G_q \right) \times \left(N \Big/ \coprod_{q \neq p} G_q \right)$$

for some $N \leq G$. Consequently, since $N \supseteq \coprod_{q \neq p} G_q$, it is not hard to detect that $G = G_t N = G_p N$ with $G_t \cap N = \coprod_{q \neq p} G_q$. Hence $G_p \cap N \subseteq \left(\coprod_{q \neq p} G_q \right)_p = 1$. Thereby, $G = G_p \times N$ and we are finished.

Theorem 2 [7]**.** *Suppose G is an abelian μ-elementary W-group for some ordinal μ. Then the R-isomorphism $RH \cong RG$ for an arbitrary group H implies that $H \Big/ \coprod_{q \neq p} H_q \cong G \Big/ \coprod_{q \neq p} G_q$. In particular, $H \cong G$, provided that G is p-mixed.*

We recall that the mixed abelian group G is said to be *p-splitting* if G_p separates as its direct factor.

So, we have at our disposal all the machinery necessary to deduce the following attainment.

Theorem 3 (Isomorphism)**.** *Suppose G is a p-splitting abelian μ-elementary W-group for some ordinal μ. Then $RH \cong RG$ as R-algebras for any group H if and only if the following conditions hold valid*:

(1) *H is p-splitting abelian*;

(2) $H_p \cong G_p$;

(3) $R(H/H_p) \cong R(G/G_p)$ *as R-algebras*.

Proof. Necessity. Property (1) follows directly from Proposition 1 together with Theorem 2. Point (2) follows also by Theorem 2 since

$$G_p \cong \left(\coprod_{q\neq p} G_q \times G_p\right)\Big/\coprod_{q\neq p} G_q = G_t\Big/\coprod_{q\neq p} G_q = \left(G\Big/\coprod_{q\neq p} G_q\right)_t$$

$$\cong \left(H\Big/\coprod_{q\neq p} H_q\right)_t = H_t\Big/\coprod_{q\neq p} H_q$$

$$= \left(\coprod_{q\neq p} H_q \times H_p\right)\Big/\coprod_{q\neq p} H_q \cong H_p.$$

Finally, the isomorphism (3) follows by virtue of one of the following articles [11], ([9], [10]) or ([1], [4]).

Sufficiency. Write $G \cong G_p \times G/G_p$ and $H \cong H_p \times H/H_p$. Therefore, $RG \cong RG_p \oplus_R R(G/G_p) \cong RH_p \otimes_R R(H/H_p) \cong RH,$ as stated.

This finishes the proof.

In certain instances for R the condition (3) may be expressed only in terms of R and G. In fact, we are now prepared to prove the following.

Corollary 4 (Invariants)**.** *Let G be a p-splitting abelian* μ*-elementary W-group for some ordinal* μ *and R be an algebraically closed field of* $char(R) = p \neq 0$. *Then* $RH \cong RG$ *as R-algebras for some other group H if and only if the following dependencies are fulfilled*:

(1) *H is p-splitting abelian*;

(2) $H_p \cong G_p$;

(3) $H/H_t \cong G/G_t$;

(4) $|H_t/H_p| = |G_t/G_p|$.

Proof. Appealing to Theorem 3, what suffices to show is that $R(G/G_p) \cong R(H/H_p)$ uniquely when $G/G_t \cong H/H_t$ and $|G_t/G_p| =$

$|H_t/H_p|$. Indeed, since $G_t/G_p = (G/G_p)_t$, we derive that $G/G_t \cong G/G_p/G_t/G_p = G/G_p/(G/G_p)_t$. By a reason of symmetry, the same is true for H/H_t. Consequently, we wish apply [11] and [12] to substantiate our claim. This completes the proof.

We recollect that the mixed abelian group G is termed *splitting* provided that G_t separates as its direct factor. Moreover, imitating [14], the set $s_q(R) = \{i \in \mathbb{N} : R(\eta_i) \neq R(\eta_{i+1})\}$, where η_i is the primitive q^i-th root of unity taken in the algebraic closure of R, is said to be the *spectrum* of R with respect to the prime number q whenever R is a field.

And so, we are now ready to illustrate the following.

Corollary 5 (Invariants)**.** *Let G be a splitting abelian μ-elementary W-group for some ordinal μ such that G_t/G_p is finite and let R be an arbitrary field of $char(R) = p > 0$. Then $RH \cong RG$ as R-algebras for another group H if and only if the following relations are realized*:

(1) *H is splitting abelian*;

(2) $H_p \cong G_p$;

(3) $H/H_t \cong G/G_t$;

(4) $|H_t/H_p| = |G_t/G_p|$;

(5) $|H_t^{q^i}/H_p| = |G_t^{q^i}/G_p|, \forall i \in s_q(R), \forall q \neq p.$

Proof. As in the previous consequence, it suffices to demonstrate that $R(G/G_p) \cong R(H/H_p)$ precisely when $G/G_t \cong H/H_t$ and $|H_t^{q^i}/H_p| = |G_t^{q^i}/G_p|$ for all $i \in s_q(R) \cup \{0\}$. Because of the equalities $G_t/G_p = (G/G_p)_t$ and $H_t/H_p = (H/H_p)_t$, we appeal to [14] and [4] to extract the claim. This concludes the proof.

We terminate the paper with the following

Critical Remark. In the reviewer's report [13] there is a mistake. In fact, there was claimed that [sic]: "Moreover, the text contains some errors. For example, he chooses a maximal ideal of R which contains the prime number p. But $charR = p$ and $p = 0$ in R", which claim of Mihovski sounds strange and is obviously wrong since 0 belongs to any ideal of R.

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