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ON DIFFERENTIAL SANDWICH THEOREMS FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS USING DZIOK-SRIVASTAVA OPERATOR

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Abstract

Let $q_1(z)$ and $q_2(z)$ be univalent and analytic in the open unit disk $\Delta := \{z : |z| < 1\}$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z) \prec z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \prec q_2(z),$$

where $H^{l,m}[\alpha_1]f$ is the familiar Dziok-Srivastava operator.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disk $\Delta := \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A} \subseteq \mathcal{H}$ denote the class of all analytic functions of the form

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$f(z) = z + a_2 z^2 + \cdots$. If $f, F \in \mathcal{H}$ and F is univalent in Δ, then we say that the function f is *subordinate* to F, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(\Delta) \subseteq F(\Delta)$, then F is said to be *superordinate* to f.

Let $h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.1}$$

then p is the solution of the differential superordination. An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.1) is said to be the *best subordinant*. Recently Miller and Mocanu [11] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [11], Bulboacă [3] have considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [2].

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}$, $j = 1, 2, \ldots m$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1;\ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a(a+1)(a+2)\cdots(a+n-1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$\Psi(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [6] (see also [17]) $H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the Hadamard product

$$\begin{aligned} &H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z) \\ &:= \Psi(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_m)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

It is well known [6] that

$$\begin{aligned} &\alpha_1 H^{l,m}(\alpha_1 + 1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z) \\ &= z[H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z)]' \\ &\quad + (\alpha_1 - 1)H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z). \end{aligned} \tag{1.3}$$

To have a simpler notation, we write $H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)f(z)$ as $H^{l,m}[\alpha_1]f(z)$.

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [7], the Carlson-Shaffer linear operator [4], the Ruscheweyh derivative operator [14], the generalized Bernardi-Libera-Livingston linear integral operator (cf. [1], [8], [9]) and the Srivastava-Owa fractional derivative operators (cf. [12], [13]).

The multiplier transformation of Srivastava [17] on $\mathcal{A}$ is the operator $I(r, \mu)$ on $\mathcal{A}$ defined by the following infinite series

$$I(r, \mu)f(z) := z + \sum_{k=2}^{\infty} \left(\frac{k + \mu}{1 + \mu}\right)^r a_k z^k. \tag{1.4}$$

A straightforward calculation shows that the multiplier operator satisfies

$$(1 + \mu)I(r + 1, \mu)f(z) = z[I(r, \mu)f(z)]' + \mu I(r, \mu)f(z). \tag{1.5}$$

The operator $I(r, 0)$ is the Sălăgean derivative operators [15]. The operator $I_{\mu}^{r} := I(r, \mu)$ was studied recently by Cho and Kim [5]. The operator $I_{r} := I(r, 1)$ was studied by Uralegaddi and Somanatha [18].

In this paper unless otherwise mentioned δ and γ are complex numbers.

2. Preliminaries

In our present investigation, we need the following definition and results to prove our main results.

Definition 2.1 [11, Definition 2, p. 817]**.** Let Q be the set of all functions f that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) := \{\zeta \in \partial\Delta : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.2 [10, Theorem 3.4h, p. 132]**.** *Let q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$.*

Set $\xi(z) = zq'(z)\phi(q(z))$, $h(z) = \theta\{q(z)\} + \xi(z)$. Suppose that,

(1) $\xi(z)$ is starlike univalent in Δ and

(2) $\operatorname{Re}\left\{\frac{zh'(z)}{\xi(z)}\right\} > 0$ for $z \in \Delta$.

If p is analytic in Δ with $p(\Delta) \subseteq D$, and

$$\theta(\{p(z)\} + zp'(z)\phi(p(z)) \prec \theta\{q(z)\} + zq'(z)\phi(q(z)), \tag{2.1}$$

then $p \prec q$ and q is the best dominant.

Lemma 2.3 [10]**.** *Let q be univalent in Δ with $q(0) = 1$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ and satisfying*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0. \tag{2.2}$$

If p is analytic in Δ, *with* $p(\Delta) \subseteq D$ *and*

$$\delta + \gamma \frac{zp'(z)}{p(z)} \prec \delta + \gamma \frac{zq'(z)}{q(z)},$$

then $p \prec q$ *and q is the best dominant.*

Lemma 2.4 [3]. *Let q be univalent in* Δ, ϑ *and* φ *be analytic in a domain D containing* $q(\Delta)$. *Suppose that*

(1) $\mathrm{Re}\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ *for* $z \in \Delta$, *and*

(2) $\xi(z) = zq'(z)\varphi(q(z))$ *is starlike univalent function in* Δ.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(\Delta) \subset D$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* Δ, *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.3}$$

then $q \prec p$ *and q is the best subordinant.*

Lemma 2.5 [11, Theorem 8, p. 822]. *Let q be convex univalent in* Δ *and satisfying* $\mathrm{Re}[\bar{\gamma}] > 0$. *If* $p \in \mathcal{H}[q(0), 1] \cap Q$ *and* $\delta + \gamma \frac{zp'(z)}{p(z)}$ *is univalent in* Δ, *then*

$$\delta + \gamma \frac{zq'(z)}{q(z)} \prec \delta + \gamma \frac{zp'(z)}{p(z)},$$

implies $q \prec p$ *and q is the best subordinant.*

3. Subordination and Superordination for Analytic Functions

By making use of Lemma 2.3, we obtain the following results.

Theorem 3.1. *Let* $0 \neq q(z)$ *be univalent in* Δ *with* $q(0) = 1$, *and satisfying*

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0. \tag{3.1}$$

If $f \in \mathcal{A}$ *satisfies*

$$\delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right] \prec \delta + \gamma \frac{zq'(z)}{q(z)},$$

then

$$z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \prec q(z),$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda}. \tag{3.2}$$

By taking the logarithmic derivative of $p(z)$ given by (3.2), we get

$$\frac{zp'(z)}{p(z)} = 1 + \lambda\left\{\frac{z(H^{l,m}[\alpha_1]f(z))'}{H^{l,m}[\alpha_1]f(z)} - 1\right\}. \tag{3.3}$$

By using the identity

$$z(H^{l,m}[\alpha_1]f(z))' = \alpha_1 H^{l,m}[\alpha_1 + 1]f(z) - (\alpha_1 - 1)H^{l,m}[\alpha_1]f(z),$$

and (3.2) in (3.3), we obtain

$$\frac{zp'(z)}{p(z)} = 1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}.$$

Now, our result follows as an application of Lemma 2.3. □

For $q(z) = \frac{1 + Az}{1 + Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, reduces to

Corollary 3.2. *Let q be univalent in* Δ *with* $q(0) = 1$. *If* $f \in \mathcal{A}$ *and*

$$\delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right] \prec \delta + \gamma\frac{(A - B)z}{(1 + Az)(1 + Bz)},$$

then

$$z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1 + Az}{1 + Bz}$ *is the best dominant.*

In particular, we have

$$\delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right] \prec \delta + \frac{2\gamma z}{1 - z^2},$$

implies

$$z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \prec \frac{1 + z}{1 - z},$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Taking $l = 2$, $m = 1$ and $\alpha_2 = 1$ in Theorem 3.1, we get

Corollary 3.3. *Let q be univalent in Δ with $q(0) = 1$. If $f \in \mathcal{A}$ and*

$$\delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{L[\alpha_1 + 1, \beta_1]f(z)}{L[\alpha_1, \beta_1]f(z)} - 1\right\}\right] \prec \delta + \gamma\frac{zq'(z)}{q(z)},$$

then

$$z\left(\frac{L(\alpha_1, \beta_1)f(z)}{z}\right)^{\lambda} \prec q(z),$$

where $L(\alpha_1, \beta_1)f$ is the familiar Carlson-Shaffer operator and q is the best dominant.

For $\alpha_1 = n + 1$ and $\beta_1 = 1$ in Corollary 3.3, we get the following corollary.

Corollary 3.4. *Let q be univalent in Δ. If $f \in \mathcal{A}$ and*

$$\delta + \gamma\left[1 + \lambda(n + 1)\left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right\}\right] \prec \delta + \gamma\frac{zq'(z)}{q(z)},$$

then

$$z\left(\frac{D^{n+1}f(z)}{z}\right)^{\lambda} \prec q(z),$$

where $D^n f$ is the Ruscheweyh operator and q is the best dominant.

Theorem 3.5. *Let q be convex univalent in Δ and $\operatorname{Re}[\bar{\gamma}] > 0$. If $f \in \mathcal{A}$,*

$$z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \in \mathcal{H}[1, 1] \cap Q,\ \delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right]$$

is univalent in Δ, *then*

$$\delta + \gamma \frac{zq'(z)}{q(z)} \prec \delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right],$$

implies

$$q(z) \prec z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda},$$

and q is the best subordinant.

Proof. Define the function $p(z)$ by

$$p(z) := z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda}.$$

Then a simple computation shows that

$$\delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right] = \delta + \gamma \frac{zp'(z)}{p(z)}.$$

An application of Lemma 2.5 gives the result.

Combining the results of subordination and superordination we get the following Sandwich theorem.

Theorem 3.6. *Let* q_1 *and* q_2 *be convex univalent in* Δ *satisfying* $\mathrm{Re}[\overline{\gamma}] > 0$ *and* (2.2) *respectively. If* $0 \neq z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \in \mathcal{H}[1, 1] \cap Q$, $\delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right]$ *is univalent in* Δ *and*

$$\delta + \gamma \frac{zq_1'(z)}{q_1(z)} \prec \delta + \gamma\left[1 + \lambda\alpha_1\left\{\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)} - 1\right\}\right] \prec \delta + \gamma \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec z\left(\frac{H^{l,m}[\alpha_1]f(z)}{z}\right)^{\lambda} \prec q_2(z),$$

where $q_1(z)$ *and* $q_2(z)$ *are respectively the best subordinant and best dominant.*

4. Application to Multiplier Transformation

Theorem 4.1. *Let* $0 \neq q(z)$ *be univalent in* Δ *with* $q(0) = 1$. *If* $f \in \mathcal{A}$ *and*

$$\delta + \gamma\left[1 + \lambda(1+\mu)\left\{\frac{I(r+1,\mu)f(z)}{I(r,\mu)f(z)} - 1\right\}\right] \prec \delta + \gamma\frac{zq'(z)}{q(z)},$$

then

$$z\left(\frac{I(r,\mu)f(z)}{z}\right)^{\lambda} \prec q(z),$$

and $q(z)$ *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := z\left(\frac{I(r,\mu)f(z)}{z}\right)^{\lambda}. \tag{4.1}$$

By taking the logarithmic derivative of $p(z)$ given by (4.1), we get

$$\frac{zp'(z)}{p(z)} = 1 + \lambda\left(\frac{z(I(r,\mu)f(z))'}{I(r,\mu)f(z)} - 1\right). \tag{4.2}$$

By using the identity

$$z[I(r,\mu)f(z)]' = (1+\mu)I(r+1,\mu)f(z) - \mu[I(r,\mu)f(z)]$$

and (4.1) in (4.2), we obtain

$$\frac{zp'(z)}{p(z)} = 1 + \lambda(1+\mu)\left\{\frac{I(r+1,\mu)f(z)}{I(r,\mu)f(z)} - 1\right\}.$$

Now, our result follows as an application of Lemma 2.3. □

We state the results pertaining to the superordination, using the duality between the subordination and the superordination.

Theorem 4.2. *Let* q *be convex univalent in* Δ *and* $q(0) = 1$. *If* $f \in \mathcal{A}$, $z\left(\frac{I(r,\mu)f(z)}{z}\right)^{\lambda} \in \mathcal{H}[1,1] \cap Q$, *and* $\delta + \gamma\left[1 + \lambda(1+\mu)\left\{\frac{I(r+1,\mu)f(z)}{I(r,\mu)f(z)} - 1\right\}\right]$ *is univalent in* Δ, *then*

$$\delta + \gamma\frac{zq'(z)}{q(z)} \prec \delta + \gamma\left[1 + \lambda(1+\mu)\left\{\frac{I(r+1,\mu)f(z)}{I(r,\mu)f(z)} - 1\right\}\right],$$

implies

$$q(z) \prec z\left(\frac{I(r, \mu)f(z)}{z}\right)^{\lambda},$$

and q is the best subordinant.

Combining the results of subordination and superordination, we state the following Sandwich theorem.

Theorem 4.3. *Let* q_1 *and* q_2 *be convex univalent in* Δ *satisfying* $\Re[\bar{\gamma}] > 0$ *and* (2.2) *respectively. If* $f \in \mathcal{A}$, $z\left(\frac{I(r, \mu)f(z)}{z}\right)^{\lambda} \in \mathcal{H}[1, 1] \cap Q$ *and* $\delta + \gamma\left[1 + \lambda(1 + \mu)\left\{\frac{I(r+1, \mu)f(z)}{I(r, \mu)f(z)} - 1\right\}\right]$ *is univalent in* Δ, *then*

$$\delta + \gamma\frac{zq_1'(z)}{q_1(z)} \prec \delta + \gamma\left[1 + \lambda(1 + \mu)\left\{\frac{I(r+1, \mu)f(z)}{I(r, \mu)f(z)} - 1\right\}\right] \prec \delta + \gamma\frac{zq_2'(z)}{q_2(z)},$$

implies

$$q_1(z) \prec z\left(\frac{I(r, \mu)f(z)}{z}\right)^{\lambda} \prec q_2(z),$$

where q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

For

$$q_1(z) = \frac{1 + A_1 z}{1 + B_1 z} \quad (-1 \le B_1 < A_1 \le 1),$$

$$q_2(z) = \frac{1 + A_2 z}{1 + B_2 z} \quad (-1 \le B_2 < A_2 \le 1),$$

we have the following corollary.

Corollary 4.4. *If* $f \in \mathcal{A}$,

$$z\left(\frac{I(r, \mu)f(z)}{z}\right)^{\lambda} \in \mathcal{H}[1, 1] \cap Q$$

and

$$\delta + \gamma\left[1 + \lambda(1 + \mu)\left\{\frac{I(r+1, \mu)f(z)}{I(r, \mu)f(z)} - 1\right\}\right]$$

is univalent in Δ, *then*

$$\delta + \gamma \frac{(A_1 - B_1)z}{(1 + A_1 z)(1 + B_1 z)} \prec \delta + \gamma\left[1 + \lambda(1 + \mu)\left\{\frac{I(r + 1, \mu)f(z)}{I(r, \mu)f(z)} - 1\right\}\right]$$

$$\prec \delta + \gamma \frac{(A_2 - B_2)z}{(1 + A_2 z)(1 + B_2 z)},$$

implies

$$\frac{1 + A_1 z}{1 + B_1 z} \prec z\left(\frac{I(r, \mu)f(z)}{z}\right)^{\lambda} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

The functions $\frac{1 + A_1 z}{1 + B_1 z}$ *and* $\frac{1 + A_2 z}{1 + B_2 z}$ *are respectively the best subordinant and best dominant.*

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